AN UPPER BOUND ON THE APPROXIMATION POWER OF PRINCIPAL SHIFT-INVARIANT SPACES

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Abstract. An upper bound on the L_p-approximation power ($1 \le p \le \infty$) provided by principal shift-invariant spaces is derived with only very mild assumptions on the generator. It applies to both stationary and non-stationary ladders, and is shown to apply to spaces generated by (exponential) box splines, polyharmonic splines, multiquadrics, and Gauss kernel.

1. Introduction

A space $\mathcal{S}$ of locally integrable functions defined on $\mathbb{R}^d$ is said to be **shift-invariant** if $f(\cdot - j) \in \mathcal{S}$ whenever $f \in \mathcal{S}$ and $j \in \mathbb{Z}^d$. For example, if $\phi : \mathbb{R}^d \to \mathbb{C}$ is locally integrable, then

$$\mathcal{S}_0(\phi) := \operatorname{span}\{\phi(\cdot - j) : j \in \mathbb{Z}^d\}$$

is shift-invariant, and since $\mathcal{S}_0(\phi)$ is generated by the single function ϕ, $\mathcal{S}_0(\phi)$ is said to be a **principal shift-invariant space**. When $\phi \in L_p := L_p(\mathbb{R}^d)$, $\mathcal{S}_0(\phi)$ can be enlarged to

$$\mathcal{S}_p(\phi) := \operatorname{closure}(\mathcal{S}_0(\phi); L_p), \qquad 1 \le p < \infty.$$

We define $\mathcal{S}_\infty(\phi)$ in §2 in a way which ensures that $\mathcal{S}_\infty(\phi)$ always contains $\operatorname{closure}(\mathcal{S}_0(\phi); L_\infty)$, while allowing it to be significantly larger in case ϕ has sufficient decay. When ϕ has compact support there is no difficulty in summing arbitrary infinite linear combinations of the shifts of ϕ, and so we define, in that case only,

$$\mathcal{S}_\star(\phi) := \{\sum_{j\in\mathbb{Z}^d} a_j \phi(\cdot - j) : a \in \mathbb{C}^{\mathbb{Z}^d}\}.$$

Shift-invariant (SI) spaces and principal shift-invariant (PSI) spaces are useful and pertinent to several areas of approximation theory: (a) SI spaces are pertinent to the study of splines, particularly multivariate, box splines, and exponential box splines (cf. [BHR]). (b) SI spaces are useful in the study of approximation to scattered data by translates of radially symmetric functions, as insights acquired there have led to new approaches to this

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difficult problem (cf. [BuR]). (c) SI spaces are used in sampling theory. (d-e) PSI spaces are the building blocks in both Gabor and wavelet expansions (cf. [Da]). (f) PSI spaces are pertinent to the study of subdivision schemes (cf. [D]).

For $h > 0$ and any shift-invariant space $\mathcal{S}$, we denote the dilations of $\mathcal{S}$ by

$$\mathcal{S}^h := \{s(\cdot/h) : s \in \mathcal{S}\}.$$

Let $\mathcal{S}(\phi)$ be a PSI space generated by some function ϕ. (When it is not important to specify exactly how the PSI space is generated by ϕ, as is the case here, we will simply write $\mathcal{S}(\phi)$.) The directed collection $(\mathcal{S}^h(\phi))_h$ is said to be a **stationary ladder** of PSI spaces. The term "stationary" indicates that the underlying PSI space $\mathcal{S}(\phi)$ does not depend on h. A natural generalization of this is to allow the underlying PSI space to depend on h. For this, we assume that for all $h > 0$, $\mathcal{S}(\phi_h)$ is a PSI space generated by ϕ_h. The directed collection $(\mathcal{S}^h(\phi_h))_h$ is then said to be a **non-stationary ladder** of PSI spaces. The importance of this generalization has gradually emerged in many of the above areas: exponential box splines in splines, spectral approximation orders provided by non-stationary ladders in radial basis functions, non-stationary wavelets which yield significantly more flexibility in the construction of wavelets, and non-stationary subdivision schemes which similarly increase flexibility.

A standard problem in Approximation Theory is the determination of the L_p-approximation power of the ladder of PSI spaces $(\mathcal{S}^h(\phi_h))_h$. Specifically, one seeks to determine the rate at which $\operatorname{dist}\big(f, \mathcal{S}^h(\phi_h); L_p\big)$ decays as $h \to 0$ for sufficiently smooth $f \in L_p$. If $\gamma > 0$ and $\operatorname{dist}\big(f, \mathcal{S}^h(\phi_h); L_p\big) = O(h^\gamma)$ for all sufficiently smooth $f \in L_p$, then $(\mathcal{S}^h(\phi_h))_h$ is said to provide L_p-**approximation order** γ. If $\gamma \geq 0$ and $\operatorname{dist}\big(f, \mathcal{S}^h(\phi_h); L_p\big) = o(h^\gamma)$ for all sufficiently smooth $f \in L_p$, then $(\mathcal{S}^h(\phi_h))_h$ is said to provide L_p-**density order** γ. Since we are concerned with upper bounds on approximation power, and since statements about upper bounds clearly become stronger as we restrict the class of "smooth functions", we will take the phrase "for all sufficiently smooth $f \in L_p$" in the definitions above to mean "for all f satisfying $\widehat{f} \in C_c^\infty$", where $\widehat{f}$ is the Fourier transform of f defined by

$$\widehat{f}(x) := \int_{\mathbb{R}^d} f(t) e^{-ix\cdot t}\, dt, \qquad \text{and}$$
$$C_c^\infty := \{f \in C^\infty(\mathbb{R}^d) : \operatorname{supp} f \text{ is compact}\}.$$

(As one may expect, the space $\widehat{C_c^\infty} := \{\widehat{f} : f \in C_c^\infty\}$ is included in the definition of "smooth functions" in all lower bound results presently in the literature.)

The first upper bound result on the approximation order of SI spaces is contained in the work of Strang and Fix [SF]. There, it was shown that if ϕ has compact support, then, for $p = 2, \infty$, the stationary ladder $(\mathcal{S}_\star^h(\phi))_h$ provides "controlled" L_p-approximation order k only if $\widehat{\phi}(0) \neq 0$ and one of the following two equivalent conditions holds:

$$\forall f \in \Pi_{k-1}\ \exists\, g \in \Pi_{k-1} \text{ such that } f = \sum_{j\in\mathbb{Z}^d} g(j)\phi(\cdot - j); \tag{1.1}$$

$$D^\alpha \widehat{\phi}(2\pi j) = 0 \text{ for all } |\alpha| < k, j \in \mathbb{Z}^d\backslash 0. \tag{1.2}$$

Condition (1.1) is known as polynomial reproduction, and (1.2) is known as the Strang-Fix conditions of order k. The "controlled" notion puts restrictions on the way the smooth function f could be approximated from $\mathcal{S}^h_\star(\phi)$, hence upper bounds that use this "controlled approximation" or any other restricted notion of approximation (e.g. "local" or "controlled-local" as mentioned below) are weaker than those based on the unqualified notion of approximation orders. For example, if $\widehat{\phi}(0) = 0$, then the "controlled" approximation order (as well as the "local" or "controlled-local" approximation order) is 0, while the approximation order as analysed in the present paper can still be arbitrarily high.

The Strang-Fix result was soon rediscovered in connection with box splines, introduced in [BD], [BH]. [DM] was the first paper to demonstrate the usefulness of the Strang-Fix conditions in box spline theory. There are several important results which like [SF] employ restricted notions of approximation (cf. [BJ], [LC], [JL], and [HL]). These papers give upper bounds on the "controlled", "local", or "controlled-local" L_p-approximation order of stationary ladders generated by one or finitely many basis functions having non-zero mean. In [SF] and [BJ], the basis functions were assumed to be of compact support, while in [LC], [JL], and [HL], the basis functions were assumed to decay like $O(|\cdot|^{-(d+k+\epsilon)})$ at ∞, where k is the approximation order. This assumption is important here, since it ensures that the sum $\sum_{j\in\mathbb{Z}^d} f(j)\phi(\cdot - j)$ converges on compact sets for all $f \in \Pi_k$, hence leaves the notion of polynomial reproduction (1.1) viable. [BJ] and [JL] consider p in the full range $1 \le p \le \infty$, while [LC] and [HL] consider $p = \infty$.

We mention also that upper bounds that are based on the specific structure of the spaces $(\mathcal{S}^h(\phi))_h$ are derived in [BH] (stationary, piecewise-polynomials), and [LJ] (non-stationary, piecewise-exponentials). It is worth mentioning [DR] as the first paper to demonstrate and emphasise the irrelavance of polynomial reproduction for non-stationary ladders.

In [R1], Ron was able to characterize the *actual* L_∞-approximation order of the stationary ladder $(\mathcal{S}^h_\star(\phi))_h$. He showed that if $\phi \in L_\infty$ was compactly supported and if $\sum_{j\in\mathbb{Z}^d}\phi(j) \ne 0$ then $(\mathcal{S}^h_\star(\phi))_h$ provides L_∞-approximation order k if and only if

$$\Pi_{k-1} \subset \mathcal{S}_\star(\phi). \tag{1.3}$$

He also showed that, if $\sum_{j\in\mathbb{Z}^d}\phi(j) = 0$, then (1.3) is not sufficient for L_∞-approximation order k, and left the necessity of (1.3) (for the case $\sum_{j\in\mathbb{Z}^d}\phi(j) = 0$) as an open question. Following this work, de Boor and Ron [BR] considered again the case $p = \infty$ for non-stationary ladders generated by functions ϕ_h which decay like $O(|\cdot|^{-(d+\epsilon)})$. They obtained upper and lower bounds on the L_∞-approximation order. It is important to note that the decay assumption on ϕ_h does *not* provide for the convergence on compact sets of the sum $\sum_{j\in\mathbb{Z}^d} f(j)\phi_h(\cdot - j)$ when $f \in \Pi_1$, hence (at least for $k > 1$) the notion of polynomial reproduction (1.1) is no longer viable. However, even in the absence of any meaning to polynomial reproduction, the Fourier transform analog of that property remains important. It was shown that the Strang-Fix conditions (1.2) are still necessary for stationary ladders, and that further, closely related Fourier transform conditions are necessary for non-stationary ladders, though these other conditions lead to no polynomial reproduction, even for compactly supported generators.

In [BDR], de Boor, DeVore, and Ron gave a complete characterization of the L_2-approximation order of $(\mathcal{S}_2^h(\Phi))_h$ when Φ is a finite subset of L_2, where

$$\mathcal{S}_2(\Phi) := \text{closure}\left(\sum_{\phi\in\Phi} \mathcal{S}_0(\phi); L_2\right).$$

Very recently in [J], Jia has generalized the stationary part of the results of Ron [R1] to $1 \le p \le \infty$. A characterization of the L_p-approximation order of $(\mathcal{S}_\star^h(\Phi))_h$ is given under the assumption that Φ consists of finitely many compactly supported functions in L_p and $\widehat{\phi}(0) \ne 0$ for all $\phi \in \Phi$, where

$$\mathcal{S}(\Phi) := \sum_{\phi\in\Phi} \mathcal{S}_\star(\phi).$$

When specialized to the case when $\Phi = \{\phi\}$, this characterization is shown to be equivalent to the Strang-Fix conditions (1.2). The aspect of that characterization which is most relevant to the present discussion can be stated as follows.

Theorem 1.4 [J]. *Let $1 \le p \le \infty$, $k \in \mathbb{N}$. Let $\phi \in L_p(\mathbb{R}^d)$ have compact support and satisfy $\widehat{\phi}(0) \ne 0$. Assume that ϕ does not satisfy the Strang-Fix conditions of order k, i.e. there exists $j_0 \in \mathbb{Z}^d\backslash 0$ and $|\alpha| < k$ such that $D^\alpha\widehat{\phi}(2\pi j_0) \ne 0$. Then the stationary ladder $(\mathcal{S}_\star^h(\phi))_h$ does not achieve L_p-density order $k-1$, and consequently, the L_p-approximation order of $(\mathcal{S}^h(\phi))_h$ cannot exceed $k-1$.*

In the present paper, we provide an upper bound on the L_p-approximation power of the non-stationary ladder $(\mathcal{S}_p^h(\phi_h))_h$ in terms of the Fourier transform of ϕ_h. In contrast to [R1], [BR] and [BDR], we do not assume $p = 2, \infty$, but rather, the full range $1 \le p \le \infty$. Also, we do not assume that the basis functions ϕ_h are compactly supported as in [R1] and [J]; we allow the basis functions ϕ_h to decay in a rather mild fashion. Under fortuitous circumstances (see condition $(*)$ of Theorems 3.1, 4.1, and 4.2) our results can even handle the troublesome case $\widehat{\phi}_h(0) = 0$ which in the past has only been handled well by [BDR]. One interesting feature of our stationary results is that when they apply, they show not just that one function in $\widehat{C_c^\infty}$ cannot be approximated better than a certain rate, but that a large class of functions cannot be approximated better than this rate. The following theorem serves as an illustration. To keep things clean, we consider the stationary case, and we assume the basis function ϕ satisfies the mild decay condition $\phi \in L_1$ and the regularity condition $\widehat{\phi}(0) \ne 0$.

Theorem 1.5. *Let $1 \le p \le \infty$. Let $\phi \in L_p \cap L_1$ satisfy $\widehat{\phi}(0) \ne 0$ and assume that all derivatives up to order $k-1$ of $\widehat{\phi}$ are continuous in a neighborhood of $2\pi j_0$ for some $j_0 \in \mathbb{Z}^d\backslash 0$ and $k \in \mathbb{N}$. If the Strang-Fix conditions of order k fail at $2\pi j_0$, i.e. there exists $|\alpha| < k$ such that $D^\alpha\widehat{\phi}(2\pi j_0) \ne 0$, then the stationary ladder $(\mathcal{S}_p^h(\phi))_h$ does not achieve L_p-density order $k-1$, and consequently, the L_p-approximation order of $(\mathcal{S}_p^h(\phi))_h$ cannot exceed $k-1$. In fact,*

$$\text{dist}\left(f, \mathcal{S}_p^h(\phi); L_p\right) \ne o(h^{k-1}) \text{ as } h \to 0$$

for all $f \in W_p^{k-1}\backslash 0$ *(if* $1 \le p \le 2$*),* $f \in W_2^{k-1} \cap L_p^0\backslash 0$ *(if* $2 < p \le \infty$*), where* W_p^k *is the usual Sobolev space.*

Proof. cf. §5.

In other words, for $1 \le p \le \infty$, the Strang-Fix conditions of order k (to the extent that they are meaningful) are necessary for the stationary ladder $(\mathcal{S}_p^h(\phi))_h$ to provide L_p-approximation order k under the assumptions that $\phi \in L_1 \cap L_p$ and $\widehat{\phi}(0) \ne 0$. That the approximation order moves all the way back to $k-1$ when the Strang-Fix conditions of order k fail to hold, is due to the smoothness assumed of $\widehat{\phi}$ near $2\pi j_0$. In the sequel, no such smoothness will be required of $\widehat{\phi}$, and accordingly, the approximation orders will not be confined to the integers.

The following notations are used throughout this paper. The natural numbers are denoted by $\mathbb{N} := \{1, 2, 3, \dots\}$, while the non-negative integers are denoted by $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$. The unit cube in $\mathbb{R}^d$, $(-1/2\,..\,1/2]^d$, is denoted by C. $B := \{x \in \mathbb{R}^d : |x| < 1\}$ denotes the open unit ball in $\mathbb{R}^d$, where

$$|x| := \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}, \qquad x \in \mathbb{R}^d.$$

The bounded functions which decay to 0 at ∞ are denoted by

$$L_\infty^0 := \{f \in L_\infty : |f(x)| \to 0 \text{ as } |x| \to \infty\}.$$

It is notationally expedient to define

$$L_p^0 := L_p(\mathbb{R}^d) \qquad \text{for } 1 \le p < \infty.$$

The semi-discrete convolution is defined formally by

$$\phi *' c := \sum_{j \in \mathbb{Z}^d} c(j)\phi(\cdot - j).$$

We denote the distance from the function f to a set of functions A, as measured in the X norm, by

$$\operatorname{dist}(f, A; X) := \inf_{s \in A} \|f - s\|_X .$$

We employ the convention that 0 times anything is 0; in particular, $0/0 := 0$. $\sigma : \mathbb{R}^d \to [0\,..\,1]$ is taken to be an even function in $C_c^\infty(B)$ which satisfies $\sigma(x) = 1$ for all $x \in B/2$. Here $C_c^\infty(\Omega) := \{f \in C_c^\infty : \operatorname{supp} f \subseteq \Omega\}$. We denote the conjugate exponent of $p \in [1\,..\,\infty]$ by p' defined by $\frac{1}{p} + \frac{1}{p'} = 1$. The inverse Fourier transform of a tempered distribution f is denoted $f^\vee$.

2. Preliminaries

We define now $\mathcal{S}_\infty(\phi)$ for $\phi \in L_\infty$. Since our purpose is to establish an upper bound on the approximation power of $(\mathcal{S}^h(\phi))_h$, it is desirable that $\mathcal{S}_\infty(\phi)$ be as large as possible. Note that closure$(\mathcal{S}_0(\phi); L_\infty)$ $(\subset L_\infty^0$ if $\phi \in L_\infty^0)$ is usually too small to approximate functions $f \in L_\infty\backslash L_\infty^0$. However, if ϕ has sufficient decay at ∞, then the sum $\phi *' c$ is meaningful in that it converges uniformly on compact sets to a bounded function whenever $c \in \ell_\infty := \ell_\infty(\mathbb{Z}^d)$. It is desirable that our definition of $\mathcal{S}_\infty(\phi)$ include functions of the form $\phi *' c$, $c \in \ell_\infty$, whenever ϕ has sufficient decay.

Definition 2.1. For $\phi \in L_\infty$, we define $\mathcal{S}_\infty(\phi)$ to be the largest (i.e. union of all) shift-invariant set $\mathcal{S} \subset L_\infty$ which satisfies

$$\operatorname{dist}(f, \mathcal{S}; L_\infty) = \operatorname{dist}(f, \mathcal{S}_0(\phi); L_\infty), \quad \forall\, f \in L_\infty^0.$$

Note that the set $\mathcal{S}_\infty(\phi)$ always contains the space closure$(\mathcal{S}_0(\phi); L_\infty)$. In case ϕ has sufficient decay at ∞, then $\mathcal{S}_\infty(\phi)$ also contains the space $\{\phi *' c : c \in \ell_\infty\}$ as recorded below:

Proposition 2.2. *If $\phi \in L_\infty$ satisfies*

$$\sum_{j \in \mathbb{Z}^d} \|\phi\|_{L_\infty(j+C)} < \infty,$$

then $\{\phi *' c : c \in \ell_\infty\} \subseteq \mathcal{S}_\infty(\phi)$.

Proof. cf. §5.

In order to explain the approach taken in the sequel, the following definition is needed (compare with [BDR; th.2.14] and [K; th.2.6]).

Definition 2.3. Let $\eta : \mathbb{R}^d \to \mathbb{C}$ be a measurable function. We denote by $\mathbb{S}(\eta)$ the collection of all locally integrable functions $s : \mathbb{R}^d \to \mathbb{C}$ which, when viewed as distributions, are tempered, and which satisfy

$$(i) \quad \widehat{s} \text{ is regular on } \mathbb{R}^d;$$
$$(ii) \quad \widehat{s} = \eta\tau \text{ for some } 2\pi\mathbb{Z}^d\text{-periodic function } \tau,$$

where a tempered distribution μ is said to be **regular** on an open set $\Omega \subseteq \mathbb{R}^d$ if μ can be identified on Ω with a function which is locally integrable on Ω.

The relevance of the above definition to the task of establishing an upper bound on the L_p-approximation power of the ladder $(\mathcal{S}_p^h(\phi_h))_h$ is shown in the following:

Proposition 2.4. *Let $1 \le p \le \infty$, and for $h \in (0\,.\,.\,1]$, let $\phi_h \in L_p$ be such that $\widehat{\phi_h}$ is regular on $\mathbb{R}^d$; hence, $\widehat{\phi_h}$ can be identified with a locally integrable function $\eta_h : \mathbb{R}^d \to \mathbb{C}$. Then, for all $f \in L_p^0$,*

$$\operatorname{dist}\left(f, \mathcal{S}_p^h(\phi_h); L_p\right) \ge \operatorname{dist}\left(f, \mathbb{S}^h(\eta_h); L_p\right).$$

Consequently, any upper bound on the L_p-approximation power of $(\mathbb{S}^h(\eta_h))_h$ is an upper bound on the L_p-approximation power of $(\mathcal{S}_p^h(\phi_h))_h$.

Proof. The assumptions on ϕ_h yield immediately that $\mathcal{S}_0(\phi_h) \subseteq \mathbb{S}(\eta_h)$. The proposition is thus a consequence of the fact that

$$\operatorname{dist}\left(f, \mathcal{S}_p^h(\phi_h); L_p\right) = \operatorname{dist}\left(f, \mathcal{S}_0^h(\phi_h); L_p\right) \qquad \text{for all } f \in L_p^0.$$

$\square$

In the present paper, we will in fact be deriving upper bounds on the L_p-approximation power of the non-stationary ladder $(\mathbb{S}^h(\eta_h))_h$ in terms of the measurable functions η_h. In addition to the scenario of Proposition 2.4, there is another important situation (arising often in the context of Radial Basis functions) which motivates our consideration of $(\mathbb{S}^h(\eta_h))_h$.

For $h > 0$, let ϕ_h be a locally integrable function which, when viewed as a distribution, is tempered, and assume that $\widehat{\phi}_h$ is regular on $\mathbb{R}^d \backslash 0$; hence, $\widehat{\phi}_h$ can be identified on $\mathbb{R}^d \backslash 0$ with a function $\eta_h : \mathbb{R}^d \to \mathbb{C}$ which is locally integrable on $\mathbb{R}^d \backslash 0$. A typical approach to defining a PSI space generated by ϕ_h is to first find a function ψ_h, in either $\mathcal{S}_0(\phi_h)$ or some super-space of $\mathcal{S}_0(\phi_h)$, such that $\psi_h \in L_p$, and then forming the PSI space $\mathcal{S}_p(\psi_h)$. ψ_h is referred to as a "localization of ϕ_h". In the literature, ψ_h generally satisfies two additional conditions:

$$\begin{aligned} &\widehat{\psi}_h \text{ is regular on } \mathbb{R}^d; \\ &\widehat{\psi}_h = \eta_h \tau_h \text{ for some } 2\pi\mathbb{Z}^d\text{-periodic function } \tau_h. \end{aligned} \tag{2.5}$$

Note that $\mathcal{S}_0(\psi_h) \subseteq \mathbb{S}(\eta_h)$ whenever $\psi_h \in L_p$ is a localization of ϕ_h satisfying (2.5). Hence for all $f \in L_p^0$,

$$\operatorname{dist}\left(f, \mathcal{S}_p^h(\psi_h); L_p\right) = \operatorname{dist}\left(f, \mathcal{S}_0^h(\psi_h); L_p\right) \geq \operatorname{dist}\left(f, \mathbb{S}^h(\eta_h); L_p\right).$$

This proves the following:

Proposition 2.6. *Let $1 \leq p \leq \infty$, and let ϕ_h and η_h be as in the above paragraph. If $\psi_h \in L_p$ satisfies (2.5), then for all $f \in L_p^0$,*

$$\operatorname{dist}\left(f, \mathcal{S}_p^h(\psi_h); L_p\right) \geq \operatorname{dist}\left(f, \mathbb{S}^h(\eta_h); L_p\right).$$

Consequently, any upper bound on the L_p-approximation power of $(\mathbb{S}^h(\eta_h))_h$ is an upper bound on the L_p-approximation power of $(\mathcal{S}^h(\psi_h))_h$ which does not depend on which localization of ϕ_h is chosen – so long as (2.5) holds.

With Propositions 2.4 and 2.6 in mind, we proceed now toward establishing upper bounds on the L_p-approximation power of the ladder $(\mathbb{S}^h(\eta_h))_h$ in terms of the family $(\eta_h)_h$. Theorem 3.1, although stated in the non-stationary context, is intended primarily for stationary ladders. Under a certain regularity assumption on the family $(\eta_h)_h$, a condition is given which ensures that for $1 \leq p \leq \infty$, the non-stationary ladder $(\mathbb{S}^h(\eta_h))_h$ does not provide L_p-density order $\gamma \geq 0$. The reason we focus here on the density order as opposed to the approximation order is that when a lower bound on the L_p-approximation order of some ladder is known, e.g. it provides L_p-approximation of order γ, then the *best* way to show that this lower bound is optimal is to show that the ladder does not provide L_p-density order γ. Theorem 3.1 attempts to provide a means for verifying that a given stationary ladder (or an especially structured non-stationary ladder) does not provide density order γ. Following its statement, conditions on a family of L_1-functions $(\phi_h)_h$ are given in Proposition 3.4 which, when satisfied, ensure that the regularity assumption of Theorem 3.1 is satisfied. The theorem is then applied to some PSI spaces generated by (exponential) box splines, polyharmonic splines, and multiquadrics.

Our most general results are recorded in Theorem 4.1 ($1 \le p \le 2$) and Theorem 4.2 ($2 < p \le \infty$). They are intended for non-stationary or stationary ladders which cannot be handled by Theorem 3.1. Under a certain regularity assumption on the family $(\eta_h)_h$ (which is weaker than that of Theorem 2.1) an upper bound on the L_p-approximation power of the ladder $(\mathbb{S}^h(\eta_h))_h$ is given. Following the statements of these theorems, an analog of Proposition 3.4 is stated which similarly gives easily verifiable conditions on a family of L_1-functions $(\phi_h)_h$ which when satisfied ensure that the regularity assumptions of Theorems 4.1 and 4.2 are satisfied. All of the theorems and propositions are proved in §5.

3. The Stationary Case

In this section, we state a somewhat specialized version of our results and apply it to the examples of (exponential) box splines, polyharmonic splines, and multiquadrics.

Theorem 3.1. *Let $1 \le p \le \infty$. Let $\eta_h : \mathbb{R}^d \to \mathbb{C}$, $h \in (0\,..\,1]$ be a family of measurable functions which satisfy, for some $j_0 \in \mathbb{Z}^d \backslash 0$, $\epsilon \in (0\,..\,\pi)$, $c < \infty$, and for all sufficiently small $h \in (0\,..\,1]$,*

$$(*) \qquad \left\| \left(\sigma(\,\cdot/\epsilon) \frac{\eta_h(\,\cdot + 2\pi j_0)}{\eta_h} \right)^{\vee} \right\|_{L_1} \le c.$$

Assume that there exists $\gamma \ge 0$, and a non-trivial locally bounded measurable function $\rho : \mathbb{R}^d \to \mathbb{C}$ such that

$$\left\| \frac{\eta_h(h\,\cdot + 2\pi j_0)}{\eta_h(h\,\cdot)} - h^\gamma \rho \right\|_{L_\infty(nB)} = o(h^\gamma) \text{ as } h \to 0, \qquad \text{for all } n > 0. \tag{3.2}$$

Then $(\mathbb{S}^h(\eta_h))_h$ does not provide L_p-density order γ and consequently, the L_p-approximation order of $(\mathbb{S}^h(\eta_h))_h$ cannot exceed γ. In fact, $\operatorname{dist}\big(f, \mathbb{S}^h(\eta_h); L_p\big) \ne o(h^\gamma)$ *as $h \to 0$ for all $f \in L_p^0 \cap L_q$ which satisfy*

$$\rho \widehat{f} \ne 0 \qquad \text{say, as a distribution;} \tag{3.3}$$

$$\left\| \widehat{f} \right\|_{L_{q'}(\mathbb{R}^d \backslash h^{-1} B)} = o(h^\gamma) \text{ as } h \to 0, \tag{3.4}$$

where $q := \min\{p, 2\}$.

Proof. cf. §5

Note that in the stationary case, condition (3.2) is satisfied if there exists a non-trivial homogeneous function ρ of order γ such that

$$\left\| \frac{\eta(\,\cdot + 2\pi j_0)}{\eta} - \rho \right\|_{L_\infty(hB)} = o(h^\gamma) \text{ as } h \to 0. \tag{3.5}$$

Condition $(*)$ is somewhat mysterious and seemingly difficult to verify. To make it less so, we show that in the context of Proposition 2.4, it is implied by other easily verifiable conditions.

Proposition 3.6. *For $h \in (0\,..\,1]$, let $\phi_h \in L_1$ be normalized so that $\lim_{h\to 0}\widehat{\phi}_h(0) = 1$. Assume that*

$$(i) \quad \sup_{h\in(0..1]} \|\phi_h\|_{L_1} < \infty;$$
$$(ii) \quad \lim_{n\to\infty} \sup_{h\in(0..1]} \|\phi_h\|_{L_1(\mathbb{R}^d\backslash nB)} = 0.$$

Then there exists $\epsilon \in (0\,..\,\pi]$ and $c < \infty$ such that, with $\eta_h := \widehat{\phi}_h$, condition $(*)$ of Theorem 3.1 is satisfied for all $j_0 \in \mathbb{Z}^d\backslash 0$.

Proof. cf. §5

In particular, in the stationary case (i.e. $\eta_h = \widehat{\phi}$), if we assume simply that $\phi \in L_1$ and $\widehat{\phi}(0) \neq 0$, then condition $(*)$ of Theorem 3.1 is satisfied for all $j_0 \in \mathbb{Z}^d\backslash 0$.

Example 3.7. (Exponential) Box Splines: Let Ξ be a multiset of directions in $\mathbb{R}^d\backslash 0$ and assume that the directions in Ξ span $\mathbb{R}^d$. For each $\xi \in \Xi$, let $\lambda_\xi \in \mathbb{C}$. The family of (exponential) box splines ϕ_h, $h \geq 0$, is then defined by

$$\widehat{\phi}_h := \prod_{\xi\in\Xi} \omega_\xi^h, \qquad \text{where } \omega_\xi^h(x) := \int_0^1 e^{(h\lambda_\xi - i\xi\cdot x)t}\,dt.$$

It was shown in [BR] ($p = \infty$) that $(\mathcal{S}^h(\phi_h))_h$ does not provide L_∞-density order k', where k' is defined by:

$$k' := \min\{\#K_j : j \in \mathbb{Z}^d\backslash 0\};$$
$$K_j := \{\xi \in \Xi : \xi\cdot j \in \mathbb{Z}\backslash 0\}.$$

It was shown by Ron in [R3] ($p = 2$) that $(\mathcal{S}_2^h(\phi_h))_h$ does not provide L_2-density order k'; in fact, $\operatorname{dist}\left(f, \mathcal{S}_2^h(\phi_h); L_2\right) \neq o(h^{k'})$ as $h \to 0$ for all nontrivial $f \in W_2^{k'}$. (For a detailed discussion of currently known lower bounds on the approximation order, the reader is refered to section 3 of [R3].) We will show by applying Theorem 3.1 that for $1 \leq p \leq \infty$, $(\mathcal{S}_p^h(\phi_h))_h$ does not provide L_p-density order k' and consequently, the L_p-approximation order of $(\mathcal{S}_p^h(\phi_h))_h$ cannot exceed k'. In fact, $\operatorname{dist}\left(f, \mathcal{S}_p^h(\phi_h); L_p\right) \neq o(h^{k'})$ as $h \to 0$ for all nontrivial $f \in L_p^0 \cap L_q$ which satisfy (3.4) (with $\gamma := k'$). Note that this recovers what was previously known for the case $p = 2$. We fail to recover what was previously known for the exceptional case $p = \infty$ only because $\mathcal{S}_\infty(\phi_h)$ as defined here is smaller than $\mathcal{S}(\phi_h)$ as defined in [BR].

In order to prove the above claim, let $1 \leq p \leq \infty$, and put $\eta_h := \widehat{\phi}_h$, $\gamma := k'$. It is known that $(\phi_h)_{h\in(0..1]}$ is a uniformly bounded family of compactly supported functions whose support is independent of h (cf. [BR]). It thus follows by Proposition 2.4 that

$$\operatorname{dist}\left(f, \mathcal{S}_p^h(\phi_h); L_p\right) \geq \operatorname{dist}\left(f, \mathbb{S}^h(\eta_h); L_p\right), \quad \forall\, f \in L_p^0.$$

Now, since $\lim_{h\to 0}\widehat{\phi}_h(0) = 1$, it follows by Proposition 3.6[1] that there exists $\epsilon \in (0\,..\,\pi]$ and $c < \infty$ such that condition $(*)$ of Theorem 3.1 is satisfied for all $j_0 \in \mathbb{Z}^d\backslash 0$. Let $j_0 \in \mathbb{Z}^d\backslash 0$

[1] Note that conditions (i) and (ii) of Proposition 3.6 follow from the above mentioned fact that the functions ϕ_h as well as their support are bounded independently of $h \in (0\,..\,1]$.

be such that $\#K_{j_0} = k'$. We now consider (3.2). Fix $n > 0$. Following [BR], note that $\|1 - \eta_h(h\,\cdot)\|_{L_\infty(nB)} \to 0$ as $h \to 0$, and for each $\xi \in \Xi \backslash K_{j_0}$, we have $\omega_\xi^0(2\pi j_0) \neq 0$ and

$$\left\|\omega_\xi^h(h\cdot + 2\pi j_0) - \omega_\xi^0(2\pi j_0)\right\|_{L_\infty(nB)} \to 0 \text{ as } h \to 0.$$

On the other hand, if $\xi \in K_{j_0}$ then

$$\sup_{x \in nB}\left|h^{-1}\omega_\xi^h(hx + 2\pi j_0) - \frac{\lambda_\xi - i\xi\cdot x}{-i\xi\cdot 2\pi j_0}\right| \to 0 \text{ as } h \to 0.$$

Defining

$$\begin{aligned}
N_h &:= \frac{1}{\eta_h(h\,\cdot)}\prod_{\xi \in \Xi\backslash K_{j_0}} \omega_\xi^h(h\cdot + 2\pi j_0);\\
Z_h &:= \prod_{\xi \in K_{j_0}} \omega_\xi^h(h\cdot + 2\pi j_0);\\
\rho(x) &:= \left(\prod_{\xi \in \Xi\backslash K_{j_0}} \omega_\xi^0(2\pi j_0)\right)\prod_{\xi \in K_{j_0}} \frac{\lambda_\xi - i\xi\cdot x}{-i\xi\cdot 2\pi j_0},
\end{aligned}$$

We conclude from the above that

$$h^{-k'}\frac{\eta_h(h\cdot + 2\pi j_0)}{\eta_h(h\,\cdot)} = h^{-k'}N_h Z_h \xrightarrow[h\to 0]{} \rho \qquad \text{in } L_\infty(nB)$$

which establishes (3.2). Lastly, note that since ρ is a nonzero polynomial, condition (3.3) holds whenever f is a non-trivial function in $L_p^0 \cap L_q$. The intended result now follows with an application of Theorem 3.1.

We will use the following lemma in the next example.

Lemma 3.8 [Wiener's Lemma]. *If $\phi \in L_1$ and $\widehat{\phi}(0) \neq 0$, then there exists $\epsilon > 0$ such that*

$$\left\|\left(\frac{\sigma(\,\cdot/\epsilon)}{\widehat{\phi}}\right)^\vee\right\|_{L_1} < \infty.$$

Proof. cf. §5

Example 3.9. Polyharmonic Splines ($a = 0$) and Multiquadrics ($a > 0$): Let $\gamma > 0$, $a \geq 0$, and define $\phi := (|\cdot|^2 + a^2)^{(\gamma-d)/2}$ (if $\gamma - d \notin 2\mathbb{Z}_+$), or $\phi := (|\cdot|^2 + a^2)^{(\gamma-d)/2}\log(|\cdot|^2 + a^2)$ (if $\gamma - d \in 2\mathbb{Z}_+$). Then according to [GS], $\widehat{\phi}$ can be identified on $\mathbb{R}^d\backslash 0$ with

$$\eta := b\,|\cdot|^{-\gamma}\,\widetilde{K}_{\gamma/2}(a\,|\cdot|),$$

where $b = b(d,\gamma)$ is some non-zero constant, $\widetilde{K}_\nu(t) := t^\nu K_\nu(t)$, and K_ν is the modified Bessel function of order ν [AS]. In case $a = 0$, we will be assuming that $\gamma > d/p'$ in order to ensure that ϕ is locally in L_p.

It is known that ϕ can be localized to a function $\psi \in L_p$, satisfying (2.5) and $\widehat{\psi}(0) \neq 0$, and that $(\mathcal{S}_p^h(\psi))_h$ provides L_p-approximation order γ (cf. [Bu], [BuR]). As for upper bounds on the approximation power, the case $p = \infty$ is considered in [BR]. It is shown that the dilates of a (large) shift-invariant space generated by ψ do not achieve L_∞-density order γ whenever ψ is a localization of ϕ satisfying (2.5) and additionally

$$\sum_{j \in \mathbb{Z}^d} |\psi(x-j)| < \text{const for all } x \in \mathbb{R}^d;$$
$$\widehat{\psi}(0) \neq 0.$$

The case $p = 2$ is considered by Ron in [R2]. It is there shown that if $\psi \in L_2$ is any localization of ϕ satisfying (2.5), then $(\mathcal{S}_2^h(\psi))_h$ does not achieve L_2-density order γ. In fact, $\operatorname{dist}\left(f, \mathcal{S}_2^h(\psi); L_2\right) \neq o(h^\gamma)$ for all non-trivial f in the Sobolev space W_2^γ. We will show, by applying Theorem 3.1 that for $1 \leq p \leq \infty$, if $\psi \in L_p$ is any localization of ϕ satisfying (2.5), then $(\mathcal{S}_p^h(\psi))_h$ does not achieve L_p-density order γ and consequently, the L_p-approximation order of $(\mathcal{S}_p^h(\psi_h))_h$ cannot exceed γ. In fact, $\operatorname{dist}\left(f, \mathcal{S}_p^h(\psi); L_p\right) \neq o(h^\gamma)$ as $h \to 0$ for all nontrivial $f \in L_p^0 \cap L_q$ which satisfy (3.4). Note that this recovers what was previously known for the case $p = 2$. We fail to recover what was previously known for the exceptional case $p = \infty$ only because $\mathcal{S}_\infty(\phi_h)$ as defined here is smaller than $\mathcal{S}(\phi_h)$ as defined in [BR].

In order to verify the above claim, let $1 \leq p \leq \infty$, and let $\psi \in L_p$ satisfy (2.5). Put $\eta_h := \eta$ for all $h \in (0\,..\,1]$. Recall from Proposition 2.6 that

$$\operatorname{dist}\left(f, \mathcal{S}_p^h(\psi); L_p\right) \geq \operatorname{dist}\left(f, \mathbb{S}^h(\eta); L_p\right), \quad \forall\, f \in L_p^0.$$

Let j_0 be any element of $\mathbb{Z}^d \backslash 0$ and define

$$\rho := \frac{\widetilde{K}_{\gamma/2}(a\,|2\pi j_0|)}{|2\pi j_0|^\gamma\, \widetilde{K}_{\gamma/2}(0)}\, |\cdot|^\gamma .$$

That (3.5) holds follows from the fact that $\widetilde{K}_{\gamma/2}$ is continuous on $[0\,..\,\infty)$ and $\widetilde{K}_{\gamma/2}(0) \neq 0$. Since ρ is non-zero almost everywhere, (3.3) holds whenever $f \in L_p^0 \cap L_q \backslash 0$. We turn now to condition $(*)$ of Theorem 3.1. There exist (see [AS]) entire functions A_1, A_2, and A_3 with $A_1(0) \neq 0$ such that

$$b\widetilde{K}_{\gamma/2}(a\,|x|) = A_1(|x|^2) + A_2(|x|^2)\,|x|^\gamma + A_3(|x|^2)\,|x|^2 \log|x|\,.$$

Thus, for $\epsilon > 0$ sufficiently small and $x \in \mathbb{R}^d \backslash 0$,

$$\begin{aligned}
\sigma(x/\epsilon)\frac{\eta(x+2\pi j_0)}{\eta(x)} &= \frac{\sigma(x/\epsilon)\eta(x+2\pi j_0)\,|x|^\gamma}{A_1(|x|^2) + A_2(|x|^2)\,|x|^\gamma + A_3(|x|^2)\,|x|^2\log|x|} \\
&= \frac{\sigma(x/\epsilon)\left(\sigma(x/2\pi)\eta(x+2\pi j_0)\,|x|^\gamma\right)}{\sigma(x/2\pi)\left(A_1(|x|^2) + A_2(|x|^2)\,|x|^\gamma + A_3(|x|^2)\,|x|^2\log|x|\right)}.
\end{aligned}$$

Since $\eta \in C^\infty(\mathbb{R}^d \backslash 0)$, $A_1(0) \neq 0$, and with Lemma 3.8 in view, it is easy to see that condition $(*)$ now follows from the fact that $(\sigma(\cdot/\epsilon)\,|\cdot|^\gamma)^\vee \in L_1$, and $\left(\sigma(\cdot/\epsilon)\,|\cdot|^2\log|\cdot|\right)^\vee \in L_1$ for all $\epsilon > 0$. Applying Theorem 3.1 yields the intended result.

4. The Non-Stationary Case

In this section we state an upper bound on the L_p-approximation power of the non-stationary ladder $(\mathbb{S}^h(\eta_h))_h$ under a certain regularity assumption on the family $(\eta_h)_h$. In order to avoid restrictive assumptions on the ladder $(\mathcal{S}_p^h(\phi_h))_h$ (like those adopted in the previous section), we will not attempt to show that our upper bound is realized by a large class of smooth approximands; rather, we show simply that there exists at least one smooth function f which cannot be approximated faster than a certain rate. We then apply the result to a non-stationary ladder of PSI spaces generated by the Gauss kernel. We treat separately the case $1 \le p \le 2$ and the case $2 < p \le \infty$.

Theorem 4.1. *Let $1 \le p \le 2$, and for $h \in (0\,..\,1]$, let $\eta_h : \mathbb{R}^d \to \mathbb{C}$ be measurable. Assume that there exists $j_0 \in \mathbb{Z}^d \backslash 0$, $\epsilon \in (0\,..\,\pi)$, and $c < \infty$ such that for all sufficiently small $h \in (0\,..\,1]$,*

$$(*) \qquad \left\| \frac{\eta_h(\,\cdot + 2\pi j_0)}{\eta_h} \right\|_{L_\infty(h\epsilon B)} \le c.$$

Then there exists a sufficiently smooth function f (i.e. $\widehat{f} \in C_c^\infty$) such that

$$\operatorname{dist}\left(f, \mathbb{S}^h(\eta_h); L_p\right) \ge \left\| \frac{\eta_h(h\,\cdot + 2\pi j_0)}{\eta_h(h\,\cdot)} \right\|_{L_{p'}(\epsilon B/2)}$$

for sufficiently small $h \in (0\,..\,1]$. In particular, $(\mathbb{S}^h(\eta_h))_h$ provides L_p-approximation order $\gamma > 0$ (resp. L_p-density order $\gamma \ge 0$) only if

$$\left\| \frac{\eta_h(h\,\cdot + 2\pi j_0)}{\eta_h(h\,\cdot)} \right\|_{L_{p'}(\epsilon B/2)} = O(h^\gamma) \text{ (resp. } = o(h^\gamma)) \text{ as } h \to 0.$$

Proof. cf. §5

Theorem 4.2. *Let $2 < p \le \infty$, and for $h \in (0\,..\,1]$, let $\eta_h : \mathbb{R}^d \to \mathbb{C}$ be measurable. Assume that there exists $j_0 \in \mathbb{Z}^d \backslash 0$, $\epsilon \in (0\,..\,\pi)$, and $c < \infty$ such that for all sufficiently small $h \in (0\,..\,1]$,*

$$(*) \qquad \left\| \left(\sigma(\,\cdot/h\epsilon) \frac{\eta_h(\,\cdot + 2\pi j_0)}{\eta_h} \right)^{\vee} \right\|_{L_1} \le c.$$

Then for all $\nu \in C_c^\infty(\epsilon B/4)$, there exists a sufficiently smooth function f (i.e. $\widehat{f} \in C_c^\infty$) such that

$$\operatorname{dist}\left(f, \mathbb{S}^h(\eta_h); L_p\right) \ge \left\| \left(\frac{\eta_h(h\,\cdot + 2\pi j_0)}{\eta_h(h\,\cdot)} \right) * \nu \right\|_{L_2(\epsilon B/4)}$$

for all sufficiently small $h \in (0\,..\,1]$. In particular, $(\mathbb{S}^h(\eta_h))_h$ provides L_p-approximation order $\gamma > 0$ (resp. L_p-density order $\gamma \ge 0$) only if

$$\left\| \left(\frac{\eta_h(h\,\cdot + 2\pi j_0)}{\eta_h(h\,\cdot)} \right) * \nu \right\|_{L_2(\epsilon B/4)} = O(h^\gamma) \text{ (resp. } = o(h^\gamma)) \text{ as } h \to 0$$

for all $\nu \in C_c^\infty(\epsilon B/4)$.

Proof. cf. §5

Note that condition $(*)$ of Theorem 4.1 is weaker than that of Theorem 4.2 which in turn is weaker than that of Theorem 3.1. As a result, an analog of Proposition 3.6 for the present theorems can be proven under weaker assumptions on the family $(\phi_h)_h$. Indeed:

Proposition 4.3. *For* $h \in (0\,..\,1]$, *let* $\phi_h \in L_1$ *be normalized so that* $\lim_{h\to 0} \widehat{\phi}_h(0) = 1$. *Assume that*

$$(i) \quad \sup_{h\in(0..1]} \|\phi_h\|_{L_1} < \infty;$$

$$(ii) \quad \lim_{n\to\infty} \sup_{h\in(0..1]} \|\phi_h\|_{L_1(\mathbb{R}^d\setminus h^{-1}nB)} = 0.$$

Then there exists $\epsilon \in (0\,..\,\pi]$ *and* $c < \infty$ *such that, with* $\eta_h := \widehat{\phi}_h$, *condition* $(*)$ *of Theorems 4.1 and 4.2 is satisfied for all* $j_0 \in \mathbb{Z}^d\backslash 0$.

Proof. cf. §5

Note that the only difference between the hypothesis of Proposition 4.3 and that of Proposition 3.6 is condition (ii). In the following example, we take advantage of this and are thus able to apply Theorems 4.1 and 4.2.

Example 4.4. Gauss Kernel: Let $\gamma > 0$ be fixed, and for $h \in (0\,..\,1/e]$ define ϕ_h by

$$\widehat{\phi}_h(x) := e^{-\gamma \log(1/h)|x|^2/4\pi^2}, \qquad x \in \mathbb{R}^d.$$

This particular family was studied in [BeL] for integer values of γ and $p = \infty$ where almost optimal lower bounds on the approximation power were derived. Furthermore, the general results of [BR] apply here (for all $\gamma > 0$) and show that, for $p = \infty$, the optimal approximation order is γ. We will show, by applying Theorems 4.1 and 4.2 that for all $1 \le p \le \infty$, $(\mathcal{S}_p^h(\phi_h))_h$ does not provide L_p-density order γ, and consequently, the L_p-approximation order of $(\mathcal{S}_p^h(\phi_h))_h$ cannot exceed γ.

For that, put $\eta_h := \widehat{\phi}_h$, and note that by Proposition 2.4,

$$\operatorname{dist}\left(f, \mathcal{S}_p^h(\phi_h); L_p\right) \ge \operatorname{dist}\left(f, \mathbb{S}^h(\eta_h); L_p\right), \qquad f \in L_p^0.$$

Since, $\phi_h(x) = (\log(1/h))^{-d/2}\phi_{1/e}(x/\sqrt{\log(1/h)})$ it follows that the hypothesis of Proposition 4.3 is satisfied (while that of Proposition 3.6 is not) and therefore, there exists $\epsilon \in (0\,..\,\pi]$ and $c < \infty$ such that condition $(*)$ of Theorems 4.1 and 4.2 are satisfied for all $j_0 \in \mathbb{Z}^d\backslash 0$. Put $j_0 := (1, 0, 0, \ldots, 0)$, and let $x \in \epsilon B$. A straightforward calculation then reveals that $h^\gamma h^{h\gamma\epsilon/\pi} \le \frac{\widehat{\phi}_h(hx+2\pi j_0)}{\widehat{\phi}_h(hx)} \le h^\gamma h^{-h\gamma\epsilon/\pi}$. Therefore, there exists $A_1, A_2 > 0$ such that for all $h \in (0\,..\,1/e]$ and $x \in \epsilon B$,

$$A_1 h^\gamma \le \frac{\widehat{\phi}_h(hx + 2\pi j_0)}{\widehat{\phi}_h(hx)} \le A_2 h^\gamma.$$

The stated conclusion now follows easily from Theorems 4.1 and 4.2.

5. The Proofs

Before proving the above-stated results, we state the Hausdorff-Young Theorem and then prove two lemmas.

Theorem 5.1. *Let $1 \le p \le 2$, and let $f \in L_p$. Then $\widehat{f} \in L_{p'}$ and*

$$\left\| \widehat{f} \right\|_{L_{p'}} \le (2\pi)^{d/p'} \left\| f \right\|_{L_p}.$$

Proof. cf. [Ka; p.142].

Lemma 5.2. *Let $1 \le p \le 2$. Let $\eta : \mathbb{R}^d \to \mathbb{C}$ be a measurable function, and suppose that there exists $j_0 \in \mathbb{Z}^d \backslash 0$ and $\delta \in (0 \mathinner{.\,.} \pi)$ such that*

$$(*) \quad c_1(\eta, j_0, \delta) := \left\| \frac{\eta(\cdot + 2\pi j_0)}{\eta} \right\|_{L_\infty(\delta B)} < \infty.$$

Let $f \in L_p$. Then for all $h > 0$,

$$\left\| \frac{\eta(h \cdot + 2\pi j_0)}{\eta(h \cdot)} \widehat{f} \right\|_{L_{p'}(h^{-1}\delta B)} \le (2\pi)^{d/2} \left(1 + c_1(\eta, j_0, \delta)\right) \operatorname{dist}\left(f, \mathbb{S}^h(\eta); L_p\right) + \left\| \widehat{f} \right\|_{L_{p'}(\mathbb{R}^d \backslash h^{-1}\delta B)}.$$

Proof. Let $h > 0$, and let $s \in \mathbb{S}^h(\eta) \cap L_p$. Let τ be a $\frac{2\pi}{h}\mathbb{Z}^d$-periodic function which satisfies $\widehat{s} = \eta(h \cdot)\tau$. Note that

$$(5.3) \qquad \left\| \chi_{h^{-1}\delta B} \widehat{f} - \widehat{s} \right\|_{L_{p'}} \ge \left\| \widehat{s} \right\|_{L_{p'}(h^{-1}(2\pi j_0 + \delta B))} = \left\| \eta(h \cdot + 2\pi j_0)\tau \right\|_{L_{p'}(h^{-1}\delta B)}.$$

Hence,

$$\begin{aligned}
&\left\| \frac{\eta(h \cdot + 2\pi j_0)}{\eta(h \cdot)} \widehat{f} \right\|_{L_{p'}(h^{-1}\delta B)} \\
&\le \left\| \frac{\eta(h \cdot + 2\pi j_0)}{\eta(h \cdot)} (\widehat{f} - \widehat{s}) \right\|_{L_{p'}(h^{-1}\delta B)} + \left\| \eta(h \cdot + 2\pi j_0)\tau \right\|_{L_{p'}(h^{-1}\delta B)} \\
&\le \left\| \widehat{f} - \widehat{s} \right\|_{L_{p'}(h^{-1}\delta B)} \left\| \frac{\eta(\cdot + 2\pi j_0)}{\eta} \right\|_{L_\infty(\delta B)} + \left\| \chi_{h^{-1}\delta B} \widehat{f} - \widehat{s} \right\|_{L_{p'}} \qquad \text{by (5.3)} \\
&\le \left(1 + c_1(\eta, j_0, \delta)\right) \left\| \widehat{f} - \widehat{s} \right\|_{L_{p'}} + \left\| \chi_{h^{-1}\delta B} \widehat{f} - \widehat{f} \right\|_{L_{p'}} \\
&\le \left(1 + c_1(\eta, j_0, \delta)\right) (2\pi)^{d/p'} \left\| f - s \right\|_{L_p} + \left\| \widehat{f} \right\|_{L_{p'}(\mathbb{R}^d \backslash h^{-1}\delta B)},
\end{aligned}$$

where the last inequality follows from the Hausdorff-Young Theorem. Taking the infimum over all $s \in \mathbb{S}^h(\eta) \cap L_p$ completes the proof. $\square$

Lemma 5.4. *Let $2 < p \le \infty$. Let $\eta : \mathbb{R}^d \to \mathbb{C}$ be a measurable function, and suppose that there exists $j_0 \in \mathbb{Z}^d \backslash 0$ and $\delta \in (0\,..\,\pi)$ such that*

$$(*) \quad c_2(\eta, j_0, \delta) := \left\| \left(\sigma(\cdot/\delta) \frac{\eta(\cdot + 2\pi j_0)}{\eta} \right)^{\vee} \right\|_{L_1} < \infty.$$

Let $f \in L_p \cap L_2$. Then for all $h > 0$ and $\nu \in C_c^\infty(h^{-1}\delta B/4)$,

$$\begin{aligned} &\left\| \left(\frac{\eta(h \cdot + 2\pi j_0)}{\eta(h\,\cdot)} \widehat{f} \right) * \nu \right\|_{L_2(h^{-1}\delta B/4)} \\ &\qquad \le c_3(\nu)\,(1 + c_2(\eta, j_0, \delta))\, \operatorname{dist}\left(f, \mathbb{S}^h(\eta); L_p\right) + \|\nu\|_{L_1} \left\| \widehat{f} \right\|_{L_2(\mathbb{R}^d \backslash h^{-1}\delta B)}, \end{aligned}$$

where $c_3(\nu) := (2\pi)^{d/2} \sum_{j \in \mathbb{Z}^d} \|\widehat{\nu}\|_{L_\infty(j+C)}$.

Proof. Let $h > 0$ and $\nu \in C_c^\infty(h^{-1}\delta B/4)$. Note that

$$(5.5) \qquad \|\widehat{g} * \nu\|_{L_2} \le c_3(\nu)\, \|g\|_{L_p}, \quad \forall\, g \in L_p.$$

Indeed, if $g \in L_p$, then

$$\begin{aligned} \|\widehat{g} * \nu\|_{L_2} = (2\pi)^{3d/2} \|g\, \nu^\vee\|_{L_2} &\le (2\pi)^{3d/2} \sum_{j \in \mathbb{Z}^d} \|g\, \nu^\vee\|_{L_2(j+C)} \\ &\le (2\pi)^{3d/2} \sum_{j \in \mathbb{Z}^d} \|g\|_{L_2(j+C)} \|\nu^\vee\|_{L_\infty(j+C)} \\ &\le (2\pi)^{3d/2} \sum_{j \in \mathbb{Z}^d} \|g\|_{L_p(j+C)} \|\nu^\vee\|_{L_\infty(j+C)} \\ &\le (2\pi)^{d/2} \|g\|_{L_p} \sum_{j \in \mathbb{Z}^d} \|\widehat{\nu}\|_{L_\infty(j+C)} = c_3(\nu)\, \|g\|_{L_p}. \end{aligned}$$

Now let $s \in \mathbb{S}^h(\eta) \cap L_p$, and let τ be a $\frac{2\pi}{h}\mathbb{Z}^d$-periodic function which satisfies $\widehat{s} = \eta(h\,\cdot)\tau$. Since $\delta < \pi$,

$$\operatorname{supp}\left((\chi_{h^{-1}\delta B} \widehat{f}) * \nu \right) \bigcap h^{-1}\left(2\pi j_0 + \delta B/4\right) = \emptyset.$$

Hence,

$$\begin{aligned} (5.6) \quad \left\| \left(\chi_{h^{-1}\delta B} \widehat{f} - \widehat{s} \right) * \nu \right\|_{L_2} &\ge \|\widehat{s} * \nu\|_{L_2(h^{-1}(2\pi j_0 + \delta B/4))} \\ &= \|(\eta(h \cdot + 2\pi j_0)\tau) * \nu\|_{L_2(h^{-1}\delta B/4)}. \end{aligned}$$

Now,

$$
\begin{aligned}
&\left\| \left(\frac{\eta(h\cdot +2\pi j_0)}{\eta(h\cdot)} \widehat{f} \right) * \nu \right\|_{L_2(h^{-1}\delta B/4)} \\
&\le \left\| \left(\frac{\eta(h\cdot +2\pi j_0)}{\eta(h\cdot)} (\widehat{f} - \widehat{s}) \right) * \nu \right\|_{L_2(h^{-1}\delta B/4)} + \|(\eta(h\cdot +2\pi j_0)\tau) * \nu\|_{L_2(h^{-1}\delta B/4)} \\
&\le \left\| \left(\sigma(h\cdot/\delta) \frac{\eta(h\cdot +2\pi j_0)}{\eta(h\cdot)} (\widehat{f} - \widehat{s}) \right) * \nu \right\|_{L_2} + \left\| \left(\chi_{h^{-1}\delta B} \widehat{f} - \widehat{s} \right) * \nu \right\|_{L_2} \quad \text{by (5.6)} \\
&\le \left\| \left(\sigma(h\cdot/\delta) \frac{\eta(h\cdot +2\pi j_0)}{\eta(h\cdot)} (\widehat{f} - \widehat{s}) \right) * \nu \right\|_{L_2} \\
&\qquad\qquad + \left\| \left(\widehat{f} - \widehat{s} \right) * \nu \right\|_{L_2} + \left\| \left(\chi_{h^{-1}\delta B} \widehat{f} - \widehat{f} \right) * \nu \right\|_{L_2} \\
&\le c_3(\nu) \left\| \left(\sigma(h\cdot/\delta) \frac{\eta(h\cdot +2\pi j_0)}{\eta(h\cdot)} (\widehat{f} - \widehat{s}) \right)^{\vee} \right\|_{L_p} + c_3(\nu) \|f - s\|_{L_p} \\
&\qquad\qquad + \|\nu\|_{L_1} \left\| \chi_{h^{-1}\delta B} \widehat{f} - \widehat{f} \right\|_{L_2} \quad \text{by (5.5)} \\
&\le c_3(\nu) \left(1 + \left\| \left(\sigma(h\cdot/\delta) \frac{\eta(h\cdot +2\pi j_0)}{\eta(h\cdot)} \right)^{\vee} \right\|_{L_1} \right) \|f - s\|_{L_p} \\
&\qquad\qquad + \|\nu\|_{L_1} \left\| \widehat{f} \right\|_{L_2(\mathbb{R}^d \setminus h^{-1}\delta B)} \\
&= c_3(\nu)\left(1 + c_2(\eta, j_0, \delta)\right) \|f - s\|_{L_p} + \|\nu\|_{L_1} \left\| \widehat{f} \right\|_{L_2(\mathbb{R}^d \setminus h^{-1}\delta B)}.
\end{aligned}
$$

Taking the infimum over all $s \in \mathbb{S}^h(\eta) \cap L_p$ completes the proof. $\square$

Remark 5.7. In the above Lemmas, $\mathbb{S}(\eta)$ can be replaced with $\mathbb{S}(\eta;\delta)$ defined to be the collection of all locally integrable functions $s : \mathbb{R}^d \to \mathbb{C}$ which when viewed as distributions are tempered, and which satisfy

(*i*) $\widehat{s}$ is regular on $2\pi\mathbb{Z}^d + \delta B$;

(*ii*) $\widehat{s} = \eta\tau$ on $2\pi\mathbb{Z}^d + \delta B$ for some $2\pi\mathbb{Z}^d$-periodic function τ.

Remark 5.8. In Lemma 5.4, condition $(*)$ is unnecessarily strong when $p \neq \infty$. It can be replaced by

$$(*') \quad c_2(\eta, j_0, \delta) := \left\| \sigma(\cdot/\delta) \frac{\eta(\cdot + 2\pi j_0)}{\eta} \right\|_{\mathcal{M}_p} < \infty,$$

where $\mathcal{M}_p$ is the algebra of bounded multipliers of L_p (see [L]).

We show now that Theorems 4.1 and 4.2 are immediate consequences of Lemmas 5.2 and 5.4.

Proof of Theorem 4.1. WLOG assume that $(*)$ holds for all $h \in (0\,.\,.\,1]$. Put $\widehat{f} := (2\pi)^{d/2}(1+c)\sigma(\,\cdot/\epsilon)$, and let $h \in (0\,.\,.\,1]$. In order to apply Lemma 5.2 put $\eta := \eta_h$ and $\delta := h\epsilon$. Note that the hypothesis of Lemma 5.2 is satisfied and

$$c_1(\eta, j_0, \delta) = c_1(\eta_h, j_0, h\epsilon) \le c.$$

Thus, by Lemma 5.2,

$$\begin{aligned}
&(2\pi)^{d/2}(1+c)\left\|\frac{\eta_h(h\cdot+2\pi j_0)}{\eta_h(h\,\cdot)}\right\|_{L_{p'}(\epsilon B/2)} \le \left\|\frac{\eta_h(h\cdot+2\pi j_0)}{\eta_h(h\,\cdot)}\widehat{f}\right\|_{L_{p'}(h^{-1}\delta B)}\\
&\le (2\pi)^{d/2}\,(1+c)\,\mathrm{dist}\left(f, \mathbb{S}^h(\eta); L_p\right) + \left\|\widehat{f}\right\|_{L_{p'}(\mathbb{R}^d\setminus h^{-1}\delta B)}\\
&= (2\pi)^{d/2}\,(1+c)\,\mathrm{dist}\left(f, \mathbb{S}^h(\eta_h); L_p\right),
\end{aligned}$$

which proves the Theorem. □

Proof of Theorem 4.2. WLOG assume that condition $(*)$ holds for all $h \in (0\,.\,.\,1]$. Let $\nu \in C_c^\infty(\epsilon B/4)$, put $\widehat{f} := c_3(\nu)(1+c)\sigma(\,\cdot/\epsilon)$, and let $h \in (0\,.\,.\,1]$. In order to apply Lemma 5.4 put $\eta := \eta_h$ and $\delta := h\epsilon$. Note that the hypothesis of Lemma 5.4 is satisfied with

$$c_2(\eta, j_0, \delta) = c_2(\eta_h, j_0, h\epsilon) \le c.$$

Thus, by Lemma 5.4,

$$\begin{aligned}
&c_3(\nu)(1+c)\left\|\left(\frac{\eta_h(h\cdot+2\pi j_0)}{\eta_h(h\,\cdot)}\right)*\nu\right\|_{L_2(\epsilon B/4)}\\
&= \left\|\left(\frac{\eta_h(h\cdot+2\pi j_0)}{\eta_h(h\,\cdot)}\widehat{f}\right)*\nu\right\|_{L_2(h^{-1}\delta B/4)}\\
&\le c_3(\nu)\,(1+c)\,\mathrm{dist}\left(f, \mathbb{S}^h(\eta); L_p\right) + \|\nu\|_{L_1}\left\|\widehat{f}\right\|_{L_2(\mathbb{R}^d\setminus h^{-1}\delta B)}\\
&= c_3(\nu)\,(1+c)\,\mathrm{dist}\left(f, \mathbb{S}^h(\eta_h); L_p\right),
\end{aligned}$$

which proves the theorem. □

Theorem 3.1 is also a consequence of Lemmas 5.2 and 5.4.

Proof of Theorem 3.1. WLOG assume that condition $(*)$ holds for all $h \in (0\,.\,.\,1]$. We consider first the case $1 \le p \le 2$. Note that $q = p$. Let $f \in L_p$ satisfy (3.3) and (3.4). Since $\rho\widehat{f} \ne 0$, there exists $n > 0$ such that

$$\left\|\rho\widehat{f}\right\|_{L_{p'}(nB)} > 0. \tag{5.9}$$

Assume $0 < h < \frac{\epsilon}{2n}$ and put $\eta := \eta_h$, $\delta := \epsilon/2$. It follows from $(*)$ that

$$\left\|\sigma(\,\cdot/\epsilon)\frac{\eta(\,\cdot+2\pi j_0)}{\eta}\right\|_{L_\infty} \le c.$$

Hence the hypothesis of Lemma 5.2 is satisfied with $c_1(\eta, j_0, \delta) \leq c$. Now since $nB \subseteq h^{-1}\delta B$,

$$
\begin{aligned}
&h^\gamma \left\|\rho\widehat{f}\right\|_{L_{p'}(nB)} \\
&\leq \left\|\left(h^\gamma\rho - \frac{\eta(h\cdot+2\pi j_0)}{\eta(h\,\cdot)}\right)\widehat{f}\right\|_{L_{p'}(nB)} + \left\|\frac{\eta(h\cdot+2\pi j_0)}{\eta(h\,\cdot)}\widehat{f}\right\|_{L_{p'}(nB)} \\
&\leq \left\|h^\gamma\rho - \frac{\eta(h\cdot+2\pi j_0)}{\eta(h\,\cdot)}\right\|_{L_\infty(nB)} \left\|\widehat{f}\right\|_{L_{p'}(nB)} + \left\|\frac{\eta(h\cdot+2\pi j_0)}{\eta(h\,\cdot)}\widehat{f}\right\|_{L_{p'}(h^{-1}\delta B)} \\
&\leq (2\pi)^{d/2}(1+c)\operatorname{dist}\left(f, \mathbb{S}^h(\eta_h); L_p\right) + o(h^\gamma),
\end{aligned}
$$

by (3.2), Lemma 5.2, and (3.4). Therefore by (5.9),

$$\operatorname{dist}\left(f, \mathbb{S}^h(\eta_h); L_p\right) \neq o(h^\gamma).$$

We consider now the case $2 < p \leq \infty$. Note that $q = 2$. Let $f \in L_p^0 \cap L_2$ satisfy (3.3) and (3.4). Since $\widehat{\sigma}(x) \neq 0$ for almost all $x \in \mathbb{R}^d$ and $\rho\widehat{f} \neq 0$, it follows that there exists $n > 1$ such that

$$\left\|(\rho\widehat{f}) * \sigma\right\|_{L_2(nB)} > 0. \tag{5.10}$$

Assume that $0 < h < \frac{\epsilon}{4n}$, and put $\eta := \eta_h$, $\delta := \epsilon$, and $\nu := \sigma$. Note that $\nu \in C_c^\infty(h^{-1}\delta B/4)$, and the hypothesis of Lemma 5.4 is satisfied with $c_2(\eta, j_0, \delta) \leq c$. Now since $nB \subset h^{-1}\delta B/4$,

$$
\begin{aligned}
&h^\gamma \left\|\left(\rho\widehat{f}\right) * \sigma\right\|_{L_2(nB)} \\
&\leq \left\|\left(\left(h^\gamma\rho - \frac{\eta(h\cdot+2\pi j_0)}{\eta(h\,\cdot)}\right)\widehat{f}\right) * \sigma\right\|_{L_2(nB)} + \left\|\left(\frac{\eta(h\cdot+2\pi j_0)}{\eta(h\,\cdot)}\widehat{f}\right) * \sigma\right\|_{L_2(nB)} \\
&\leq \|\sigma\|_{L_1} \left\|h^\gamma\rho - \frac{\eta(h\cdot+2\pi j_0)}{\eta(h\,\cdot)}\right\|_{L_\infty((n+1)B)} \left\|\widehat{f}\right\|_{L_2((n+1)B)} \\
&\qquad\qquad + \left\|\left(\frac{\eta(h\cdot+2\pi j_0)}{\eta(h\,\cdot)}\widehat{f}\right) * \nu\right\|_{L_2(h^{-1}\delta B/4)} \\
&\leq c_3(\nu)(1+c)\operatorname{dist}\left(f, \mathbb{S}^h(\eta_h); L_p\right) + o(h^\gamma),
\end{aligned}
$$

by (3.2), Lemma 5.4, and (3.4). Therefore by (5.10),

$$\operatorname{dist}\left(f, \mathbb{S}^h(\eta_h); L_p\right) \neq o(h^\gamma).$$

□

Theorem 1.5 is a consequence of Theorem 3.1.

Proof of Theorem 1.5. WLOG assume that $\widehat{\phi}(0) = 1$. Put $\eta := \widehat{\phi}$. Since $\phi \in L_1$, it follows by Proposition 2.4 that

$$\operatorname{dist}\left(f, \mathcal{S}_p^h(\phi); L_p\right) \geq \operatorname{dist}\left(f, \mathbb{S}^h(\eta); L_p\right), \quad \forall\, f \in L_p^0,\ h > 0.$$

By Proposition 3.6, there exists $\epsilon \in (0\,..\,\pi)$ and $c < \infty$ such that condition $(*)$ of Theorem 3.1 holds. Since $\widehat{\phi}$ is smooth near $2\pi j_0$ and the Strang-Fix conditions of order k fail at $2\pi j_0$, there exists a non-trivial homogeneous polynomial ρ of degree $\leq k-1$ such that (3.5) holds with $\gamma := \deg\, \rho \leq k-1$. Let $f \in W_p^{k-1}\backslash 0$ (if $1 \leq p \leq 2$), $f \in W_2^{k-1} \cap L_p^0 \backslash 0$ (if $2 < p \leq \infty$). Since ρ is non-zero almost everywhere, condition (3.3) of Theorem 3.1 is satisfied. It is a straightforward matter to verify that the smoothness assumptions made on f ensure that condition (3.4) is also satisfied. The proof is now completed by applying Theorem 3.1. $\square$

Proof of Proposition 2.2. Since $\mathcal{S}_0(\phi) \subset X := \{\phi *' c : c \in \ell_\infty\}$, it suffices to show that

$$\operatorname{dist}\left(f, \mathcal{S}_0(\phi); L_\infty\right) \leq \operatorname{dist}\left(f, X; L_\infty\right), \quad \forall\, f \in L_\infty^0. \tag{5.11}$$

So let $f \in L_\infty^0$. Let $c \in \ell_\infty$, $\epsilon > 0$. We will show that

$$\operatorname{dist}\left(f, \mathcal{S}_0(\phi); L_\infty\right) < \|\phi *' c - f\|_{L_\infty} + \epsilon.$$

Claim. There exists $N \in \mathbb{N}$ and $0 < r_0 < r_1 < \cdots < r_N < \infty$ such that if $c_0 := c$ and for $1 \leq n \leq N$,

$$c_n(j) := \begin{cases} c_{n-1}(j), & |j| \leq r_n; \\ \frac{N-n}{N} c(j), & |j| > r_n, \end{cases}$$

then for $n = 0, 1, \ldots, N$,

$$\|\phi *' c_n - f\|_{L_\infty} < \|\phi *' c - f\|_{L_\infty} + \epsilon. \tag{5.12}$$

proof. Put $M := \sum_{j \in \mathbb{Z}^d} \|\phi\|_{L_\infty(j+C)}$. Then

$$\|\phi *' a\|_{L_\infty} \leq M \|a\|_{\ell_\infty}, \quad \forall\, a \in \ell_\infty.$$

Let $N \in \mathbb{N}$ be such that $N > 2M \|c\|_{\ell_\infty} / \epsilon$. Let $r_0 > 0$ be so large that $\|f\|_{L_\infty(\mathbb{R}^d \backslash r_0 B)} < \epsilon/10$. Clearly, (5.12) holds for $n = 0$. Proceeding inductively, assume that r_n has been chosen for all $n \leq k < N$ and that (5.12) holds for all $n \leq k$. Consider $n = k+1$. There exists $\widetilde{r}_{k+1} > r_k$ so large that

$$\|c\|_{\ell_\infty} \sum_{|j| \leq r_k} \|\phi(\,\cdot - j)\|_{L_\infty(\mathbb{R}^d \backslash \widetilde{r}_{k+1} B)} < \epsilon/5. \tag{5.13}$$

There exists $r_{k+1} > \widetilde{r}_{k+1}$ so large that

$$\|c\|_{\ell_\infty} \sum_{|j| > r_{k+1}} \|\phi(\,\cdot - j)\|_{L_\infty(\widetilde{r}_{k+1} B)} < \epsilon + \|\phi *' c - f\|_{L_\infty} - \|\phi *' c_k - f\|_{L_\infty} \tag{5.14}$$

Now,

$$\begin{aligned}
&\|\phi *' c_{k+1} - f\|_{L_\infty(\widetilde{r}_{k+1}B)} \\
&\le \|\phi *' (c_{k+1} - c_k)\|_{L_\infty(\widetilde{r}_{k+1}B)} + \|\phi *' c_k - f\|_{L_\infty(\widetilde{r}_{k+1}B)} \\
&\le \|c\|_{\ell_\infty} \sum_{|j|>r_{k+1}} \|\phi(\,\cdot - j)\|_{L_\infty(\widetilde{r}_{k+1}B)} + \|\phi *' c_k - f\|_{L_\infty} \\
&< \|\phi *' c - f\|_{L_\infty} + \epsilon \qquad \text{by (5.14).}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\|\phi *' c_{k+1} - f\|_{L_\infty(\mathbb{R}^d \setminus \widetilde{r}_{k+1}B)} \\
&\le \|\phi *' (c_{k+1} - c_k)\|_{L_\infty(\mathbb{R}^d \setminus \widetilde{r}_{k+1}B)} + \|\phi *' c_k - f\|_{L_\infty(\mathbb{R}^d \setminus \widetilde{r}_{k+1}B)} \\
&\le M\,\|c_{k+1} - c_k\|_{\ell_\infty} + \left\|\phi *' \left(c_k - \left(\frac{N-k}{N}\right) c\right)\right\|_{L_\infty(\mathbb{R}^d \setminus \widetilde{r}_{k+1}B)} \\
&\qquad\qquad + \frac{N-k}{N}\,\|\phi *' c - f\|_{L_\infty(\mathbb{R}^d \setminus \widetilde{r}_{k+1}B)} + \frac{k}{N}\,\|f\|_{L_\infty(\mathbb{R}^d \setminus \widetilde{r}_{k+1}B)} \\
&\le \frac{M}{N}\,\|c\|_{\ell_\infty} + \|c\|_{\ell_\infty} \sum_{|j|\le r_k} \|\phi(\,\cdot - j)\|_{L_\infty(\mathbb{R}^d \setminus \widetilde{r}_{k+1}B)} + \|\phi *' c - f\|_{L_\infty} + \epsilon/10 \\
&< \|\phi *' c - f\|_{L_\infty} + \epsilon \qquad \text{by (5.13) and the choice of } N.
\end{aligned}$$

Therefore, (5.12) holds for $n = k+1$, and the claim is proven.

Note that $c_N(j) = 0$ for all $|j| > r_N$. Hence $\phi *' c_N \in \mathcal{S}_0(\phi)$ and thus by the claim,

$$\operatorname{dist}(f, \mathcal{S}_0(\phi); L_\infty) \le \|\phi *' c_N - f\|_{L_\infty} < \|\phi *' c - f\|_{L_\infty} + \epsilon.$$

Since $f \in L_\infty^0$, $c \in \ell_\infty$ and $\epsilon > 0$ were arbitrary, we have proven (5.11). $\square$

Proof of Proposition 3.6. WLOG we may assume that $\widehat{\phi}_h(0) = 1$ for all $h \in (0\,..\,1]$. By (i),

$$c_1 := \sup_{h\in(0..\,1]} \|\phi_h\|_{L_1} < \infty.$$

By (ii), there exists $r \in (0\,..\,\infty)$ such that

$$\sup_{h\in(0..\,1]} \|\phi_h\|_{L_1(\mathbb{R}^d \setminus rB)} \le \frac{1}{\|\widehat{\sigma}\|_{L_1}}.$$

There exists $\epsilon \in (0\,..\,\pi]$ such that

$$\int_{\mathbb{R}^d} |\widehat{\sigma}(x) - \widehat{\sigma}(x - 2\epsilon y)|\,dx < \frac{1}{c_1} \text{ for all } y \in rB.$$

Let $h \in (0\,..\,1]$. Note that,

$$
\begin{aligned}
\left\| \left(\sigma(\,\cdot/2\epsilon)(1-\widehat{\phi}_h)\right)^\vee \right\|_{L_1} &= (2\epsilon)^d \left\|\sigma^\vee(2\epsilon\,\cdot) - \sigma^\vee(2\epsilon\,\cdot) * \phi_h\right\|_{L_1} \\
&= (2\pi)^{-d}(2\epsilon)^d \int_{\mathbb{R}^d} \left| \widehat{\sigma}(2\epsilon x) - \int_{\mathbb{R}^d} \widehat{\sigma}(2\epsilon x - 2\epsilon y)\phi_h(y)\,dy \right| dx \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \left(\widehat{\sigma}(x) - \widehat{\sigma}(x-2\epsilon y)\right)\phi_h(y)\,dy \right| dx, \qquad \text{since } \widehat{\phi}_h(0) = 1, \\
&\le (2\pi)^{-d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left|\widehat{\sigma}(x) - \widehat{\sigma}(x-2\epsilon y)\right| dx \right) |\phi_h(y)|\,dy, \qquad \text{by Fubini}, \\
&\le (2\pi)^{-d} \int_{rB} c_1^{-1} |\phi_h(y)|\,dy + (2\pi)^{-d} \int_{\mathbb{R}^d \backslash rB} 2\left\|\widehat{\sigma}\right\|_{L_1} |\phi_h(y)|\,dy \\
&\le (2\pi)^{-d} + (2\pi)^{-d} 2 < \frac{1}{2}.
\end{aligned}
\tag{5.15}
$$

It follows from (5.15) that $\left\|\sigma(\,\cdot/2\epsilon)(1-\widehat{\phi}_h)\right\|_{L_\infty} < 1/2$. Consequently,

$$
\frac{\sigma(\,\cdot/\epsilon)}{\widehat{\phi}_h} = \sigma(\,\cdot/\epsilon)\left(1 + \frac{\sigma(\,\cdot/2\epsilon)(1-\widehat{\phi}_h)}{1-\sigma(\,\cdot/2\epsilon)(1-\widehat{\phi}_h)}\right) = \sigma(\,\cdot/\epsilon)\left(1 + \sum_{n=1}^{\infty}\left(\sigma(\,\cdot/2\epsilon)(1-\widehat{\phi}_h)\right)^n\right).
$$

And so by (5.15), it follows that

$$
\left\| \left(\frac{\sigma(\,\cdot/\epsilon)}{\widehat{\phi}_h}\right)^\vee \right\|_{L_1} \le \left\|\sigma^\vee\right\|_{L_1} \left(1 + \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\right) = 2\left\|\sigma^\vee\right\|_{L_1}. \tag{5.16}
$$

Hence, for all $j_0 \in \mathbb{Z}^d \backslash 0$,

$$
\left\| \left(\sigma(\,\cdot/\epsilon)\frac{\widehat{\phi}_h(\,\cdot + 2\pi j_0)}{\widehat{\phi}_h}\right)^\vee \right\|_{L_1} \le \left\|\phi_h\right\|_{L_1} \left\| \left(\frac{\sigma(\,\cdot/\epsilon)}{\widehat{\phi}_h}\right)^\vee \right\|_{L_1} \le 2c_1 \left\|\sigma^\vee\right\|_{L_1} =: c.
$$

Therefore, condition $(*)$ of Theorem 3.1 holds for all $j_0 \in \mathbb{Z}^d \backslash 0$. Thus completing the proof. $\square$

Note that Lemma 3.8 is proved at (5.16).

Proof of Proposition 4.3. Since condition $(*)$ of Theorem 4.1 is weaker than that of Theorem 4.2, it suffices to show that condition $(*)$ of Theorem 4.2 is satisfied. WLOG we may assume that $\widehat{\phi}_h(0) = 1$ for all $h \in (0\,..\,1]$. By (i),

$$
c_1 := \sup_{h \in (0\,..\,1]} \left\|\phi_h\right\|_{L_1} < \infty.
$$

By (ii), there exists $r \in (0\,..\,\infty)$ such that

$$\sup_{h\in(0\,..\,1]} \|\phi_h\|_{L_1(\mathbb{R}^d\backslash h^{-1}rB)} \le \frac{1}{\|\widehat{\sigma}\|_{L_1}}.$$

For $h \in (0\,..\,1]$, put $\widehat{\mu}_h := \widehat{\phi}_h(h\,\cdot)$. Then $\mu_h = h^{-d}\phi_h(\cdot/h)$ and hence for all $h \in (0\,..\,1]$,

$$\|\mu_h\|_{L_1} = \|\phi_h\|_{L_1} \le c_1;$$
$$\|\mu_h\|_{L_1(\mathbb{R}^d\backslash rB)} = \|\phi_h\|_{L_1(\mathbb{R}^d\backslash h^{-1}rB)} \le \frac{1}{\|\widehat{\sigma}\|_{L_1}}.$$

Since we also have that $\widehat{\mu}_h(0) = 1$ for all $h \in (0\,..\,1]$, it follows by Proposition 3.6 that there exists $\epsilon \in (0\,..\,\pi]$ and $c < \infty$ such that for $h \in (0\,..\,1]$ and $j_0 \in \mathbb{Z}^d\backslash 0$,

$$\left\| \left(\sigma(\cdot/h\epsilon) \frac{\widehat{\phi}_h(\cdot + 2\pi j_0)}{\widehat{\phi}_h} \right)^{\vee} \right\|_{L_1} = \left\| \left(\sigma(\cdot/\epsilon) \frac{\widehat{\mu}_h(\cdot + 2\pi j_0)}{\widehat{\mu}_h} \right)^{\vee} \right\|_{L_1} \le c.$$

Thus completing the proof. $\square$

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References

[AS] Abramowitz, M. and I. A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, 1970.

[BeL] Beatson, R. K., and W. A. Light, *Quasi-interpolation in the absence of polynomial reproduction*, Numerical Methods of Approximation Theory (D. Braess L.L. Schumaker, eds.), Birkhäuser-Verlag, 1992, pp. 21–39.

[BD] de Boor, C. and R.A. DeVore, *Approximation by smooth multivariate splines*, Trans. of Amer. Math. Soc. **276** (1983), 775–788.

[BDR] de Boor, C., R.A. DeVore, and A. Ron, *Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$*, Trans. Amer. Math. Soc. (to appear).

[BH] de Boor, C. and K. Höllig, *B-splines from parallelepipeds*, J. d'Anal. Math. **42** (1982/3), 99–115.

[BHR] de Boor, C., K. Höllig and S. Riemenschneider, *Box Splines*, Springer-Verlag, New York, 1993.

[BJ] de Boor, C. and R.-Q. Jia, *Controlled approximation and a characterization of the local approximation order*, Proc. Amer. Math. Soc. **95** (1985), 547–553.

[BR] de Boor, C. and A. Ron, *Fourier analysis of approximation orders from principal shift-invariant spaces*, Constructive Approximation **8** (1992), 427–462.

[Bu] Buhmann, M. D., *On quasi-interpolation with radial basis functions*, J. Approx. Th. **72** (1993), 103–130.

[BuR] Buhmann, M. D. and A. Ron, *RADIAL BASIS FUNCTIONS: L_p-Approximation orders with scattered centres*, Preprint.

[DM] Dahmen, W. and C.A. Micchelli, *On the approximation order from certain multivariate spline spaces*, J. Austral. Math. Soc. Ser B **26** (1984), 233–246.

[Da] Daubechies, I., *Ten lectures on wavelets*, Society for Industrial and Applied Mathematics, Phil. PA, 1992.

[D] Dyn, N., *Analysis of uniform binary subdivision schemes for curve design*, Constr. Approx. **7 no.2** (1991), 127–147.

[DR] Dyn, N. and A. Ron, *Local approximation by certain spaces of multivariate exponential-polynomials, approximation order of exponential box splines and related interpolation problems*, Trans. Amer. Math. Soc. **319** (1990), 381–404.

[GS] Gelfand, I. M. and G. E. Shilov, *Generalized Functions*, vol. 1, Academic Press, 1964.

[HL] Halton, E. J. and W. A. Light, *On local and controlled approximation order*, J. Approx. Theory **72** (1993), no. 3, 268–277.

[J] Jia, R.-Q., *The Toeplitz Theorem and its Applications to Approximation Theory and Linear PDE's*, ms. (1993).

[JL] Jia, R.-Q., and J. Lei, *Approximation by Multiinteger Translates of Functions Having Global Support*, J. Approx. Theory **72** (1993), 2–23.

[Ka] Katzneslon, Y., *An introduction to Harmonic Analysis*, Wiley, 1968.

[K] Kyriazis, G.C., *Approximation From Shift-Invariant Spaces*, Constr. Approx. (to appear).

[L] Larsen, R., *The multiplier problem*, Springer, 1969.

[LJ] Lei, J. and R.-Q. Jia, *Approximation by piecewise exponentials*, SIAM J. Math. Anal. **22** (1991), no. 6, 1776–1789.

[LC] Light, W. A. and E. W. Cheney, *Quasi-interpolation with translates of a function having noncompact support*, Constr. Approx. **8** (1992), 35–48.

[R1] Ron, A., *A characterization of the approximation order of multivariate spline spaces*, Studia Math. **98** (1991), no. 1, 73–90.

[R2] Ron, A., *The L_2-Approximation Orders of Principal Shift-Invariant Spaces Generated by a Radial Basis Function*, Numerical Methods of Approximation Theory (D. Braess and L. L. Schumaker, eds.), vol. 9, 1991, pp. 245–268.

[R3] Ron, A., *APPROXIMATION ORDERS OF AND APPROXIMATION MAPS FROM LOCAL PRINCIPAL SHIFT-INVARIANT SPACES*, CMS TSR #92-2, University of Wisconsin-Madison, May 1993.

[SF] Strang, G. and G. Fix, *A Fourier analysis of the finite element variational method*, Constructive Aspects of Functional Analysis (G. Geymonat, ed.), C.I.M.E., 1973, pp. 793–840.