NONLINEAR INTERPOLATION BY SPLINES, PSEUDOSPLINES, AND ELASTICA

G. Birkhoff
Harvard University
Cambridge, Massachusetts

H. Burchard and D. Thomas
Mathematics Department
General Motors Research Laboratories
Warren, Michigan

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ABSTRACT

Many efficient linear methods are known for interpolation to plane curves which turn through an angle of less than 180°. Among these methods, linear "spline" interpolation is especially versatile. We describe below various methods for interpolating to general plane curves, including interpolation by (nonlinear) mechanical splines and "elastica", and discuss their relation to an adaptation of linear splines proposed by Fowler and Wilson.

NONLINEAR SPLINE INTERPOLATION

"Linearized splines", defined by piecewise cubic polynomial functions $y = f(x)$ of class C^2, have recently been effectively used to represent smooth plane curves whose tangent direction θ changes by less than 180°. At least two proposals ([1], [3]) have been made for modifying this approach to define "nonlinear spline interpolation" between points on arbitrary smooth plane curves. The purpose of this note is to relate these proposals to true "nonlinear spline interpolation" by mechanical splines, and to interpolation by more general "elastica". In all cases, we suppose given $n + 1$ points $P_i = (x_i, y_i)$, $i = 0, 1, \cdots, n$, on a smooth plane curve Γ, and we interpolate a smooth approximating curve through these points.

The most natural way to do this is by trying to minimize the strain energy $(B/2)\int K^2\,ds$ of a mechanical spline of "stiffness" B passing through these points -- K denoting the curvature and s the arc-length. Local minima of this functional describe the positions of stable equilibrium of a mechanical spline constrained to pass through these points, but otherwise free to deform or slip. For such a spline, the Euler-Lagrange variational equations are, if dots refer to derivatives with respect to arc-length,