A Bound on the Approximation Order of Surface Splines

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Abstract

The functions $\phi_m := |\cdot|^{2m-d}$ if d is odd, and $\phi_m := |\cdot|^{2m-d}\log|\cdot|$ if d is even, are known as *surface splines*, and are commonly used in the interpolation or approximation of smooth functions. We show that if one's domain is the unit ball in $\mathbb{R}^d$, then the approximation order of the translates of ϕ_m is at most m. This is in contrast to the case when the domain is all of $\mathbb{R}^d$ where it is known that the approximation order is exactly $2m$.

1 Introduction

The area of Radial Basis Function Approximation is motivated by the practical problem of approximating a smooth function $f : \mathbb{R}^d \to \mathbb{C}^d$ which is known only at finitely many scattered points $\Xi \subset \mathbb{R}^d$. A standard technique for approximation is that one starts with a radially symmetric function $\phi \in C(\mathbb{R}^d)$, and then seeks an interpolant of $f_{|\Xi}$ from the space spanned by the Ξ-translates of ϕ. In other words, one seeks

$$s \in span\{\phi(\cdot - \xi) : \xi \in \Xi\}$$

such that $s(\xi) = f(\xi)$ for all $\xi \in \Xi$. Of course there is, in general, no guarantee that such a function s exists, but conditions on ϕ which ensure the existence of such an s are known. However, for many interesting choices of the function ϕ, the above setup needs to be modified slightly in order to guarantee the existence of the interpolant s.

Definition 1 *Let $\phi \in C(\mathbb{R}^d)$ and $m \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. For finite pointsets $\Xi \subset \mathbb{R}^d$, we denote by*

$$S(\phi; \Xi, m)$$

the collection of all functions s of the form

$$s = \sum_{\xi\in\Xi} \lambda_\xi \phi(\cdot - \xi) + p,$$

where $p \in \Pi_{m-1} := \{$*polynomials of* $\deg \leq m-1\}$ *and the coefficients* $\{\lambda_\xi\}$ *satisfy*

$$\sum_{\xi \in \Xi} \lambda_\xi q(\xi) = 0 \textit{ for all } q \in \Pi_{m-1}.$$

Micchelli [8] has given general conditions on ϕ which ensure that for all f, there exists a unique $s \in S(\phi; \Xi, m)$ such that $s_{|_\Xi} = f_{|_\Xi}$. Examples of suitable ϕ are the *surface splines* given by

$$\phi_m := \begin{cases} \rho^{2m-d} & \text{if} \quad d \text{ is odd} \\ \rho^{2m-d} \log \rho & \text{if} \quad d \text{ is even} \end{cases}, \; m > d/2.$$

Here, $\rho : \mathbb{R}^d \to \mathbb{R}$ is given by $\rho(x) := |x|$, $x \in \mathbb{R}^d$. The surface splines, ϕ_m, arise naturally in the solution of a variational problem. Duchon [6] showed that of all functions which interpolate $f_{|_\Xi}$, the one which minimizes a certain semi-norm is the one which belongs to $S(\phi_m; \Xi, m)$. The results of the present work apply primarily to this example.

In order to raise the issue of approximation, let us suppose that we desire to approximate f on an open bounded domain $\Omega \subset \mathbb{R}^d$. One hopes that the interpolant s approximates f better and better as the pointset Ξ becomes 'dense' in Ω. To make this notion precise, let us measure the density of Ξ in Ω by

$$\delta := \delta(\Xi; \Omega) := \sup_{x \in \Omega} \min_{\xi \in \Xi} |x - \xi|.$$

In other words, δ is the smallest value for which $\Omega \subset \Xi + \delta B$, where $B := \{x \in \mathbb{R}^d : |x| < 1\}$ is the open unit ball in $\mathbb{R}^d$.The notion of 'approximation order' is typically used to describe the asymptotic rate at which the error decays when f, the function to be approximated, is smooth. Let us say that interpolation from $S(\phi; \Xi, m)$ provides *approximation of order* γ *in* Ω if

$$\|f - s\|_{L_\infty(\Omega)} = O(\delta^\gamma) \quad \text{as} \quad \delta \to 0$$

for all sufficiently smooth functions f, where $s \in S(\phi; \Xi, m)$ is the unique interpolant to $f_{|_\Xi}$. A basic problem is that of determining, for a given ϕ, the largest such γ; that is, determining the *approximation order* of interpolation from $S(\phi; \Xi, m)$. In the literature there are two rather distinct approaches to this problem.

One of the approaches starts by writing the interpolant $s \in S(\phi; \Xi, m)$ as the solution of a variational problem. The error can then be estimated by careful consideration of the variational problem. In the case of the surface spline ϕ_m, what is known is that if Ω is a compact subset of $\mathbb{R}^d$ satisfying a uniform interior cone condition, then interpolation from $S(\phi_m; \Xi, m)$ provides approximation of order $m - d/2$ (cf. [12], [11], [10]). That is

$$\|f - s\|_{L_\infty(\Omega)} = O(\delta^{m-d/2}) \text{ as } \delta := \delta(\Xi, \Omega) \to 0,$$

where $s \in S(\phi_m; \Xi, m)$ interpolates f at Ξ.

The other approach started under the simplifying assumptions $\Omega = \mathbb{R}^d$ and $\Xi = h\mathbb{Z}^d$. Of course these assumptions violate our initial setup, but it was hoped that the results and insights gained under these assumptions would shed light on the original problem. The reader is referred to the surveys [4],[1],[9] for descriptions of these works. For the sake of the present discussion, we mention only that for the surface spline, ϕ_m, it is known that interpolation from $S(\phi_m; h\mathbb{Z}^d, 0)$[1] provides approximation of order $2m$ in $\Omega = \mathbb{R}^d$.

Later, Buhmann, Dyn and Levin [2] were able to dismiss the assumption $\Xi = h\mathbb{Z}^d$, while still retaining the assumption $\Omega = \mathbb{R}^d$ (see also [5],[3]). By and large they obtained the same approximation orders as were obtained under the simplifying assumption $\Xi = h\mathbb{Z}^d$. It should be pointed our that their work was done not in the context of interpolation from $S(\phi; \Xi, 0)$, but rather in the context of approximation from $S(\phi; \Xi, 0)$. The difference being that one doesn't insist that the chosen $s \in S(\phi; \Xi, 0)$ actually interpolate $f_{|_\Xi}$. With regard to the surface spline, ϕ_m, they showed that one can approximate from $S(\phi_m; \Xi, 0)$ with approximation of order $2m$ in $\Omega = \mathbb{R}^d$. From their results, one can derive the following concerning the case when Ω is bounded: If $\widetilde{\Omega}$ is a compact subset of Ω (necessarily at a positive distance from the boundary of Ω), then as Ξ becomes dense in Ω, one can approximate from $S(\phi_m; \Xi, 0)$ with approximation of order $2m$ in $\widetilde{\Omega}$. In other words, it is possible to choose $s \in S(\phi_m; \Xi, 0)$ so that

$$\|f - s\|_{L_\infty(\widetilde{\Omega})} = O(\delta^{2m}) \text{ as } \delta := \delta(\Xi, \Omega) \to 0$$

for all sufficiently smooth f. This of course leaves open the question as to how well f can be approximated near the boundary of Ω.

The purpose of the present work is to examine this boundary question primarily in the context of the surface spline ϕ_m. Since we intend only to give an upper bound on the approximation order, we will consider only the nice case when the domain Ω is taken as the unit ball B. We will show that the approximation order of interpolation from $S(\phi_m; \Xi, m)$ is at most m. Note that this lies strictly between the values $m - d/2$ and $2m$ mentioned above. Our bound on the approximation order applies not just to the error of interpolation from $S(\phi_m; \Xi, m)$, but in fact to the error of *best approximation* from $S(\phi_m; \Xi, m)$ (or more generally, from $S(\phi_m; \Xi, \ell)$, $\ell = 0, 1, \ldots, m$). The arguments also apply in case one measures the error in the $L_p(B)$ norm. Here is the result:

Theorem 1 *For all $1 \le p \le \infty$ and $\ell = 0, 1, \ldots, m$, there exists $f \in C^\infty(\mathbb{R}^d)$ such that*

$$E(f, S(\phi_m; \Xi, \ell); L_p(B)) \ne o(\delta^{m+1/p}) \quad as \quad \delta \to 0.$$

Here, $E(f, V; X)$ denotes the error of best approximation, measured in the X norm, from the space V to the function f:

$$E(f, V; X) := \inf_{v \in V} \|f - v\|_X .$$

[1] $S(\phi; \Xi, 0)$ is often used in place of $S(\phi; \Xi, m)$ when $\Omega = \mathbb{R}^d$ or when one is not interpolating f.

Throughout this paper we use standard multi-index notation:

$$D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}, \ \alpha \in \mathbb{Z}_+^d.$$

The Laplacian operator is denoted by

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}.$$

For $\alpha \in \mathbb{Z}_+^d$, we define $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_d$, while for $x \in \mathbb{R}^d$ we define $|x| := \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$. The natural basis for $\mathbb{R}^d$ is denoted as $\{e_1, e_2, \ldots, e_d\}$. We employ the function $r : (0..\infty) \to \mathbb{R}$ defined by $r(t) := t$. For $t \in \mathbb{R}$, the greatest integer $\le t$ is denoted by $\lfloor t \rfloor$.

2 A reduction of the problem

Our plan for proving Theorem 1 is to exhibit (the existence of) a function $f \in C^\infty(\mathbb{R}^d)$ and a family of pointsets $\{\Xi_h\}_{h\in(0..1/2]}$ such that $\delta(\Xi_h; B) = O(h)$ as $h \to 0$, but

$$E(f, S(\phi_m; \Xi_h, \ell); L_p(B)) \neq o(h^{m+1/p}) \text{ as } h \to 0. \tag{1}$$

Let $\{\Xi_h\}_{h\in(0..1/2]}$ be any family of finite pointsets satisfying

$$\begin{aligned} \Xi_h &\subset (1-h)B \text{ and} \\ \delta(\Xi_h; B) &= O(h) \text{ as } h \to 0. \end{aligned}$$

As hinted in the introduction, when examining an approximation to f from $S(\phi_m; \Xi_h, \ell)$, we will focus our attention not on the entire domain B, but rather on just the boundary layer $B \backslash (1-h)B$. So rather than (1), we will in fact be showing

$$E(f, S(\phi_m; \Xi_h, \ell); L_p(B \backslash (1-h)B)) \neq o(h^{m+1/p}) \text{ as } h \to 0.$$

We can further reduce our focus to just the interval $[1-h..1]$ by employing the spherical averaging operator $R : C(\mathbb{R}^d) \to C([0..\infty))$ given by

$$Ru(t) := \frac{1}{c} \int_{|x|=1} u(tx)\, d\sigma(x), \text{ where } c := \int_{|x|=1} 1\, d\sigma(x).$$

Here, σ is the measure associated with spherical coordinates (see [7]). In words, $Ru(t)$ is the average of u over the sphere of radius t with center at the origin.

Lemma 2 *Let $\phi \in C(\mathbb{R}^d)$. If there exists $g \in C^\infty(\mathbb{R})$ such that*

$$E(g, R[S(\phi; \Xi_h, \ell)]; L_p[1-h..1]) \neq o(h^{k+1/p}) \text{ as } h \to 0, \tag{2}$$

then there exists $f \in C^\infty(\mathbb{R}^d)$ such that

$$E(f, S(\phi; \Xi_h, \ell); L_p(B)) \neq o(h^{k+1/p}) \quad \text{as} \quad h \to 0.$$

Proof. Let $1 \leq p \leq \infty$, and assume that for all $f \in C^\infty(\mathbb{R}^d)$,

$$E(f, S(\phi; \Xi_h, \ell); L_p(B)) = o(h^{k+1/p}) \quad \text{as} \quad h \to 0.$$

Let $g \in C^\infty(\mathbb{R})$. Since we are only concerned with g on the interval $[1/2..1]$, we may assume WLOG that g is identically 0 on a neighborhood of 0. Hence, $f := g \circ \rho \in C^\infty(\mathbb{R}^d)$. For the case $1 \leq p < \infty$, we note that if $s \in S(\phi; \Xi_h, \ell)$, then

$$\begin{aligned}\|g - Rs\|^p_{L_p[1-h..1]} &= \int_{1-h}^1 \left| \int_{|x|=1} (f(tx) - s(tx)) \tfrac{1}{c} d\sigma(x) \right|^p dt \\ &\leq \int_{1-h}^1 \int_{|x|=1} |f(tx) - s(tx)|^p \tfrac{1}{c} d\sigma(x) dt, \text{ by Jensen's inequality,} \\ &\leq 2^{d-1} \int_0^1 \int_{|x|=1} |f(tx) - s(tx)|^p \, t^{d-1} \tfrac{1}{c} d\sigma(x) dt = \tfrac{2^{d-1}}{c} \|f - s\|^p_{L_p(B)}.\end{aligned}$$

Taking p^{th} roots proves

$$\|g - Rs\|_{L_p[1-h..1]} \leq \frac{2^{(d-1)/p}}{c^{1/p}} \|f - s\|_{L_p(B)} \tag{3}$$

for the case $1 \leq p < \infty$. That (3) holds as well for the case $p = \infty$, can be seen by taking the limit as $p \to \infty$. The proof of the lemma is now completed by taking, in (3), the infimum over all $s \in S(\phi; \Xi_h, m)$. ■

Although $S(\phi_m; \Xi_h, \ell)$ has a rather complicated structure, the structure of $R[S(\phi_m; \Xi_h, \ell)]$ (the spherical averages of the functions in $S(\phi_m; \Xi_h, \ell)$) is surprisingly simple. We will show that there is an m-dimensional space $V_{m,\ell}$ (independent of Ξ) whose restriction to the interval $[1-h..1]$ always contains the restriction of $R[S(\phi_m; \Xi_h, \ell)]$ to the same interval. The upshot is that we can replace the (apparantly h-dependent) space $R[S(\phi_m; \Xi_h, \ell)]$ with the (h-independent) space $V_{m,\ell}$. This allows us to show, in a rather simple fashion, that there exists $g \in C^\infty(\mathbb{R})$ such that (2) holds with $k = m$.

Theorem 3 *Let $\phi \in C(\mathbb{R}^d)$. If there exists a k-dimensional space $V \subset C[1/2..1]$ such that*

$$R[S(\phi; \Xi_h, \ell)]\big|_{[1-h..1]} = V\big|_{[1-h..1]} \text{ for all } h \in (0..1/2],$$

then there exists $f \in C^\infty(\mathbb{R}^d)$ such that

$$E(f, S(\phi; \Xi_h, \ell); L_p(B)) \neq o(h^{k+1/p}) \quad \text{as} \quad h \to 0.$$

Proof. Assume to the contrary that

$$E(f, S(\phi; \Xi_h, \ell); L_p(B)) = o(h^{k+1/p}) \quad \text{for all } f \in C^\infty(\mathbb{R}^d).$$

Then by Lemma 2,

$$E(g, R[S(\phi; \Xi_h, \ell)]; L_p[1-h..1]) = o(h^{k+1/p}) \text{ for all } g \in C^\infty(\mathbb{R}).$$

Since $R[S(\phi; \Xi_h, \ell)]_{|[1-h..1]} = V_{|[1-h..1]}$, it follows that

$$E(g, \widetilde{V}; L_p[0..h]) = o(h^{k+1/p}) \text{ for all } g \in C^\infty(\mathbb{R}),$$

where $\widetilde{V} := \{v(1-\cdot) : v \in V\}$. Let $v_{n,h} \in \widetilde{V}$ be such that

$$\|r^n - v_{n,h}\|_{L_p[0..h]} = o(h^{k+1/p}) \text{ as } h \to 0, \text{ for } n = 0, 1, \ldots, k.$$

Let p' be the exponent conjugate to p. Note that even if $p > 1$, we still have

$$\begin{aligned} \|r^n - v_{n,h}\|_{L_1[0..h]} &\leq \|1\|_{L_{p'}[0..h]} \|r^n - v_{n,h}\|_{L_p[0..h]}, \text{ by Holder's inequality,} \\ &= h^{1/p'} \|r^n - v_{n,h}\|_{L_p[0..h]} = o(h^{k+1/p+1/p'}) = o(h^{k+1}). \end{aligned}$$

Since $\dim \widetilde{V} < k+1$, there exists scalars $c_{n,h}$ such that

$$\sum_{n=0}^{k} c_{n,h} v_{n,h} = 0 \text{ and } \max_{0 \leq n \leq k} |c_{n,h}| = 1, \ \forall\, h \in (0..1].$$

Hence,

$$\begin{aligned} \left\| \sum_{n=0}^{k} c_{n,h} r^n \right\|_{L_1[0..h]} &= \left\| \sum_{n=0}^{k} c_{n,h} \left(r^n - v_{n,h}\right) \right\|_{L_1[0..h]} \\ &\leq \sum_{n=0}^{k} |c_{n,h}| \, \|r^n - v_{n,h}\|_{L_1[0..h]} = o(h^{k+1}). \end{aligned}$$

Define

$$M := \min_{0 \leq n \leq k} E\left(r^n, span\{1, r, \ldots, r^{n-1}, r^{n+1}, \ldots, r^k\}; L_1[0..1]\right).$$

Since $\{1, r, \ldots, r^k\}$ is linearly independent in $L_1[0..1]$, it follows that $M > 0$. Note that

$$\inf \left\{ \left\| \sum_{n=0}^{k} a_n r^n \right\|_{L_1[0..1]} : \max_{0 \leq n \leq k} |a_n| \geq 1 \right\} \geq M$$

Hence,

$$\begin{aligned}\left\| \sum_{n=0}^{k} c_{n,h} r^n \right\|_{L_1[0..h]} &= h \left\| \sum_{n=0}^{k} c_{n,h} (hr)^n \right\|_{L_1[0..1]} \\ &= h^{k+1} \left\| \sum_{n=0}^{k} h^{n-k} c_{n,h} r^n \right\|_{L_1[0..1]} \geq h^{k+1} M\end{aligned}$$

because $\max_{0 \leq n \leq k} h^{n-k} |c_{n,h}| \geq 1$. Thus we have a contradiction. ∎

3 The space $V_{m,\ell}$

Definition 2 *For $n = 0, 1, \ldots, m$, define $v_n : (0..\infty) \to \mathbb{R}$ by*

$$v_n(t) := (\Delta^n \phi_m)(t e_1).$$

For $0 \leq \ell \leq m$, let $V_{m,\ell}$ be the subspace of $C^\infty(0..\infty)$ given by

$$V_{m,\ell} := span\{v_n : \ell \leq 2n \leq 2m-2\} + span\{1, r^2, r^4, \ldots, r^{2\lfloor(\ell-1)/2\rfloor}\}.$$

That $\dim V_{m,\ell} \leq m$ can be easily seen. Indeed, if ℓ is even, then

$$\begin{aligned}\dim V_{m,\ell} &\leq \#\{n : \ell/2 \leq n \leq m-1\} + \#\{0, 1, \ldots, \lfloor(\ell-1)/2\rfloor\} \\ &= (m - \ell/2) + (\ell/2) = m;\end{aligned}$$

while if ℓ is odd, then

$$\begin{aligned}\dim V_{m,\ell} &\leq \#\{n : \ell \leq 2n \leq 2m-2\} + \#\{0, 1, \ldots, \lfloor(\ell-1)/2\rfloor\} \\ &= \#\{n : (\ell+1)/2 \leq n \leq m-1\} + \#\{0, 1, \ldots, (\ell-1)/2\} \\ &= (m - (\ell+1)/2) + (\ell+1)/2 = m.\end{aligned}$$

Now, in view of Theorem 3, in order to prove Theorem 1 it suffices to show the following

Theorem 4 *For all $h \in (0..1/2]$,*

$$R[S(\phi_m; \Xi_h, \ell)]\big|_{[1-h..1]} = V_{m,\ell}\big|_{[1-h..1]}.$$

In order to prove 4, we need the following lemmata:

Lemma 5 *Let $0 \le a < b \le \infty$. If $g \in C^{2m}(a..b)$, then $\Delta^m(g \circ \rho) = 0$ on the annulus $\{x \in \mathbb{R}^d : a < |x| < b\}$ if and only if*

$$g \in span\{1, r^2, ..., r^{2m-2}\}$$
$$+ \begin{cases} span\{r^{2-d}, r^{4-d}, ..., r^{2m-d}\} & \text{if } d \text{ is odd} \\ span\{r^{2-d}, r^{4-d}, ..., r^{-2}, \log r, r^2 \log r, ..., r^{2m-d} \log r\} & \text{if } d \text{ is even} \end{cases}$$

Proof. It is a straightforward matter to verify firstly that

$$\Delta(g \circ \rho) = g''(\rho) + \frac{d-1}{\rho} g'(\rho),$$

and secondly that

$$\Delta^m(g \circ \rho) = g^{(2m)}(\rho) + p_{2m-1}(\rho) g^{(2m-1)}(\rho) + \cdots + p_0(\rho) g(\rho),$$

for some functions $p_0, \ldots, p_{2m-1} \in C^\infty(0..\infty)$. Hence, we can appeal to the classical differential equations theory to conclude that the solution space of the linear DEQ

$$g^{(2m)} + p_{2m-1} g^{(2m-1)} + \cdots + p_0 g = 0$$

has dimension $2m$. Since the proposed space of solutions has dimension $2m$, it suffices to simply show that these are solutions of the equation. For $m = 1$, this can be verified by inspection. Continuing by induction, assume the Theorem is true for all $m' < m$. Consider m. It suffices to show that

1) $\Delta(\rho^{2m-2}) \in span\{\rho^{2(m-1)-2}\}$
2) If d is odd or $2m - d < 0$, then $\Delta(\rho^{2m-d}) \in span\{\rho^{2(m-1)-d}\}$
3) If d is even and $2m = d$, then $\Delta(\rho^{2m-d} \log \rho) \in span\{\rho^{2(m-1)-d}\}$
4) If d is even and $2m - d > 0$, then $\Delta(\rho^{2m-d} \log \rho) \in span\{\rho^{2(m-1)-d} \log \rho, \rho^{2(m-1)-2}\}$

Each of these can be easily verified. ∎

Lemma 6 *Let $u \in C(\mathbb{R}^d)$, and for $t > 0$ define $f_t : \mathbb{R}^d \to \mathbb{R}$ by*

$$f_t(\xi) := R[u(\cdot - \xi)](t).$$

If $u \in C^\infty(\mathbb{R}^d \backslash 0)$, then $f_t \in C^\infty(tB)$ and

$$D^\alpha f_t = \frac{(-1)^{|\alpha|}}{c} \int_{|x|=1} (D^\alpha u)(tx - \cdot)\, d\sigma(x) \text{ on } tB \tag{4}$$

for all $\alpha \in \mathbb{Z}_+^d$.

Proof. We begin by noting that

$$f_t(\xi) = R[u(\cdot - \xi)](t) = \frac{1}{c}\int_{|x|=1} u(tx-\xi)\, d\sigma(x).$$

Clearly $f_t \in C(tB)$ and (4) holds for $\alpha = 0$. Proceeding by induction assume that $f_t \in C^k(tB)$ and (4) holds for all $|\alpha| \le k$. Let $\alpha, \beta \in \mathbb{Z}_+^d$ be such that $|\alpha| = k$ and $|\beta| = 1$. Let $y \in tB$. Then for h of sufficiently small magnitude,

$$\begin{aligned}
&\frac{D^\alpha f_t(y+h\beta) - D^\alpha f_t(y)}{h} \\
= \; & \frac{(-1)^k}{c}\int_{|x|=1} \frac{D^\alpha u(tx-y-h\beta) - D^\alpha u(tx-y)}{h} d\sigma(x) \\
\to \; & \frac{(-1)^{k+1}}{c}\int_{|x|=1} D^{\alpha+\beta} u(tx-y)\, d\sigma(x) \text{ as } h \to 0,
\end{aligned}$$

by the Dominated Convergence Theorem. Therefore,

$$D^\beta D^\alpha f_t = \frac{(-1)^{|\alpha+\beta|}}{c}\int_{|x|=1} (D^{\alpha+\beta}u)(tx-\cdot)\, d\sigma(x) \text{ on } tB.$$

Note that $D^\beta D^\alpha f_t$ is continuous on tB. Therefore, $f_t \in C^{k+1}(tB)$ and (4) holds for all $|\alpha| \le k+1$. ■

Lemma 7 *If $p \in \Pi_k$, then $Rp \in span\{1, r^2, r^4, \ldots, r^{2\lfloor k/2 \rfloor}\}$.*

Proof. Since Π_k is spanned by homogeneous polynomials and since R is linear, it suffices to show that if $q \in \Pi_k$ is homogeneous of order n, then $Rp \in span\{r^n\}$ if n is even and $Rp = 0$ if n is odd. So let $q \in \Pi_k$ be homogeneous of order n and consider first the case n even. If $t > 0$, then

$$Rq(t) = \frac{1}{c}\int_{|x|=1} q(tx)\, d\sigma(x) = t^n \frac{1}{c}\int_{|x|=1} u(x)\, d\sigma(x) = t^n Rq(1).$$

Hence $Rq \in span\{r^n\}$. On the other hand, if n is odd, then

$$\begin{aligned}
Rq(t) &= \frac{1}{c}\int_{|x|=1} q(tx)\, d\sigma(x) \\
&= \frac{1}{c}\int_{|x|=1} q(t(-x))\, d\sigma(x), \text{ since } \sigma \text{ is a symmetric measure,} \\
&= -\frac{1}{c}\int_{|x|=1} q(tx)\, d\sigma(x) = -Rq(t).
\end{aligned}$$

Hence $Rq = 0$. ■

Proof. (*of Theorem 4*) Let $s \in S(\phi_m; \Xi_h, \ell)$, say

$$s = \sum_{\xi \in \Xi_h} \lambda_\xi \phi_m(\cdot - \xi) + p$$

for some $p \in \Pi_{\ell-1}$ and coefficients satisfying

$$\sum_{\xi \in \Xi} \lambda_\xi q(\xi) = 0 \text{ for all } q \in \Pi_{\ell-1}. \tag{5}$$

Then $Rs = \sum_{\xi \in \Xi_h} \lambda_\xi R[\phi_m(\cdot - \xi)] + Rp$. By Lemma 7,

$$Rp \in span\{1, r^2, r^4, \ldots, r^{2\lfloor(\ell-1)/2\rfloor}\}. \tag{6}$$

We turn now to analysing $R[\phi_m(\cdot - \xi)]$. As in Lemma 6, for $t > 0$, define

$$f_t(\xi) := R[\phi_m(\cdot - \xi)](t), \;\; \xi \in \mathbb{R}^d.$$

Since $\phi_m \in C^\infty(\mathbb{R}^d \backslash 0)$, we know, by Lemma 6, that $f_t \in C^\infty(tB)$ and

$$D^\alpha f_t = \frac{(-1)^{|\alpha|}}{c} \int_{|x|=1} (D^\alpha \phi_m)(tx - \cdot)\, d\sigma(x) \text{ on } tB \text{ for all } \alpha \in \mathbb{Z}_+.$$

Hence, since $\Delta^m \phi_m = 0$ on $\mathbb{R}^d \backslash 0$, it follows that $\Delta^m f_t = 0$ on tB. Therefore, by Lemma 5 (note that $f_t \in C^\infty(tB)$ has no singularity at 0), $f_t \in span\{1, \rho^2, \ldots, \rho^{2m-2}\}$. Hence there exist $c_0(t), c_1(t), \ldots, c_{m-1}(t)$ such that

$$f_t = \sum_{n=0}^{m-1} c_n(t) \rho^{2n} \text{ on } tB. \tag{7}$$

The coefficients $c_n(t)$ can be determined in a straightforward fashion: First, one shows that

$$(\Delta^k \rho^{2n})(0) = \begin{cases} 0 & \text{if } \; k \neq n \\ a_n & \text{if } \; k = n \end{cases},$$

where $a_n := \prod_{j=0}^{n-1} 2(n-j)(2(n-j)+d-2)$ is positive for $n \in \mathbb{Z}_+$. Applying Δ^n to (7) and evaluating at 0 yields $(\Delta^n f_t)(0) = a_n c_n(t)$. Hence

$$\begin{aligned} c_n(t) &= \frac{1}{a_n}(\Delta^n f_t)(0) = \frac{1}{a_n c} \int_{|x|=1} (\Delta^n \phi_m)(tx)\, d\sigma(x) \\ &= \frac{1}{a_n} \Delta^n \phi_m(te_1) \text{ , since } \Delta^n \phi_m \text{ is radially symmetric,} \\ &= \frac{1}{a_n} v_n(t). \end{aligned}$$

Therefore,

$$R[\phi_m(\cdot - \xi)](t) = \sum_{n=0}^{m-1} \frac{1}{a_n} v_n(t) |\xi|^{2n}, \ |\xi| < t.$$

Now, if $t \in [1-h..1]$, then $|\xi| < t$ for all $\xi \in \Xi_h$. Hence,

$$\begin{aligned} \sum_{\xi \in \Xi_h} \lambda_\xi R[\phi_m(\cdot - \xi)](t) &= \sum_{n=0}^{m-1} \frac{1}{a_n} v_n(t) \sum_{\xi \in \Xi_h} \lambda_\xi |\xi|^{2n} \\ &= \sum_{\ell \le 2n \le 2m-2} \frac{1}{a_n} v_n(t) \sum_{\xi \in \Xi_h} \lambda_\xi |\xi|^{2n} \text{ , by (5).} \end{aligned}$$

Thus $\sum_{\xi \in \Xi_h} \lambda_\xi R[\phi_m(\cdot - \xi)]_{|_{[1-h..1]}} \in span\{v_{n}{}_{|_{[1-h..1]}} : \ell \le 2n \le 2m-2\}$ which, in view of (6), completes the proof. ■

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