Best Approximation Properties of Spline Functions of Odd Degree

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Introduction. In [4], interpolation by cubic spline functions is discussed, and some best approximation properties of the cubic spline fit are described. This note extends the results of [4], in a somewhat modified form, to spline functions of odd degree $m = 2k - 1$, $k \geqq 2$.

Definition. A *spline function* of *degree m* with *joints* $\xi_1 < \xi_2 < \cdots < \xi_n$ is defined as a function $F(x)$ with the following two properties ([2], p. 67):

a. In each of the intervals $(-\infty, \xi_1), [\xi_1, \xi_2), \cdots, [\xi_n, \infty)$, $F(x)$ is a polynomial of degree m;
b. $F(x)$ has continuous derivatives through the $(m - 1)^{\text{st}}$, or, for short, $F(x) \in C^{m-1}$.

The class of functions $F(x)$ with these properties will be denoted by $S_m(\xi_1, \cdots, \xi_n)$.

The following lemma establishes the existence and uniqueness of a spline function of degree $(2k - 1)$ with $(n - 1)$ joints which coincides with a given function $f(x)$ at $(n + 1)$ prescribed points. The lemma is a consequence of Theorem 2 in [3], p. 258.

Lemma 1. *Let $f(x)$ be any function of class $C^k[a, b]$. For each choice of $n + 1$ abscissae x_i, $a = x_0 < x_1 < \cdots < x_n = b$, there exists exactly one spline function in $S_{2k-1}(x_1, \cdots, x_{n-1})$, denoted by $\bar{s}(x)$, such that*

$$\bar{s}(x_i) = f(x_i), \qquad i = 0, \cdots, n, \tag{1}$$

$$\bar{s}^{(k+j)}(x_i) = 0, \qquad i = 0, n; \qquad j = 0, \cdots, k - 2, \tag{2}$$

where $\bar{s}^{(m)}(x)$ denotes the m^{th} derivative of $\bar{s}(x)$.

It has been known for some time (*cf.*, *e.g.* [2], p. 67) that in the case $k = 2$ of cubic splines the interpolating function $\bar{s}(x)$ minimizes $\int_a^b [u''(x)]^2\, dx$ among all functions $u(x) \in C^2$ which coincide with $f(x)$ at the points x_i, $i = 0, \cdots, n$. The

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cubic spline function $\bar{s}(x)$ gives therefore, approximately, the shape of a thin beam or "spline", which is forced to go through the points $\{x_i, f(x_i)\}$, $i = 0, \cdots, n$. This result can be seen by considering the integral $\int [u''(x)]^2\, dx$ as a linearized approximation to the strain energy of a thin beam, which is $\int u''^2/(1 + u'^2)^{5/2}\, dx$. Thus $\bar{s}(x)$ minimizes the strain energy subject to the geometrical constraints stated (*cf.* [1], p. 92–98). The corresponding nonlinear problem was first considered by L. Euler and D. Bernoulli.

The inner product

$$(f, g)_k = \int_a^b f^{(k)}(x) g^{(k)}(x)\, dx \tag{3}$$

is defined for any two functions f, g which have square-integrable k^{th} derivatives on $[a, b]$. It defines a pseudo-norm

$$\|f\|_k = [(f, f)_k]^{1/2} \tag{4}$$

on the linear space $C^k[a, b]$, in which $\|f\|_k = 0$ if and only if $f(x)$ is a polynomial of degree $(k - 1)$ or less.

Theorem 1. *Among all the functions $u(x) \in C^k[a, b]$, which satisfy* (1′) $u(x_i) = f(x_i)$, $i = 0, \cdots, n$, *the presudo-norm $\|u\|_k$ is minimized by $\bar{s}(x)$.*

In this sense, the spline function $\bar{s}(x)$ is the smoothest function interpolating $f(x)$ at the points x_i, $i = 0, \cdots, n$.

This theorem is a direct consequence of the following lemma.

Lemma 2. *If $f(x) \in C^k[a, b]$, and $\bar{s}(x) \in S_{2k-1}(x_1, \cdots, x_n)$ satisfies* (1) *and* (2), *then*

$$\|f\|_k^2 - \|\bar{s}\|_k^2 = \|f - \bar{s}\|_k^2. \tag{5}$$

Proof. Let $\eta(x) \equiv f(x) - \bar{s}(x)$. The right-hand side of (5) may be written as

$$\|\eta\|_k^2 = \|f\|_k^2 - \|\bar{s}\|_k^2 - 2(\eta, \bar{s})_k. \tag{6}$$

By successive integration by parts, one has

$$(\eta, \bar{s})_k = \left[\sum_{j=0}^{k-2} (-1)^j \eta^{(k-1-j)}(x) \bar{s}^{(k+j)}(x)\right]\Bigg|_a^b + (-1)^{k-1} \int_a^b \eta'(x) \bar{s}^{(2k-1)}(x)\, dx. \tag{7}$$

The first term of the right-hand side of (7) vanishes because of (2). Since $\bar{s}^{(2k-1)}(x)$ is a constant in each of the intervals (x_i, x_{i+1}), $i=0, \cdots, n-1$, $(x_0=a, x_n=b)$, one has for the second term

$$\begin{aligned}\int_a^b \eta'(x)\bar{s}^{(2k-1)}(x)\, dx &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \eta'(x)\bar{s}^{(2k-1)}(x)\, dx \\ &= \sum_{i=0}^{n-1} [\eta(x_{i+1}) - \eta(x_i)]\bar{s}^{(2k-1)}(i\text{-}^{\text{th}}\text{ interval}),\end{aligned} \tag{8}$$

and this, by (1), is zero.

Remark. Lemmas 1 and 2 remain true, if condition (2) is replaced by

$$\bar{s}^{(j)}(x_i) = f^{(j)}(x_i), \qquad i = 0, n; \qquad j = 1, \cdots, k - 1. \tag{2′}$$

Lemma 2'. *If $f(x) \in C^k[a, b]$, and $\bar{s}(x) \in S_{2k-1}(x_1, \cdots, x_{n-1})$ satisfies* (1) *and* (2') *then*

$$||f||_k^2 - ||\bar{s}||_k^2 = ||f - \bar{s}||_k^2. \tag{5}$$

For the remainder of this note, let $\bar{s}(x) \in S_{2k-1}(x_1, \cdots, x_{n-1})$ denote the unique spline function of degree $2k - 1$ which satisfies (1) and (2'), and hence (5).

Theorem 1'. *Among the functions $u(x) \in C^k[a, b]$, which satisfy* (1) *and* (2'), (*with $\bar{s}(x)$ replaced by $u(x)$*), *the norm $||u||_k$ is minimum for $\bar{s}(x)$.*

Lemma 2' not only implies Theorem 1', but provides a *characterization* of the best approximation $s^*(x)$ to $f(x) \in C^k[a, b]$ by spline functions $s(x) \in S_{2k-1}(x_1, \cdots, x_{n-1})$ with respect to the measure of approximation

$$||f - s||_k. \tag{9}$$

A best approximation $s^*(x) \in S_{2k-1}(x_1, \cdots, x_{n-1})$ has to satisfy

$$||f - s^*||_k \leqq ||f - s||_k, \quad \text{for all } s \in S_{2k-1}(x_1, \cdots, x_{n-1}). \tag{10}$$

Since $||f||_k = 0$ if and only if $f(x)$ is a polynomial of degree $(k - 1)$, *i.e.*, $f(x) \equiv P_{k-1}(x)$, best approximations are not unique; $(s(x) + P_{k-1}(x))$ is a best approximation, if $s(x)$ is.

Theorem 2. *For $f(x) \in C^k[a, b]$,*

$$s^*(x) = \bar{s}(x) + P_{k-1}(x), \tag{11}$$

i.e.. the spline function $\bar{s}(x)$ interpolating $f(x)$ at the points x_i, $i = 0, \cdots, n$, and satisfying (2'), *is a best approximation to $f(x)$ by spline functions in $S_{2k-1}(x_1, \cdots, x_{n-1})$ with respect to the measure of approximation* (9).

Proof. Let $s(x)$ be any function in $S_{2k-1}(x_1, \cdots, x_{n-1})$. In Lemma 2', replace $f(x)$ by $(f(x) - s(x))$. Then $(\bar{s}(x) - s(x))$ is the corresponding unique function in $S_{2k-1}(x_1, \cdots, x_{n-1})$ satisfying (1) and (2'), so that

$$||f - \bar{s}||_k^2 = ||f - s||_k^2 - ||\bar{s} - s||_k^2. \tag{12}$$

Hence

$$||f - \bar{s}|| \leqq ||f - s||,$$

with equality holding if and only if $||\bar{s} - s||_k = 0$,—*i.e.*, when $s(x) = \bar{s}(x) + P_{k-1}(x)$.

References

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