Numerical Analysis

A student taking the exam for "breadth" must answer three (3) questions and these must be handed in after two hours. A student taking the exam for "depth" must answer six (6) questions, with at least three (3) chosen from problems 6-10.

1. In a graphics program destined for a very small machine, the distance

$$d := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

between two points (x_1, y_1) and (x_2, y_2) needs to be computed, but, for reasons of speed and memory requirements, the direct use of the squareroot function is not desirable.

How could you compute d to 'graphic accuracy', say within 1%, using just arithmetic operations (and comparisons)? Could you do it in about 10 floating point operations?

(Hint: Assuming $|x_1 - x_2| \geq |y_1 - y_2|$, one has $d = |x_1 - x_2|\sqrt{1 + r}$ with $r \in [0, 1]$.)

2. Consider the matrix

$$A := \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ -1 & -7 & 0 & 1 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ \frac{1}{2} & -1 & 0 & 11 & 0 \\ 0 & 1 & 0 & 0 & -3 \end{bmatrix}$$

(a) Prove that all eigenvalues of A are real and simple.
(b) Prove that A is diagonalizable (under similarity).

3. We are given three points (x_1, y_1, c_1), (x_2, y_2, c_2), (x_3, y_3, c_3) in 3-space and wish to find a (bivariate) polynomial p for which

$$p(x_j, y_j) = c_j, \qquad j = 1, 2, 3.$$

(a) Explain why it is not always possible to solve this problem with a *linear* polynomial (i.e., $p(x, y) = \alpha_0 + \alpha_1 x + \alpha_2 y$ for some constants α_0, α_1, α_2).
(b) Is it always possible to solve the problem with a quadratic polynomial (i.e., $p(x, y) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2$ for some constants α_0, α_1, α_2, α_3, α_4, α_5)?

4. Discuss a matrix bisection method for finding the eigenvalues of a real symmetric matrix.

5. What is the best you can say about the number $\int_0^1 f(x)dx$ if all you know about f is that (i) $f(0) = 0$, (ii) $f(1) = 1$, (iii) $\sup_{0 \leq x \leq 1} |Df(x)| \leq 2$? (Here, Df denotes the first derivative of f.)

6.
(a) Prove that Simpson's rule is exact for cubic polynomials and conclude from this that the error in the composite Simpson's rule is $O(h^4)$ when applied to any sufficiently smooth function.
(b) Prove that the composite Simpson's rule converges for all continuous functions.

7.

(a) Define the term 'approximate inverse' (of a bounded linear map).

(b) Give an instance in numerical calculations in which an approximate inverse is essential.

8. Consider the boundary value problem

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - (1 + \frac{1}{2}\sin x)u = 1 + x^2 \qquad \text{in } \Omega \tag{8.1}$$

$$u = 0 \qquad \text{on } \partial\Omega \tag{8.2}$$

Let Ω be the unit square, i.e., $0 < x, y < 1$. Consider the finite difference equation

$$U_{k+1,j} + U_{k-1,j} + U_{k,j+1} + U_{k,j-1} - [4 + h^2(1 + \frac{1}{2}\sin kh)]U_{kj} = (1 + (kh)^2)h^2$$

for $1 \leq k, j \leq N$, and

$$U_{k,0} = U_{k,N+1} = U_{0,j} = U_{N+1,j} = 0,$$

with $h := 1/(N+1)$.

(a) Discuss the convergence of U_{kj} to $u(x, y)$.

(b) Obtain a good bound for the finite difference solution, i.e., an M for which

$$|U_{kj}| \leq M \qquad \text{for all } k, j, h.$$

9. Show that the finite difference approximation

$$\begin{aligned} w_m^n &= v_m^n + \frac{1}{8}\Delta t\, b\, \delta^2 v_m^n \\ v_m^{n+1} &= v_m^n + \Delta t\, b\, \delta^2 w_m^n \end{aligned}$$

for the heat equation $u_t = bu_{xx}$ (with $b > 0$) is stable for $b\Delta \leq 2(\Delta x)^2$.

10. Consider the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(0) = y_0. \tag{10.1}$$

Construct a two-step linear multistep method for its (approximate) solution which is (i) 2nd order accurate, and (ii) unstable.