Numerical Analysis

A student taking the exam for "breadth" must answer three (3) questions and these must be handed in after two hours. A student taking the exam for "depth" must answer six (6) questions, with at least three (3) chosen from problems 6-10.

1.

(a) Describe the Gauß-Seidel iteration for the linear system

$$\begin{aligned} x + .1y &= 5 \\ 8x + y &= 2 \end{aligned}$$

and analyze its convergence.

(b) How can the ∞-norm of the iteration matrix be used, in this case, to determine the convergence/divergence of the method?

(c) Would the Jacobi iteration converge for the above system?

2. Consider the following $n \times n$ tridiagonal matrix:

$$A = \begin{pmatrix} 7 & 2 & 0 & 0 & . & . & . & & & 0 \\ 2 & 7 & 2 & 0 & 0 & . & . & . & & 0 \\ 0 & 2 & 7 & 2 & 0 & . & . & . & & 0 \\ . & . & . & . & . & . & . & . & & . \\ 0 & . & . & . & & & 0 & 2 & 7 & 2 \\ 0 & . & . & . & & & & 0 & 2 & 7 \end{pmatrix}$$

Provide bounds for the ∞-condition number and the 2-condition number of this matrix A.
Hint: Recall that

$$\|x\|_\infty = \max\{|x_k| : \ k = 1, ..., n\},$$

$$\|x\|_2 = \sqrt{\sum_{k=1}^{n} |x_k|^2},$$

and the condition number of A is $\|A\|\|A^{-1}\|$, with e.g., $\|A\| = \sup \frac{\|Ax\|}{\|x\|}$

3.

(a) Give an example of a real invertible symmetric matrix A and an initial guess $x_0 \neq 0$, for which the following version

$$\widetilde{x}_{k+1} = Ax_k, \quad x_{k+1} = \frac{\widetilde{x}_{k+1}}{\|\widetilde{x}_{k+1}\|}$$

of the power method does not converge.

(b) State and prove a necessary and sufficient condition on the real invertible symmetric matrix A for the power method to converge, *regardless of the initial guess* $x_0 \neq 0$.

4. The following is an algorithm for the evaluation of the cubic polynomial

$$p(x) = ax^3 + bx^2 + cx + d$$

at the infinite sequence of points

$$a_j = a + jh, \ j = 0, 1, 2, ...$$

(where a and h are some fixed numbers):

Step 1: Precompute (in any possible way) the numbers
$\alpha = p(a_3)$
$\beta = h\ p[a_2, a_3]$
$\gamma = 2h^2\ p[a_1, a_2, a_3]$
$\delta = 6h^3\ p[a_0, a_1, a_2, a_3]$.
(This requires the values $p(a_0)$, $p(a_1)$, $p(a_2)$ and $p(a_3)$.)
Step 2: For $j = 4, 5, 6, ...$ do:
$\gamma \longleftarrow \gamma + \delta$
$\beta \longleftarrow \beta + \gamma$
$\alpha \longleftarrow \alpha + \beta$
$p(a_j) \longleftarrow \alpha$

Show that the numbers $p(a_4), p(a_5), ...$ obtained in this way are indeed the required values of p. (**Hint:** show that the number β and γ obtained in each iteration of **Step 2** are $h\ p[a_{j-1}, a_j]$ and $2h^2\ p[a_{j-2}, a_{j-1}, a_j]$ resp.)

5. In order to find a solution for

$$x^2 = 1 + \ln x,$$

we use the fixed-point iteration

$$x_{k+1} = \frac{1 + \ln x_k}{x_k},$$

with $x_0 = 1.5$.
(a) Prove that $\lim_{k\to\infty} x_k = 1$.
(b) Prove that the convergence is faster than that of the bisection method.
(c) Is $x_0 = .6$ a better initial guess?

6. Show that the composite trapezoid rule

$$T_N f \approx \int_a^b f(x)\, dx,$$

with N equally spaced points on $[a, b]$, converges for any function f continuous on $[a, b]$.

7. Consider the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - (1 + \frac{1}{2}\sin x)u = 1 + x^2 \qquad \text{in } \Omega \tag{7.1}$$

$$u = 0 \qquad \text{on } \partial\Omega \tag{7.2}$$

Let Ω be the unit square, i.e., $0 < x, y < 1$. Consider the finite difference equation

$$U_{k+1,j} + U_{k-1,j} + U_{k,j+1} + U_{k,j-1} - [4 + h^2(1 + \frac{1}{2}\sin kh)]U_{kj} = (1 + (kh)^2)h^2$$

for $1 \le k, j \le N$, and

$$U_{k,0} = U_{k,N+1} = U_{0,j} = U_{N+1,j} = 0,$$

with $h := 1/(N+1)$.

(a) Discuss the convergence of U_{kj} to $u(x,y)$.

(b) Obtain a good bound for the finite difference solution, i.e., an M for which

$$|U_{kj}| \le M \qquad \text{for all } k, j, h.$$

8. Consider the boundary value problem

$$y'' - 100y = 1,$$

$$y(0) = 0, \ y(1) = 1.$$

Describe in detail how you would solve this problem numerically using the "shooting method". Your discussion should include some discussion of the reasons why (or why not) you might use "parallel shooting".

9. Consider, for the initial value problem

$$y' = -y, \ y(0) = 1, \tag{9.1}$$

each of the following two multistep methods:

(a)

$$Y_{k+1} - \frac{3}{2}Y_k + \frac{1}{2}Y_{k-1} = -\frac{1}{2}hY_k,$$

$$Y_0 = 1, \ Y_1 = 1 - h.$$

(b)

$$Y_{k+1} + Y_k - 2Y_{k-1} = -3hY_k,$$

$$Y_0 = 1, \ Y_1 = 1 - h.$$

Discuss the consistency, stability, numerical stability and convergence of these methods.

If you had to solve this problem *numerically* using one of these methods, which would you choose?

10. Show that the finite difference scheme

$$w_m^n = v_m^n + \frac{1}{8}\Delta t \; b \; \delta^2 v_m^n$$
$$v_m^{n+1} = v_m^n + \Delta t \; b \; \delta^2 w_m^n$$

for the heat equation $u_t = bu_{xx}$ (with $b > 0$) is stable for $b\Delta t \le 2(\Delta x)^2$.