Fall 2003

1. This problem is concerned with the partial differential equation $u_t + u_x = 0$ on the interval $[0..1]$ with booundary data specified at the left-hand side, $u(t, 0) = b(t)$. Consider the forward-time, backward space scheme,

$$u_m^{n+1} = (1-\lambda)u_m^n + \lambda u_{m-1}^n$$

written in matrix form as

$$\begin{pmatrix} u_0^{n+1} \\ u_1^{n+1} \\ \vdots \\ u_M^{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \lambda & 1-\lambda & 0 & \dots & 0 \\ 0 & \lambda & 1-\lambda & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda & 1-\lambda \end{pmatrix} \begin{pmatrix} u_0^n \\ u_1^n \\ \vdots \\ u_M^n \end{pmatrix} + \begin{pmatrix} b(t_n) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $\lambda = \Delta t/\Delta x$.

By analyzing the eigenvalues and other properties of the matrix determine the (von Neumann) stability of the scheme when λ is a constant.

Solution Call the matrix C_M where the number of rows and columns is $M + 1$. Stability requires that there be a uniform bound on C_M^n independent of M for $0 \le n\Delta t \le T$. for each value of T.

The diagonal elements of C_M^n, other than the first 0, are $(1-\lambda)^n$ and so this quantity must be bounded. For $(1-\lambda)^n$ to be bounded we must have $|1-\lambda| \le 1$. (Actually, the upper limit can be $1 + K\Delta t$ for some constant K but then λ is not constant.) So we have $0 \le \lambda \le 2$, so far.

But we need more to get the uniform estimate, by looking at the vector $v_\pm = (1, -1, 1, -1, 1, \dots)$ we can compute $C_M^n v_\pm$ and see that the last element is like $\pm(1-2\lambda)^n$ when $n \le M$. So, we also need that $1 - 2\lambda$ is bounded by 1. This gives $0 \le \lambda \le 1$.

This conditions gives the maximumm norm of C_M to be 1. (This is the maximum of the row sums.) Thus in the maximum norm C_M^n is bounded by 1 for all M and n, hence all equivalent norms are bounded, including the 2-norm. This is the von Neumann stability condition.

2. This question deals with a discretization of the boundary value problem for the nonlinear partial differential equation $\nabla^2 u = \ln(1 + u)$ on the unit square with values of u specified on th e boundary. Consider using the discrete five-point Laplacian ∇_h^2 to approximate the differential Laplacian.

Prove that if a solution exists to $\nabla_h^2 u = \ln(1+u)$ with data specified on the boundary, then the solution must be unique.

Solution Using equal spacing for both x and y directions the finite difference scheme for this problem can be written as

$$u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right) - \frac{h^2}{4}\ln(1 + u_{i,j})$$

(At least, this is the most reasonable of schemes to use.)

If we assume that $v_{i,j}$ is also a solution of the scheme, then the difference $w_{i,j} = u_{i,j} - v_{i,j}$ satisfies

$$w_{i,j} = \frac{1}{4}\left(w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1}\right) - \frac{h^2}{4} w_{i,j} P_{i,j}$$

where

$$P_{i,j} = \frac{\ln(1 + u_{i,j}) - \ln(1 + v_{i,j})}{u_{i,j} - v_{i,j}}$$

is a **positive** quantity. Thus we have

$$w_{i,j} = \frac{1}{4(1 + h^2 P_{i,j})}\left(w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1}\right)$$

Thus at each point, $w_{i,j}$ is a true sub-average of its neighbors. Now we can use the usual argument for maximum principles. If $w_{i,j}$ were a positive maximum at an interior point, it can not be a sub-average of its neighbors. A similar argument applies to a negative minimum. This contradiction, shows that the only possible solution is for $w_{i,j}$ to be identically 0. That is, there is a unique solution if there is a solution.

3. The differential equation

$$y'(t) = \sqrt{y(t)}$$

with initial condition $y(0) = 0$ has the solution

$$y(t) = \frac{1}{4}t^2$$

The Euler scheme for this equation

$$Y_{k+1} - Y_k = h\sqrt{Y_k}$$

with initial condition $Y_0 = 0$ has the solution $Y_k = 0$ for all k.

Discuss why the solution of the finite difference scheme does not converge to the given solution of the differential equation. Mention appropriate theorems that give convergence results for finite difference schemes applied to differential equations.

Solution The reason that the usual theorems for convergence of differential equations do not apply here is that the differential equation does not satisfy a Lipschitz condition or similar condition. The ode does not have a unique solution, in fact, $y(t) = 0$ is a solution to the ode, so the solution of the scheme does converge (i.e., is) a solution of the ode.