Dependency relations among the shifts of a multivariate refinable distribution

Thomas A. Hogan[1] and Rong-Qing Jia[2]

Abstract. Refinable functions are an intrinsic part of subdivision schemes and wavelet constructions. The relevant properties of such functions must usually be determined from their refinement masks. In this paper, we provide a characterization of linear independence for the shifts of a multivariate refinable distribution in terms of its (finitely supported) refinement mask.

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Authors' affiliations and addresses:

[1] Department of Mathematics
Vanderbilt University
Nashville, TN 37240
U.S.A.
hogan@math.vanderbilt.edu

[2] Department of Mathematical Sciences
University of Alberta
Edmonton
Canada T6G 2G1
jia@xihu.math.ualberta.ca

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T. A. Hogan, R.-Q. Jia

1. Introduction

A function ϕ is said to be **refinable** if it satisfies the following refinement equation

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(2\cdot - \alpha), \tag{1.1}$$

where a is a finitely supported sequence on $\mathbb{Z}^s$, called the **refinement mask**. Refinable functions are an intrinsic part of subdivision schemes and wavelet constructions. In general, any relevant properties of the function ϕ must be determined from the mask a. In this paper, we provide a characterization of linear independence of the shifts of a multivariate refinable function in terms its refinement mask.

Let $\ell(\mathbb{Z}^s)$ denote the linear space of all sequences on $\mathbb{Z}^s$. For a compactly supported distribution ϕ, define

$$N(\phi) := \Big\{ c \in \ell(\mathbb{Z}^s) : \sum_{\alpha \in \mathbb{Z}^s} c(\alpha)\phi(\cdot - \alpha) = 0 \Big\}.$$

Then ϕ is said to have **linearly independent** shifts if $N(\phi)$ is trivial, *i.e.*, $N(\phi) = \{0\}$. When ϕ is a compactly supported continuous function, Dahmen and Micchelli [2] showed that $N(\phi)$ is non-trivial if and only if it contains an exponential, *i.e.*, a sequence of the form $(z^\alpha)_{\alpha \in \mathbb{Z}^s}$ for some $z \in (\mathbb{C} \setminus \{0\})^s$.

Linear independence is a necessary condition for orthogonality, or even biorthogonality, of refinable functions. It is also a sufficient condition for stability. In fact, our results provide a characterization of stability for the shifts of ϕ, since we characterize all exponentials in $N(\phi)$ and ϕ has stable shifts if and only if $N(\phi)$ contains an exponential which lies on the s-dimensional unit torus (*cf.* [8]). A characterization of stability for refinable functions in $L_2(\mathbb{R}^s)$ has also been provided by Lawton, Lee, and Shen in [10] but, as our examples in Section 3 demonstrate, the results of this paper can be significantly less complicated to apply.

In the univariate case, a useful characterization of linear independence for the shifts of ϕ was given in terms of a by Jia and Wang in [9]. However, their techniques are inherently univariate. Attempts to generalize their results to functions of several variables have been made by Hogan [6] and Zhou [12]. In [6], the conditions of [9] were shown to be necessary in several variables; and they were shown to be also sufficient for functions of a certain type. These results were not satisfactory for two reasons: the proofs do not apply to general multivariate functions; and the conditions, though easy to verify in the univariate case, are more elusive in several variables. In [12], a fairly easily verifiable condition on the mask a was provided which, together with stability, characterizes linear independence. However, no satisfactory means were provided for determining the stability.

We denote by $C_c(\mathbb{R}^s)$ the normed linear space of all compactly supported continuous functions on $\mathbb{R}^s$ equipped with the norm

$$\|f\| := \max_{x\in\mathbb{R}^s} |f(x)|, \quad f \in C_c(\mathbb{R}^s).$$

For a function $f \in C_c(\mathbb{R}^s)$,

$$\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x)e^{-ix\cdot\xi}\,dx, \quad \xi \in \mathbb{R}^s,$$

where $x \cdot \xi$ denotes the usual inner product of the vectors x and ξ in $\mathbb{R}^s$. This definition naturally extends to distributions.

If the refinement mask a satisfies

$$\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = 2^s, \tag{1.2}$$

then it is known (*cf.* [1]) that Eq. (1.1) has a unique compactly supported distribution solution ϕ subject to the condition $\hat{\phi}(0) = 1$. This distribution is said to be the **normalized solution** to the refinement equation with mask a.

Given a mask a, there is, in general, no explicit expression for the solution ϕ to Eq. (1.1). Instead, the solution is approximated by iterating the **cascade operator**

$$T_a : C_c(\mathbb{R}^s) \to C_c(\mathbb{R}^s) : f \mapsto T_a f := \sum_{\beta\in\mathbb{Z}^s} a(\beta) f(2\cdot - \beta)$$

associated with a. This process is called a **subdivision scheme**.

Let h be the univariate hat function

$$h(x) := \max\{1 - |x|, 0\}, \quad x \in \mathbb{R}.$$

Define $f_0 \in C_c(\mathbb{R}^s)$ by $f_0(x_1, \ldots, x_s) := \prod_{j=1}^s h(x_j)$ and $f_n := T_a f_{n-1}$ for $n = 1, 2, 3, \ldots$. Then we say that the subdivision scheme associated with the mask a **converges** if there is a function $f \in C_c(\mathbb{R}^s)$ such that

$$\lim_{n\to\infty} \|f_n - f\| = 0.$$

If the subdivision scheme converges, then the limiting function f is the normalized solution to the refinement equation.

Throughout this paper, we assume that the subdivision scheme associated with the mask a converges. Consequently, ϕ is a continuous function with support in the convex hull of the support of a.

Suppose a is supported on the closed cell

$$\prod_{j=1}^{s} [M_j, N_j],$$

where M_j and N_j are integers and $M_j < N_j$, $j = 1, \dots, s$. Let

$$K := \mathbb{Z}^s \cap \Big(\prod_{j=1}^{s} [M_j, N_j - 1]\Big).$$

For $\alpha \in \mathbb{Z}^s$, we define

$$\phi_\alpha(x) := \begin{cases} \phi(x+\alpha) & \text{for } x \in [0,1)^s, \\ 0 & \text{for } x \in \mathbb{R}^s \setminus [0,1)^s. \end{cases}$$

Clearly, $\phi_\alpha = 0$ for $\alpha \in \mathbb{Z}^s \setminus K$ and $\phi = \sum_{k \in K} \phi_k(\cdot - k)$. Let $\Phi := (\phi_k)_{k \in K}$. Then Φ is a vector of functions supported on the unit cube $[0,1]^s$ and continuous on $[0,1)^s$.

By E we denote the set $\{0,1\}^s$ of all vertices of the unit cube $[0,1]^s$. For $\varepsilon \in E$, let A_ε be the linear operator on $\mathbb{C}^K$ given by

$$A_\varepsilon v(j) := \sum_{k \in K} a(\varepsilon + 2j - k) v(k), \quad j \in K,$$

where $v \in \mathbb{C}^K$. Now, for all $\varepsilon \in E$, if $x \in \varepsilon/2 + [0, 1/2)^s$ then, by Eq. (1.1),

$$\phi_j(x) = \phi(x+j) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(2x + 2j - \alpha) = \sum_{k \in K} a(\varepsilon + 2j - k)\phi_k(2x - \varepsilon), \quad j \in K.$$

This shows that Φ satisfies the following vector refinement equation:

$$\Phi = \sum_{\varepsilon \in E} A_\varepsilon \Phi(2 \cdot - \varepsilon). \tag{1.3}$$

We point out that, although the arguments and results of this paper require that the subdivision scheme converge and, hence, that ϕ be continuous, these results can easily be used to determine whether any refinable distribution has linearly independent shifts, as long as the mask a satisfies Eq. (1.2). To see how, define the operator

$$\mu : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s) : c \mapsto 2^{-s} \sum_{\varepsilon \in E} c(\cdot - \varepsilon).$$

Then the method used to prove Theorem 3.3 of [7] can be used to show that for any mask a satisfying Eq. (1.2), if n is a positive integer such that $2^n > \rho(\{A_\varepsilon : \varepsilon \in E\})$ then the subdivision scheme associated with $\mu^n a$ converges. (See Section 3 for a definition of $\rho(\{A_\varepsilon : \varepsilon \in E\})$.) Let ψ be the normalized solution to the refinement equation with mask $\mu^n a$. Then ψ is the n-fold convolution of ϕ with the characteristic function of the unit cube $[0,1]^s$, and it is well-known that $N(\psi) = N(\phi)$. In particular, ϕ has linearly independent shifts if and only if ψ does.

2. Results and proofs

Since the subdivision scheme is assumed to converge, 1 is a simple eigenvalue of A_0 and the other eigenvalues of A_0 are less than 1 in modulus. By Eq. (1.3), we have $\Phi(0) = A_0\Phi(0)$. So $\Phi(0) = (\phi_k(0))_{k\in K}$ is the unique eigenvector of A_0 corresponding to the eigenvalue 1 and subject to the condition $\sum_{k\in K}\phi_k(0) = 1$. By using the refinement equation again, we obtain

$$\Phi(\mathbb{1}/2) = A_{\mathbb{1}}\Phi(0),$$

where $\mathbb{1} := (1, \dots, 1) \in E$.

Let $\mathcal{A}$ be a finite collection of linear operators on a vector space V. A subspace W of V is said to be $\mathcal{A}$-invariant if it is invariant under every operator A in $\mathcal{A}$. Let u be a vector in V. The intersection of all $\mathcal{A}$-invariant subspaces of V containing u is itself $\mathcal{A}$-invariant. We call this the **minimal common invariant subspace** of the operators A in $\mathcal{A}$ generated by u.

Let V be the minimal common invariant subspace of A_ε $(\varepsilon \in E)$ generated by the vector $u := \Phi(\mathbb{1}/2) = (\phi(\mathbb{1}/2 + \beta))_{\beta\in K} \in \mathbb{C}^K$. And for any two vectors $u = (u(\beta))_{\beta\in K}$ and $v = (v(\beta))_{\beta\in K}$ in $\mathbb{C}^K$, define

$$\langle u, v\rangle := \sum\nolimits_{\beta\in K} u(\beta)v(\beta).$$

Lemma 1. *For a vector $\lambda \in \mathbb{C}^K$, the following two conditions are equivalent:*
(a) $\langle\lambda, \Phi(x)\rangle = 0$ *for all* $x \in [0, 1)^s$.
(b) $\langle\lambda, v\rangle = 0$ *for all* $v \in V$.

Remark: A variation of Lemma 1 has been provided already by Theorem 2.7 of [5]. In fact, it was that theorem which motivated our current work.

Proof. Suppose λ is a vector in $\mathbb{C}^K$ such that $\langle\lambda, \Phi(x)\rangle = 0$ for all $x \in [0, 1)^s$. Since $u = \Phi(\mathbb{1}/2)$, we have $\langle\lambda, u\rangle = 0$. The linear space V is spanned by the vectors $A_{\varepsilon_1}\cdots A_{\varepsilon_j}u$, where $j = 0, 1, \dots$ and $\varepsilon_1, \dots, \varepsilon_j \in E$. With the help of the vector refinement equation, we have $A_{\varepsilon_1}\cdots A_{\varepsilon_j}u = \Phi(x)$ for some $x \in (0, 1)^s$. Hence, $\langle\lambda, A_{\varepsilon_1}\cdots A_{\varepsilon_j}u\rangle = 0$. In other words, $\langle\lambda, v\rangle = 0$ for all $v \in V$. This shows that (a) implies (b).

To prove that (b) implies (a), let G be the set of those points $(m_1/2^n, \dots, m_s/2^n)$ for which $n = 1, 2, \dots$ and $m_1, \dots, m_s$ are odd integers between 0 and 2^n. Evidently, G is dense in the closed cube $[0, 1]^s$. We claim that $\Phi(x) \in V$ for each $x = (m_1/2^n, \dots, m_s/2^n) \in G$. This will be done by induction on n.

When $n = 1$ and $x = (m_1/2, \dots, m_s/2)$, we must have $m_1 = \cdots = m_s = 1$. Hence $x = \mathbb{1}/2$ and $\Phi(x) = u \in V$. Suppose $n > 1$ and our claim has been verified for $n - 1$. Let $x = (m_1/2^n, \dots, m_s/2^n)$, where $m_1, \dots, m_s$ are odd integers between 0 and 2^n. Set $\eta = (\eta_1, \dots, \eta_s)$, where

$$\eta_j := \begin{cases} 0 & \text{if } m_j < 2^{n-1}, \\ 1 & \text{if } m_j > 2^{n-1}, \end{cases}$$

for $j = 1, \dots, s$. By using the vector refinement equation, we have

$$\Phi(x) = \sum_{\varepsilon\in E} A_\varepsilon\Phi(2x - \varepsilon) = A_\eta\Phi(y),$$

where $y = (m_1/2^{n-1} - \eta_1, \ldots, m_s/2^{n-1} - \eta_s)$. By the induction hypothesis, we have $\Phi(y) \in V$. Since V is invariant under A_η, it follows that $\Phi(x) = A_\eta \Phi(y) \in V$. This justifies our claim that $\Phi(x) \in V$ for each $x = (m_1/2^n, \ldots, m_s/2^n) \in G$.

Let λ be a vector in $\mathbb{C}^K$ such that $\langle \lambda, v \rangle = 0$ for all $v \in V$. By what has been proved, $\langle \lambda, \Phi(x) \rangle = 0$ for all $x \in G$. But G is dense in $[0,1]^s$ and Φ is continuous on $[0,1)^s$. Therefore, we have $\langle \lambda, \Phi(x) \rangle = 0$ for all $x \in [0,1)^s$. This shows that (b) implies (a). $\square$

If $\dim(V) = \#K$, then $\langle \lambda, v \rangle = 0$ for all $v \in V$ implies $\lambda = 0$. So, by Lemma 1, a sufficient condition for the shifts of ϕ to be linearly independent is that $\dim(V) = \#K$. This condition is not however necessary in general. A simple necessary and sufficient condition is provided by the following theorem.

Theorem 2. *The shifts of ϕ are linearly independent if and only if the Laurent polynomials*

$$p_v(z) := \sum_{\beta \in K} v(\beta) z^\beta, \quad v \in V,$$

have no common zeros in $(\mathbb{C} \setminus \{0\})^s$. Moreover, $c \in N(\phi)$ if and only if $c*v = 0$ for all $v \in V$, where $c*v \in \ell(\mathbb{Z}^s)$ is defined by

$$c*v(\alpha) := \sum_{k \in K} c(\alpha - k) v(k), \quad \alpha \in \mathbb{Z}^s.$$

Proof. According to [2], the shifts of ϕ are linearly dependent if and only if there exists $z \in (\mathbb{C} \setminus \{0\})^s$ such that $\sum_{j \in \mathbb{Z}^s} z^{-j} \phi(\cdot - j) = 0$. For $z \in (\mathbb{C} \setminus \{0\})^s$, define $\lambda_z \in \mathbb{C}^K$ by $\lambda_z(k) := z^k \quad (k \in K)$ and note that for $\alpha \in \mathbb{Z}^s$ and $x \in \alpha + [0,1)^s$,

$$\begin{aligned}\sum_{j \in \mathbb{Z}^s} z^{-j} \phi(x - j) &= \sum_{j \in \mathbb{Z}^s} z^{-j} \sum_{k \in K} \phi_k(x - j - k) \\ &= \sum_{j \in \mathbb{Z}^s} z^{-j} \sum_{k \in K} z^k \phi_k(x - j) \\ &= z^{-\alpha} \langle \lambda_z, \Phi(x - \alpha) \rangle.\end{aligned}$$

So, by Lemma 1, ϕ has dependent shifts if and only if there exists $z \in (\mathbb{C} \setminus \{0\})^s$ such that $p_v(z) = \langle \lambda_z, v \rangle = 0$ for all $v \in V$.

Now, let $c \in \ell(\mathbb{Z}^s)$. Then

$$\begin{aligned}\sum_{\alpha \in \mathbb{Z}^s} c(\alpha) \phi(\cdot - \alpha) &= \sum_{\alpha \in \mathbb{Z}^s} c(\alpha) \sum_{k \in K} \phi_k(\cdot - \alpha - k) \\ &= \sum_{\alpha \in \mathbb{Z}^s} \sum_{k \in K} c(\alpha - k) \phi_k(\cdot - \alpha).\end{aligned}$$

So $c \in N(\phi)$ if and only if $\sum_{k \in K} c(\alpha - k) \phi_k = 0$ for all $\alpha \in \mathbb{Z}^s$ which, by Lemma 1, is equivalent to $c*v = 0$ for all $v \in V$. $\square$

3. Examples

We provide two examples of bivariate functions with linearly dependent shifts. In both cases, the symbol of the mask is not factorizable—so the characterizations provided in [6] are not applicable. In the first example, the function actually has stable shifts, so the dependency relations would not be identified by the techniques of [10]. It is true, however, that necessary conditions for independence were provided in [6] which would allow one to determine that these functions have dependent shifts, if one could identify the zero set of the symbol of the mask. In our second example, the symbol of the mask has no symmetric zeros. This, along with the fact that the symbol is not factorizable, makes it especially difficult to identify its pertinent zeros.

To apply our results to these examples, we will need to know that the associated subdivision schemes converge. To this end, we recall some definitions and results from [11] and [3].

Let W be a finite dimensional normed linear space with norm $\|\cdot\|$. As usual, the norm of a linear operator A on W is defined by

$$\|A\| := \max_{\|v\|=1} \{\|Av\|\}.$$

Let $\mathcal{A}$ be a finite collection of linear operators on W. For a positive integer n, we denote by $\mathcal{A}^n$ the n-fold Cartesian product of $\mathcal{A}$ with itself:

$$\mathcal{A}^n := \{(A_1, \ldots, A_n) : A_1, \ldots, A_n \in \mathcal{A}\}.$$

By convention, $\mathcal{A}^0 := \{I\}$, where I is the identity operator on W. Define

$$\|\mathcal{A}^n\| := \max\{\|A_1 \cdots A_n\| : (A_1, \ldots, A_n) \in \mathcal{A}^n\}.$$

The **(uniform) joint spectral radius** of $\mathcal{A}$ was defined in [11] to be

$$\rho(\mathcal{A}) := \lim_{n\to\infty} \|\mathcal{A}^n\|^{1/n}.$$

It is well-known that $\rho(\mathcal{A})$ is independent of which norm is used in W and that

$$\rho(\mathcal{A}) = \inf_{n\geq 1} \|\mathcal{A}^n\|^{1/n}. \tag{3.1}$$

In the case $s = 1$, the joint spectral radius was used in [3] to derive sufficient conditions to ensure continuity of the normalized solution to Eq. (1.1).

Now, define

$$W := \left\{ v \in \mathbb{C}^K : \sum\nolimits_{\alpha\in K} v(\alpha) = 0 \right\}.$$

If $\sum_{\beta\in\mathbb{Z}^s} a(\varepsilon - 2\beta) = 1$ for all $\varepsilon \in E$, then W is invariant under every A_ε, $\varepsilon \in E$. So $B_\varepsilon := A_\varepsilon|_W$ is a linear operator on W for each $\varepsilon \in E$. According to [4], the subdivision scheme associated with a converges if and only if

$$\sum\nolimits_{\beta\in\mathbb{Z}^s} a(\alpha - 2\beta) = 1 \quad \forall \alpha \in \mathbb{Z}^s$$

and

$$\rho(\{B_\varepsilon : \varepsilon \in E\}) < 1.$$

The masks of our examples clearly satisfy the first of these conditions, and we don't mention it again. By Eq. (3.1), then, the subdivision scheme converges if and only if there exists a nonnegative integer n such that

$$\|B_{\varepsilon_1}\cdots B_{\varepsilon_n}\| < 1$$

for all $(\varepsilon_1,\ldots,\varepsilon_n)\in E^n$. It is this characterization that we will make use of.

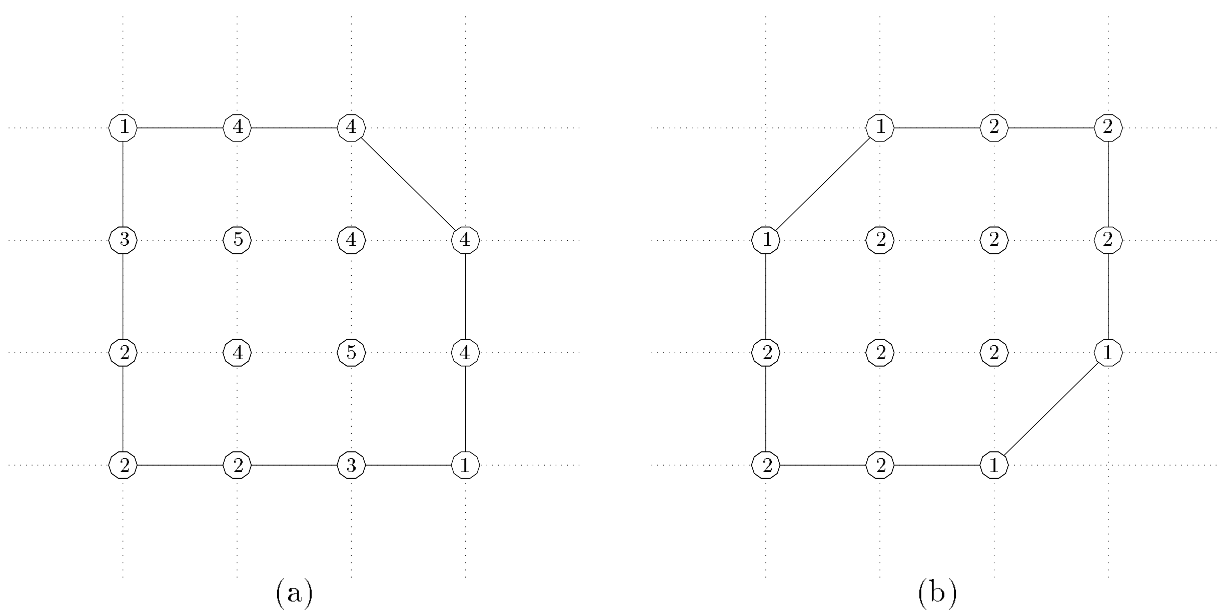
Example 1. Let ϕ be the normalized solution of the refinement equation

$$\phi = \sum_{\alpha\in\mathbb{Z}^2} a(\alpha)\phi(2\cdot-\alpha),$$

where the symbol of the mask a is given by

$$\sum_{\alpha\in\mathbb{Z}^2} a(\alpha)z^\alpha := \frac{1}{12}\Big\{z_1^3+z_2^3+2(1+z_1+z_2)+3(z_1^2+z_2^2) \\ +4(z_1z_2+z_1^3z_2+z_1^2z_2^2+z_1z_2^3+z_1^3z_2^2+z_1^2z_2^3)+5(z_1^2z_2+z_1z_2^2)\Big\}.$$

The nonzero terms of the sequence $12a$ are shown in Figure(a) along with an outline of the support of ϕ. The lower left corner of this support is at the origin $(0,0)$. In this case, $K=\{(0,0),(1,0),(2,0),(0,1),(1,1),(2,1),(0,2),(1,2),(2,2)\}$.



Figures. The non-zero terms of the mask a along with an outline of the support of the corresponding refinable function ϕ. (a) $12a$ from Example 1; (b) $6a$ from Example 2.

We first show that the subdivision scheme converges. The vectors

$$d_j(\alpha) := \begin{cases} 1 & \text{for } \alpha = (0,0) \\ -1 & \text{for } \alpha = j \\ 0 & \text{for } \alpha \in \mathbb{Z}^2 \setminus \{j,(0,0)\} \end{cases} \qquad (j \in K' := K \setminus \{(0,0)\})$$

form a basis for W. For each $\varepsilon \in E$, let B_ε be the matrix representation of $A_\varepsilon|_W$ with respect to this basis. Using `Maple`, we computed $\|B_{\varepsilon_1}B_{\varepsilon_2}B_{\varepsilon_3}\|$ for all $(\varepsilon_1,\varepsilon_2,\varepsilon_3) \in E^3$ and found that

$$\|B_{(0,1)}B_{(1,1)}B_{(0,0)}\| = \frac{407}{432}$$

was the maximum such quantity. Since this is less than 1, the subdivision scheme associated with a converges. The operator norm we used was

$$\|B\| := \max_{i\in K'} \sum_{j\in K'} |B(i,j)|,$$

corresponding to the norm $\| \sum_{j\in K'} c(j)d_j\| := \max_{j\in K'} |c(j)|$ in W.

Now, let $w \in \mathbb{C}^K$ be the eigenvector of $A_{(0,0)}$ corresponding to the eigenvalue 1 such that the sum of its components is equal to 1. Using `Maple`, we obtain

$$w = \frac{1}{15}[0,0,0,0,3,4,0,4,4]^T.$$

Moreover,

$$u := A_{(1,1)}w = \frac{1}{180}[6,17,4,17,68,32,4,32,0]^T.$$

Define

$$V_0 := \{u\}, \quad V_n := V_{n-1} \cup \bigcup_{\varepsilon\in E} A_\varepsilon V_{n-1} \quad \text{for} \quad n = 1,\, 2,\, 3,\, \ldots.$$

Then the minimal common invariant subspace of $A_{(0,0)}$, $A_{(1,0)}$, $A_{(0,1)}$, and $A_{(1,1)}$ generated by u is $V = \bigcup_{n\in\mathbb{Z}_+} \operatorname{span} V_n = \operatorname{span} V_N$, with

$$N := \min\{n \in \mathbb{Z}_+ : \dim(\operatorname{span} V_n) = \dim(\operatorname{span} V_{n+1})\}.$$

(Note that N is necessarily less than or equal to $\#K$). Using `Maple` again, we find that $\dim(\operatorname{span} V_2) = \dim(\operatorname{span} V_3) = 8$, and that a basis for $V (= \operatorname{span} V_2)$ is provided by the vectors $v_j \in \mathbb{C}^K$, $j \in J := K \setminus \{(2,2)\}$ defined by

$$v_j(k) := \begin{cases} 0 & \text{for } k \in J \setminus \{j\} \\ 2 & \text{for } k = j \\ (-2)^{5-j_1-j_2} & \text{for } k = (2,2). \end{cases}$$

Thus, the shifts of ϕ are linearly independent if and only if the polynomials

$$p_{j_1,j_2}(z_1,z_2) := 2z_1^{j_1}z_2^{j_2} + (-2)^{5-j_1-j_2}z_1^2z_2^2, \quad (j_1,j_2) \in J,$$

have no common zeros in $(\mathbb{C}\setminus\{0\})^2$.

For all nonzero z_1 and z_2, if $p_{2,1}(z_1,z_2)=0$ then $z_2=-1/2$ and if $p_{1,2}(z_1,z_2)=0$ then $z_1=-1/2$. On the other hand, $p_{j_1,j_2}(-1/2,-1/2)=0$ for all $(j_1,j_2)\in J$. So these polynomials have exactly one common zero in $(\mathbb{C}\setminus\{0\})^2$, namely $(-1/2,-1/2)$. Therefore, we conclude that the shifts of ϕ are linearly dependent. We have also determined all possible linear dependence relations of the shifts of ϕ. That is, $N(\phi)$ is spanned by the sequence c given by $c(\alpha_1,\alpha_2):=(-1/2)^{\alpha_1+\alpha_2}$, $(\alpha_1,\alpha_2)\in\mathbb{Z}^2$.

Note that since $(-1/2,-1/2)$ is not on the 2-dimensional torus,

$$\mathbb{T}^2 := \{z\in\mathbb{C}^2 : |z_1|=|z_2|=1\},$$

and $\phi\in L_\infty(\mathbb{R}^2)$, Theorem 1 of [8] implies that ϕ has ℓ_p-stable shifts for any $0<p\le\infty$.

Example 2. Let ϕ be the normalized solution of the refinement equation

$$\phi = \sum_{\alpha\in\mathbb{Z}^2} a(\alpha)\phi(2\cdot-\alpha),$$

where the symbol of the mask a is

$$\frac{1}{6}\big(z_1^2+z_1^3z_2+z_2^2+z_1z_2^3\big)+\frac{1}{3}\big(1+z_1+z_2+z_1z_2+z_1^2z_2+z_1z_2^2+z_1^2z_2^2+z_1^3z_2^2+z_1^2z_2^3+z_1^3z_2^3\big).$$

The nonzero terms of the sequence $6a$ are shown in Figure(b) along with an outline of the support of the function ϕ. K is the same as in Example 1.

With W and B_ε defined as in Example 1, but with

$$\|B\| := \max_{j\in K'}\sum_{i\in K'}|B(i,j)|$$

(which corresponds to the norm $\|\sum_{j\in K'}c(j)d_j\| := \sum_{j\in K'}|c(j)|$ in W), we find that

$$\max_{(\varepsilon_1,\varepsilon_2)\in E^2}\|B_{\varepsilon_1}B_{\varepsilon_2}\| = \|B_{(0,0)}B_{(0,1)}\| = 2/3.$$

Since this is less than 1, the subdivision scheme converges.

Now, using the same notation as in Example 1,

$$w=[0,0,0,0,1/3,1/6,0,1/6,1/3]^T,\quad u=[1/9,1/9,0,1/9,1/3,1/9,0,1/9,1/9]^T,$$

and $\dim(\operatorname{span}V_2)=\dim(\operatorname{span}V_3)=7$. The rows of the matrix

$$\begin{bmatrix}1&0&0&0&0&-1&0&0&0\\1&0&0&0&0&0&0&-1&0\\1&0&1&0&0&0&0&0&1\\1&0&0&0&1&0&0&0&1\\1&0&0&0&0&0&1&0&1\\0&-1&0&0&0&0&0&0&1\\0&0&0&-1&0&0&0&0&1\end{bmatrix}$$

form a basis for span V_2. Thus, the shifts of ϕ are linearly independent if and only if the polynomials

$$1 - z_1^2 z_2,\ 1 - z_1 z_2^2,\ 1 + z_1^2 + z_1^2 z_2^2,\ 1 + z_1 z_2 + z_1^2 z_2^2,\ 1 + z_2^2 + z_1^2 z_2^2,\ z_1^2 z_2^2 - z_1,\ z_1^2 z_2^2 - z_2$$

have no common zeros. These polynomials have two common zeros: $\frac{1}{2}(-1+\sqrt{3}i, -1+\sqrt{3}i)$ and $\frac{1}{2}(-1-\sqrt{3}i, -1-\sqrt{3}i)$. Therefore, we conclude that the shifts of ϕ are linearly dependent.

Since these common zeros actually lie on $\mathbb{T}^2$, Theorem 1 of [8] implies that the shifts of ϕ are in fact not stable. Although this could be determined using results from [10], the computations would be significantly more complicated.

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