The polynomials in the linear span of integer translates of a compactly supported function

Carl de Boor

Abstract. Algebraic facts about the space of polynomials contained in the span of integer translates of a compactly supported function are derived and then used in a discussion of the various quasi–interpolants from that span.

0. Introduction

This note was stimulated by the recent papers [CD], [CJW], and [CL] in which the authors take a new look at the space of integer translates of box splines and, in particular, introduce and highlight the *commutator* of a locally supported pp function φ of several variables. The intent of this note is to offer alternative proofs of some of these results, and to point to some connections with earlier work (e.g., [BH], [DM83], [BJ]), but also to focus more attention on the space Π_φ of *all* polynomials contained in the span of the integer translates of the box spline (or other compactly supported) φ.

The first section collects simple algebraic facts about Π_φ and the action of the linear map

$$\varphi{*}' : f \mapsto \sum_{j\in\mathbb{Z}^d} \varphi(\cdot - j) f(j)$$

on it.

The second section records that Π_φ is invariant under differentiation and translation, and brings yet another characterization of Π_φ, this time in terms of the Fourier transform of φ.

The final section makes use of these facts about Π_φ in a discussion of the various quasi–interpolants available.

Throughout, I will use standard multi–index notation. I find it convenient to use the special symbol $[\![\,]\!]^\alpha$ for the **normalized monomial of degree** α, i.e., for the map given by the rule

$$[\![\,]\!]^\alpha : \mathbb{R}^d \to \mathbb{R} : x \mapsto x^\alpha/\alpha!.$$

With this,

$$\Pi_\alpha := \operatorname{span}([\![\,]\!]^\beta)_{\beta\le\alpha}$$

denotes the space of all polynomials of degree $\le \alpha$, and

$$\Pi_k := \operatorname{span}([\![\,]\!]^\beta)_{|\beta|\le k}, \qquad \Pi_{<k} := \operatorname{span}([\![\,]\!]^\beta)_{|\beta|<k}, \qquad \Pi := \operatorname{span}([\![\,]\!]^\beta)$$

have similarly obvious meaning.

1. The polynomials

Consider the span of integer translates of a compactly supported function φ on $\mathbb{R}^d$, i.e.,

(1.1) $$S := S_\varphi := \{\varphi{*}c : c \in \mathbb{R}^{\mathbb{Z}^d}\}.$$

Here I use the convolution product notation

(1.2) $$\varphi{*}c := \sum_{j\in\mathbb{Z}^d} \varphi(\cdot - j)c(j)$$

since there is no danger of confusion with either the continuous or the discrete convolution product. I find it convenient to use the special notation

(1.3) $$\varphi{*}'f := \varphi{*}f_{|\mathbb{Z}^d} = \sum_{j\in\mathbb{Z}^d} \varphi(\cdot - j) f(j)$$

in case f is a function on $\mathbb{R}^d$, in order to stress the semidiscrete character of this product. Further, since the restriction to $\mathbb{Z}^d$ of a function on $\mathbb{R}^d$ occurs often here, I will employ the abbreviation

$$f_| := f_{|\mathbb{Z}^d}$$

for it.

The asymmetry in the semidiscrete convolution product (1.3) is not all that strong since, after all,

$$\varphi *' f = f *' \varphi \quad \text{on} \quad \mathbb{Z}^d.$$

This implies, e.g., that, for $f \in \Pi$ (hence $f *' \varphi \in \Pi$),

$$\varphi *' f = f *' \varphi \Leftrightarrow \varphi *' f \in \Pi,$$

hence

$$\Pi_\varphi := \{f \in \Pi : \varphi *' f \in \Pi\} = \{f \in \Pi : \varphi *' f = f *' \varphi\}. \tag{1.4}$$

It also implies that

$$\varphi *' f = f *' \varphi \qquad \text{for all} \qquad f \in S, \tag{1.5}$$

since, for $f = \varphi * c$,

$$\begin{aligned} \varphi *' f &= \varphi * (\varphi_| * c) \\ &= \varphi * (c * \varphi_|) \\ &= (\varphi * c) * \varphi_| = f *' \varphi. \end{aligned}$$

As a consequence, one gets the inclusion

$$\Pi \cap S \subseteq \{f \in \Pi : \varphi *' f = f *' \varphi\} = \Pi_\varphi, \tag{1.6}$$

and the conclusion that

$$\varphi *' : f \mapsto \varphi *' f$$

maps Π_φ into $\Pi \cap S$. This implies that there must be equality throughout (1.6) as soon as the linear map

$$L := \varphi *'_{|\Pi_\varphi}$$

can be shown to be 1–1. But that is easy to do under the assumption that φ is **normalized**, i.e.,

$$\sum_{j \in \mathbb{Z}^d} \varphi(j) = 1.$$

For, under this assumption,

$$\text{for } f \in \Pi_\varphi, \qquad \begin{array}{ccccc} \varphi *' f & = & f \sum_{j \in \mathbb{Z}^d} \varphi(j) & - & \sum_{j \in \mathbb{Z}^d} (f - f(\cdot - j)) \varphi(j) \\ & \in & f & + & \Pi_{<\deg f} \end{array} \tag{1.7}$$

since, for each j, $f - f(\cdot - j) \in \Pi_{<\deg f}$.

The salient facts of this discussion are gathered in the following.

Proposition 1.1. *If φ is normalized, then*

$$
\begin{aligned}
(1.8) \qquad \Pi_\varphi : &= \{f \in \Pi : \varphi *' f \in \Pi\} &=& \{f \in \Pi : \varphi *' f = f *' \varphi\} \\
&= \Pi \cap S &=& \{f \in \Pi : \varphi *' f \in f + \Pi_{<\deg f}\}.
\end{aligned}
$$

Further, $L := \varphi *'_{|\Pi_\varphi}$ is onto, and

$$U := 1 - L \tag{1.9}$$

is degree–reducing. In particular,

$$L(\Pi_\varphi \cap \Pi_\alpha) = \Pi_\varphi \cap \Pi_\alpha. \tag{1.10}$$

As a consequence, $U^k = 0$ on

$$\Pi_{\varphi,k} := \Pi_\varphi \cap \Pi_{<k}.$$

Therefore

$$(L_{|\Pi_{\varphi,k}})^{-1} = (1 + U + \cdots + U^{k-1})_{|\Pi_{\varphi,k}}. \tag{1.11}$$

Note that Π_φ *is necessarily finite dimensional*, since φ is compactly supported. Precisely, for any bounded set G, the set

$$A(G) := \{\alpha \in \mathbb{Z}^d : \varphi(\cdot - \alpha)_{|G} \neq 0\}$$

is finite, hence if G also has interior, then

$$\dim \Pi_\varphi = \dim \Pi_{\varphi|G} \leq \#A(G) < \infty.$$

The sharpest bound attainable this way for a piecewise continuous φ would be

$$\dim \Pi_\varphi \leq \max_x \#A(\{x\}). \tag{1.12}$$

In any case, this implies that

$$L^{-1} = 1 + U + U^2 + \cdots,$$

with the Neumann series actually finite.

The assumption that φ be normalized is no real restriction except when

$$\sum_{j \in \mathbb{Z}^d} \varphi(j) = 0.$$

In this case, (1.7) shows L to be degree–reducing, hence in particular, not invertible. Consequently, $\Pi \cap S$ may be strictly smaller than Π_φ. For example, with $\varphi = 1$ on $[-1, 0[$, $= -1$ on $[0, 1[$, and $= 0$ otherwise, $\Pi_\varphi = \Pi_1 \neq \Pi_0 = \Pi \cap S$.

2. Invariance

Denote by E the **multivariate shift**, i.e.,

$$E^\alpha f := f(\cdot + \alpha), \qquad \alpha \in \mathbb{Z}^d.$$

While it is obvious that $\varphi *'$ commutes with E, hence Π_φ is invariant under E, some of the other properties of Π_φ derivable from this fact may not be as immediate.

Proposition 2.1. *The linear map $L = \varphi *'_{|\Pi_\varphi}$ commutes with differentiation, hence with translation, i.e.,*

$$LD^\alpha = D^\alpha L, \forall \alpha \in \mathbb{Z}_+^d, \qquad E^y L = LE^y, \forall y \in \mathbb{R}^d. \tag{2.1}$$

Proof: Since Π_φ is a finite–dimensional polynomial subspace, there exists, for each $\alpha \in \mathbb{Z}_+^d$, a weight sequence w of finite support so that

$$D^\alpha = \sum_{\beta \in \mathbb{Z}_+^d} w(\beta) E^\beta \qquad \text{on } \Pi_\varphi. \tag{2.2}$$

(For example, with l_i the Lagrange polynomials for the points $0, \ldots, k := \max\deg \Pi_\varphi$, we have

$$p = \sum_{0 \le \beta(j) \le k} l^\beta E^\beta p(0)$$

for all $p \in \Pi_k(\mathbb{R}) \otimes \cdots \otimes \Pi_k(\mathbb{R}) \supseteq \Pi_\varphi$, hence $w(\beta) := D^\alpha l^\beta(0)$, all β, would do.) Thus, $LE = EL$ implies $LD = DL$. But this finishes the proof since

$$E^y = \sum_\alpha [\![y]\!]^\alpha D^\alpha. \tag{2.3}$$

Q.E.D.

Remark. The argument shows that any E–invariant polynomial subspace is D–invariant, hence even translation–invariant, i.e., for any linear subspace P of Π,

$$\begin{aligned} \forall \alpha \in \mathbb{Z}^d, E^\alpha P \subseteq P \ &\Rightarrow\ \forall \alpha \in \mathbb{Z}^d, D^\alpha P \subseteq P \\ &\Rightarrow\ \forall y \in \mathbb{R}^d, E^y P \subseteq P. \end{aligned} \tag{2.4}$$

Corollary. Π_φ *is D–invariant and translation–invariant.*

As a simple consequence, consider the polynomials g_α defined in [CJW] by the recurrence

$$g_\alpha(x) := x^\alpha - \sum_{j \in \mathbb{Z}^d} \varphi(j) \sum_{\beta \ne \alpha} \binom{\alpha}{\beta} (-j)^{\alpha-\beta} g_\beta(x) \tag{2.5}$$

and then shown to satisfy

$$x^\alpha = \sum_{j \in \mathbb{Z}^d} g_\alpha(j) \varphi(x - j) \tag{2.6}$$

in case $|\alpha| < m$ and $\Pi_{<m} \subset \Pi_\varphi$. In other words, $g_\alpha = L^{-1}()^\alpha$.

Since (2.6) is, offhand, the reason for our interest in the g_α, it would seem more direct to *define* the g_α by (2.6), i.e., to set

$$g_\alpha := L^{-1}()^\alpha, \tag{2.7}$$

and then to verify that necessarily (2.5) holds for these g_α, as follows:

$$()^\alpha = L^{-1}(\varphi *' ()^\alpha) = L^{-1}(()^\alpha *' \varphi) = \sum_j \varphi(j) \sum_{\beta\le\alpha} L^{-1}()^\beta \binom{\alpha}{\beta} (-j)^{\alpha-\beta}$$
$$= g_\alpha + \sum_j \varphi(j) \sum_{\beta<\alpha} g_\beta \binom{\alpha}{\beta} (-j)^{\alpha-\beta}$$

using Proposition 1 and the normalization $\sum_j \varphi(j) = 1$. This even shows the validity of (2.6) for any α for which $()^\alpha \in \Pi_\varphi$, since then also $()^\beta \in \Pi_\varphi$ for all $\beta \le \alpha$, hence the definition $g_\beta := L^{-1}()^\beta$ makes sense for all such β.

This leaves unanswered the question of whether the two definitions, (2.5) and (2.7), are equivalent, at least for the range of α for which they both make sense. It also raises the question as to the nature of the polynomials g_α defined by (2.5) when $()^\alpha \notin \Pi_\varphi$.

To answer these, recall that the **Appell sequence** for a continuous linear functional μ on $C(\mathbb{R}^d)$ with $\mu(1) = 1$ is, by definition, the sequence (g_α) determined by the conditions

$$g_\alpha \in \Pi_\alpha, \qquad \mu D^\beta g_\alpha = \delta_{\beta\alpha}.$$

There is, in fact, exactly one such sequence for given μ since the linear system

$$\mu D^\beta \Big(\sum_{\gamma\le\alpha} [\![\,]\!]^\gamma a_\gamma \Big) = \delta_{\beta\alpha}$$

for the power coefficients (a_γ) for g_α has a unit triangular coefficient matrix. Backsubstitution therefore provides the formula

$$g_\alpha = [\![\,]\!]^\alpha - \sum_{\beta\ne\alpha} \mu [\![\,]\!]^{\alpha-\beta} g_\beta,$$

whose correctness can also be verified directly by induction on α:

$$\begin{aligned} \mu D^\gamma g_\alpha &= \mu D^\gamma [\![\,]\!]^\alpha - \textstyle\sum_{\beta\ne\alpha} \mu [\![\,]\!]^{\alpha-\beta} \mu D^\gamma g_\beta \\ &= \mu [\![\,]\!]^{\alpha-\gamma} - \mu [\![\,]\!]^{\alpha-\gamma} = 0 \end{aligned}$$

for $\gamma < \alpha$, while $\mu D^\alpha g_\alpha = \mu D^\alpha [\![\,]\!]^\alpha = \mu(1) = 1$. With existence and uniqueness established, facts about the Appell sequence, such as symmetries which reflect those of μ, or that $D^\beta g_\alpha = g_{\alpha-\beta}$, follow immediately.

In our case, $\mu : f \mapsto \varphi *' f(0)$, hence, for $[\![\,]\!]^\alpha \in \Pi_\varphi$,

$$\delta_{\beta\alpha} = \mu D^\beta g_\alpha = \varphi *' (D^\beta g_\alpha)(0) = D^\beta(\varphi *' g_\alpha)(0),$$

which, together with the fact that $\varphi *' g_\alpha \in L\Pi_\alpha = \Pi_\alpha$, shows that

$$\varphi *' g_\alpha = [\![\,]\!]^\alpha. \tag{2.6$'$}$$

The resulting different normalization of g_α as compared with (2.5) or (2.7) avoids all those factorials.

Dahmen and Micchelli [DM83] consider the polynomial space

$$\{p \in \Pi : p(D)\widehat{\varphi} = 0 \text{ on } 2\pi\mathbb{Z}^d \backslash 0\}, \tag{2.8}$$

with $\widehat{\varphi}$ the Fourier transform of φ. It seems slightly more convenient to consider instead

$$\widetilde{\Pi}_\varphi := \{p \in \Pi : p(-\mathrm{i}D)\widehat{\varphi} = 0 \text{ on } 2\pi\mathbb{Z}^d \backslash 0\}.$$

They prove that any affinely invariant (i.e., translation– and scale–invariant) subspace of (2.8), hence of $\widetilde{\Pi}_\varphi$, is contained in Π_φ. But their proof can be made to show more.

Proposition 2.2. Π_φ *is the largest* E*–invariant subspace of* $\widetilde{\Pi}_\varphi$*.*

Proof: The proof in [DM83] is based on the observation that, by Poisson's summation formula `[ would need some assumptions, else mollify]`,

$$\varphi *' p(x) = \sum_\alpha \varphi(x-\alpha)p(\alpha) =: \sum_\alpha \psi(\alpha) = \sum_\alpha \widehat{\psi}(2\pi\alpha),$$

while, for any $p \in \Pi$, the function $\psi : y \mapsto \varphi(x-y)p(y)$ has the Fourier transform

$$\widehat{\psi}(\eta) = \mathbf{e}^{-\mathrm{i}x\eta}(p(x-\mathrm{i}D)\widehat{\varphi})(-\eta).$$

If now $p \in P$, with P an E–invariant (hence D–invariant) subspace of $\widetilde{\Pi}_\varphi$, then

$$\widehat{\psi}(2\pi\alpha) = (p(x-\mathrm{i}D)\widehat{\varphi})(2\pi\alpha) = \sum_\beta [\![x]\!]^\beta (D^\beta p(-\mathrm{i}D)\widehat{\varphi})(2\pi\alpha) = 0$$

for $\alpha \neq 0$, hence

$$\begin{aligned} \varphi *' p(x) &= (p(x-\mathrm{i}D)\widehat{\varphi})(0) \\ &= \sum_\alpha D^\alpha p(x) [\![-\mathrm{i}D]\!]^\alpha \widehat{\varphi}(0) \\ &= p(x)\widehat{\varphi}(0) + \sum_{|\alpha|>0} D^\alpha p(x) [\![-\mathrm{i}D]\!]^\alpha \widehat{\varphi}(0), \end{aligned} \tag{2.9}$$

showing that $\varphi *' p \in \Pi$, i.e., $p \in \Pi_\varphi$.

On the other hand, if $p \in \Pi_\varphi$, then

$$\varphi *' p = \sum_\alpha \mathbf{e}^{-2\pi\mathrm{i}\alpha()}(p(\cdot - \mathrm{i}D)\widehat{\varphi})(-2\pi\alpha)$$

is a polynomial, and this is possible only if

$$p(\cdot - \mathrm{i}D)\widehat{\varphi}(2\pi\alpha) = 0, \qquad \forall \alpha \neq 0,$$

showing that $p \in \widetilde{\Pi}_\varphi$. Q.E.D

While Π_φ has been shown in [BH] to be dilation–invariant in case φ is a box spline, it is not clear that Π_φ is necessarily dilation–invariant for arbitrary φ. For this, I note that a polynomial subspace P is dilation–invariant if and only if P **stratifies**, i.e., $P = \sum_k P \cap \Pi_k^0$, with

$$\Pi_k^0 := \mathrm{span}([\![\,]\!]^\alpha)_{|\alpha|=k}.$$

Hence, $\mathrm{span}\{[\![\,]\!]^{2,0} + [\![\,]\!]^{0,1}, [\![\,]\!]^{1,0}, 1\}$ provides a simple example of an E–invariant polynomial subspace which is not dilation–invariant.

3. Quasi–interpolants

The space Π_φ is of interest because it characterizes the local approximation order obtainable from S, or, more precisely, from the **ladder** (S_h) associated with S. To recall,

$$S_h := \sigma_h(S),$$

with

$$\sigma_h f : x \mapsto f(x/h).$$

Further, the **local approximation order** of S is the largest k for which

$$\operatorname{dist}(f, S_h) = O(h^k)$$

for all smooth f, with the distance measured in some norm, e.g., the max–norm on some bounded domain, and the support of the approximation to f within h of the support of f.

In [FS], Fix and Strang give a characterization of the local approximation order from the ladder (S_h) which, in the terms of Section 1, can be phrased thus: it is the largest k for which

$$U := 1 - \varphi{*'} \text{ is degree-reducing on } \Pi_{<k}. \tag{3.1}$$

Proposition 1.1 shows that we can state this condition more simply as

$$\Pi_{<k} \subseteq \Pi_\varphi. \tag{3.2}$$

To be precise, [FS] consider the "controlled" approximation order, which turns out to be the same as the local approximation order; cf. [BJ].

Fix and Strang use in their proof a quasi–interpolant whose construction relies on Fourier transform arguments which, in a univariate context, can already be found in Schoenberg's basic spline paper [S] and which appear in the proof of Proposition 2.2. This makes it easy to recall their construction here.

Define the quasi–interpolant Q on Π by the rule

$$Qf := \varphi{*'} Ff$$

with

$$Ff := \sum_\alpha a_\alpha (-\mathrm{i}D)^\alpha f$$

and $a_\alpha := [\![D]\!]^\alpha (1/\widehat{\varphi})(0)$ the Taylor coefficients for $1/\widehat{\varphi}$. Dahmen and Micchelli [DM83] prove that Q reproduces any affinely invariant subspace of (2.8), but, again, their argument supports a stronger claim, namely that

$$Q_{|\Pi_\varphi} = 1. \tag{3.3}$$

For, if $p \in \Pi_\varphi$, then also $Fp \in \Pi_\varphi$ since Π_φ is D–invariant, hence, by (2.9),

$$\begin{aligned}
Qp &= \sum_\alpha (D^\alpha Fp)[\![-\mathrm{i}D]\!]^\alpha \widehat{\varphi}(0) \\
&= \sum_\alpha \sum_\beta a_\beta (-\mathrm{i}D)^{\alpha+\beta} p [\![D]\!]^\alpha \widehat{\varphi}(0) \\
&= \sum_\gamma (-\mathrm{i}D)^\gamma p \sum_{\alpha+\beta=\gamma} [\![D]\!]^\beta (1/\widehat{\varphi})(0) [\![D]\!]^\alpha \widehat{\varphi}(0) \\
&= \sum_\gamma (-\mathrm{i}D)^\gamma p \delta_{0\gamma} = p.
\end{aligned}$$

The construction is finished by noting that (3.3) only depends on the action of F on Π_φ, hence a local quasi–interpolant on smooth functions which reproduces Π_φ can be obtained in the form

$$Qf := \varphi{*'}(\lambda{*}f), \tag{3.4}$$

with

$$(\lambda{*}f)(x) := \lambda f(\cdot + x), \tag{3.5}$$

and λ any locally supported linear functional which agrees on Π_φ with $p \mapsto Fp(0)$.

The construction idea in [BH] seems more direct. There the locally supported bounded linear functional (on whatever normed linear space X you may wish to carry out approximation from $S \cap X$) is constructed as an extension of the linear functional

$$p \mapsto (L^{-1}p)(0). \tag{3.6}$$

Since $L = \varphi *'_{|\Pi_\varphi}$ commutes with E, so does L^{-1}. Thus, for $p \in \Pi_\varphi$,

$$(\lambda * p)(j) = (L^{-1}p)(j) = (L^{-1}p(\cdot + j))(0),$$

hence

$$Qp = \varphi *'(L^{-1}p) = p.$$

In order to obtain a quasi–interpolant of the optimal order k, the extension λ only needs to match (3.6) on $\Pi_{<k}$. For example, one obtains the Strang–Fix quasi–interpolant by expressing the extension as a linear combination of the linear functionals

$$f \mapsto (-\mathrm{i}D)^\alpha f(0), \qquad |\alpha| < k, \tag{3.7}$$

i.e., in the form

$$\lambda f = \sum_{|\alpha|<k} a_\alpha (-\mathrm{i}D)^\alpha f(0).$$

The weights a_α are uniquely determined by the requirement that this linear functional match (3.6) on $\Pi_{<k}$ since (3.7) is maximally linearly independent over $\Pi_{<k}$. In particular,

$$a_\alpha = L^{-1}[\![\mathrm{i}\,\cdot\,]\!]^\alpha(0) = \mathrm{i}^\alpha g_\alpha(0),$$

by (2.6'). This shows, incidentally, that

$$[\![D]\!]^\alpha (1/\widehat{\varphi})(0) = \mathrm{i}^\alpha g_\alpha(0).$$

If point evaluation is continuous on X, then the linear functional λ can be written as a linear combination of evaluations at the integer points near 0. For, by (1.11),

$$(L^{-1})_{|\Pi_{<k}} = (1 + U + \cdots + U^{k-1})_{|\Pi_{<k}},$$

while, from (1.9),

$$(Uf)(j) = (c * f_|)(j),$$

with

$$c := \delta - \varphi_|$$

and δ the unit sequence, i.e., $\delta(j) = \delta_{0j}$. Hence

$$(L^{-1}p)(0) = p^{[k]}(0)$$

with $p^{[k]}$ obtained inductively in the following computation:

$$p^{[r]} := \begin{cases} 0 & \text{if } r = 0; \\ p_| + c * p^{[r-1]} & \text{if } r > 0. \end{cases} \tag{3.8}$$

This gives

$$(L^{-1}p)(0) = \sum_{j \in \mathbb{Z}^d} C(j) p(j) \qquad \text{all} \qquad p \in \Pi_{<k},$$

with the weight sequence C of finite support since c has finite support.

This construction was arrived at by different means by Chui and Diamond [CD], who added the following very useful observation. *If φ is symmetric, then U reduces the degree by at least 2*, since (1.7) can be written in the form

$$(1.7') \qquad \text{for} \qquad f \in \Pi_\varphi, \qquad \varphi *' f = f \sum_{j\in\mathbb{Z}^d} \varphi(j) + \sum_{j\in\mathbb{Z}^d} (f(\cdot + j) - 2f + f(\cdot - j))\varphi(j)/2.$$

This implies that, on $\Pi_\varphi \cap \Pi_k$, $U^{\lfloor k/2 \rfloor}$ already vanishes, hence only half the iteration (3.8) is necessary in this case.

Even for a symmetric φ, the support of the resulting λ may be far from minimal. Since we are only interested in extending a linear functional from Π_φ, a support consisting of $(\dim \Pi_\varphi)$ points is sufficient. These points can be chosen from $\mathbb{Z}^d$ since $\mathbb{Z}^d$ is total for Π. It would be interesting to find out whether they could be chosen as neighbors.

Such questions of minimal support for λ have been answered quite elegantly by Dahmen and Micchelli in case φ is a box spline. They find in [DM85] that the $(\dim \Pi_\varphi)$ integer points in the (right–continuous) support of φ are linearly independent over Π_φ, and so conclude the existence of an extension from Π_φ involving just these $(\dim \Pi_\varphi)$ point evaluations.

I note that the quasi–interpolant construction in [BJ] takes the opposite tack. Instead of constructing an appropriate λ as a linear combination of certain point evaluations, a compactly supported function $\psi \in S$ is constructed there so that $\psi *'$ already reproduces Π_φ.

References

[BH] C. de Boor and K. Höllig, B-splines from parallelepipeds, J. Analyse Math.**42**(1982/83), 99–115.

[BJ] C. de Boor and R. Q. Jia, Controlled approximation and a characterization of the local approximation order, Proc. Amer. Math. Soc.**95**(1985), 547–553.

[CD] C. K. Chui and H. Diamond, A natural formulation of quasi-interpolation by multivariate splines, Proc. Amer. Math. Soc.**99**(1987), 643–646.

[CJW] Charles K. Chui, K. Jetter, and J. D. Ward, Cardinal interpolation by multivariate splines, Math. Comp.**48(178)**(1987), 711–724.

[CL] C. K. Chui and M.-J. Lai, A multivariate analog of Marsden's identity and a quasi-interpolation scheme, Constr. Approx.**3**(1987), 111–122.

[DM83] W. Dahmen and C. A. Micchelli, Translates of multivariate splines, Linear Algebra Appl.**52**(1983), 217–234.

[DM85] W. Dahmen and C. A. Micchelli, On the solution of certain systems of partial difference equations and linear independence of translates of box splines, Trans. Amer. Math. Soc.**292**(1985), 305–320.

[FS] G. Fix and G. Strang, Fourier analysis of the finite element method in Ritz-Galerkin theory, Studies in Appl. Math.**48**(1969), 265–273.

[S] I. J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions, Parts A and B, Quart. Appl. Math.**4**(1946), 45–99, 112–141.