Interpolation from spaces spanned by monomials

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Dedicated to Mariano Gasca on the occasion of his sixtieth birthday

Abstract This is an extension and emendation of recent results on the use of Gauss elimination in multivariate polynomial interpolation and, in particular, ideal interpolation.

Let $\Pi \subset (\mathbb{F}^d \to \mathbb{F})$ be the space of all polynomials in d real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) variables. Let Q be an **n-row map on** Π, i.e., a linear map from Π to $\mathbb{F}^n$, and consider the task of solving

$$Q? = a$$

for given $a \in \mathbb{F}^n$. We assume that this problem has a solution for arbitrary $a \in \mathbb{F}^n$, i.e., that Q is onto. Then there is a standard recipe for finding all solutions, namely Gauss elimination applied to the **Gram matrix**

$$QV = (\eta_i v_j : i = 1{:}n, j \in J),$$

with the linear functionals η_i the **rows** of the row map Q, i.e., $Qf =: (\eta_i f : i = 1{:}n)$, and the polynomials v_j the **columns** of the invertible **column map**

$$V = [v_j : j \in J] : \mathbb{F}_0^J \to \Pi : a \mapsto \sum_j v_j a(j)$$

or basis for Π (indexed by some set J). Here,

$$\mathbb{F}_0^J := \{a : J \to \mathbb{F} : \#\operatorname{supp} a < \infty\},$$

hence V is well-defined.

Take for V a **monomial basis**, i.e., the column map

$$V = [()^\alpha : \alpha \in \mathbb{Z}_+^d] : \mathbb{F}_0^{\mathbb{Z}_+^d} \to \Pi : \widehat{p} \mapsto p := \sum_\alpha ()^\alpha \widehat{p}(\alpha),$$

with its columns the monomials

$$()^\alpha : \mathbb{F}^d \to \mathbb{F} : x \mapsto x^\alpha := x(1)^{\alpha(1)} \cdots x(d)^{\alpha(d)}$$

arranged in some order in which each collection of columns has a left-most one (i.e., the order must be a well-ordering). Since Q is onto, QV is of full rank, hence has exactly n **bound** columns, i.e., columns that are not weighted sums of columns to the left of it. This is a standard result of basic linear algebra in case V has finitely many columns but needs, perhaps, a proof in the present setting, of a V with infinitely many columns.

For this, let $\preceq$, $\prec$, etc, indicate the order on $\mathbb{Z}_+^d$ corresponding to the order in which the monomials appear as columns in V. Further, let

$$\beta_j := \min \Gamma_j, \quad \Gamma_j := \{\gamma : \operatorname{rank} Q[()^\alpha : \alpha \preceq \gamma] \geq j\}, \quad j = 1{:}n.$$

There is, in fact, such a minimum since Γ_j must be nonempty (due to the fact that $\operatorname{rank} QV = n$), hence, by our assumption on the columns' ordering, must have a left-most element. Further, with β_j that left-most element, column β_j is necessarily bound (since its adjunction to $Q[()^\alpha : \alpha \prec \beta_j]$ raises the rank). It follows that all the columns of the square matrix QV_β, with

$$V_\beta := [()^{\beta_j} : j = 1{:}n],$$

are bound, hence QV_β is invertible. This makes

$$F := \operatorname{ran} V_\beta$$

a **monomial** subspace (i.e., a space spanned by monomials) that is **correct for** Q in the sense that Q maps it 1-1 onto $\mathbb{F}^n$, i.e., F contains, for every $a \in \mathbb{F}^n$, exactly one f that matches the information a in the sense that $Qf = a$. Put differently, F contains, for each $p \in \Pi$, exactly one f that **agrees with** p **at** Q in the sense that $Qf = Qp$. We can write this f in the form

$$f = Pp,$$

with

$$P := V_\beta (QV_\beta)^{-1} Q$$

the linear projector on Π with $\operatorname{ran} P = F$ and $\ker P = \ker Q$.

The subspace F is *minimal*, among all monomial subspaces correct for Q, in the following sense.

Define the **$\prec$-degree** of $p = \sum_\alpha ()^\alpha \widehat{p}(\alpha) \in \Pi$ to be the multiindex

$$\delta(p) \ := \ \max \operatorname{supp} \widehat{p} = \max\{\alpha : \widehat{p}(\alpha) \neq 0\},$$

with $\delta(0)$, offhand, undefined. Define, correspondingly, for any finite-dimensional subset F_1 of Π,

$$\delta(F_1) \ := \ \max_{p \in F_1} \delta(p).$$

Also, follow [S] in using the handy abbreviation

$$p \prec q \ := \ \delta(p) \prec \delta(q), \quad p, q \in \Pi.$$

Now, among all subspaces F_1 correct for Q, our F is **$\prec$-minimal** in the sense that $\delta(F) \preceq \delta(F_1)$ for all such F_1. This follows immediately from the fact that any subspace F_1 with $\delta(F_1) \prec \delta(F)$ lies in $\operatorname{ran}[()^\alpha : \alpha \prec \beta_n]$ and, by the very choice of β_n, the rank of $Q[()^\alpha : \alpha \prec \beta_n]$ is less than n, hence Q cannot map F_1 *onto* $\mathbb{F}^n$.

Actually, F is minimal in the more subtle way that it, or its corresponding linear projector P, is **$\prec$-reducing** in the sense that

(1) $$\delta(Pp) \preceq \delta(p), \quad \forall p \in \Pi.$$

Indeed, since P is linear and

$$\delta(p) = \max_\alpha \{\delta(()^\alpha) : \widehat{p}(\alpha) \neq 0\},$$

it is sufficient to check (1) for monomials only. In the discussion, call a monomial **bound** or **free** according to whether the corresponding column of QV is bound or free (with a column **free** exactly when it is not bound, i.e., when it is the weighted sum of columns to the left of it). There are two cases. If $()^\alpha$ is bound, then (1) holds trivially for $p = ()^\alpha$ since then $p \in F$, hence $Pp = p$. In the contrary case, $()^\alpha$ is free. But this means that $Q()^\alpha$ is writable as a linear combination of columns to the left of it, hence of the bound columns to the left of it, and the corresponding linear combination of bound monomials is an interpolant from F to $p = ()^\alpha$, hence must be Pp, and that verifies (1) for this case, too.

Next, consider uniqueness of such $\prec$-minimal or $\prec$-reducing monomial spaces F. If there is some free column to the left of the nth bound column, say column α, then, as we already observed, $Q()^\alpha$ is writable as a weighted sum of the bound columns to the left of it. This implies that the space

$$F_1 := \operatorname{ran}[()^{\beta_1}, \ldots, ()^{\beta_{n-1}}, ()^{\beta_n} + ()^\alpha]$$

is also correct for Q and $\delta(F_1) = \delta(F)$, hence F_1 is also $\prec$-minimal. It is also $\prec$-reducing since the interpolant $P_1 p$ it provides for any $p := ()^\gamma$ with $\gamma \prec \beta_n$ is still Pp while, for $\gamma \succeq \beta_n$,

$$\delta(P_1 p) \preceq \delta(F_1) = \beta_n \preceq \delta(p).$$

Thus, unless the bound columns of QV are its first n columns, there are many $\prec$-reducing spaces F_1 other than F. But any such F_1 fails to be monomial, and this is as it should be because of the following

Proposition 2. *F is the unique monomial $\prec$-reducing space for Q.*

Proof: If

$$F_1 := \mathrm{ran}[()^{\gamma_1}, \ldots, ()^{\gamma_n}]$$

is a monomial $\prec$-reducing space correct for Q and $\gamma_1 \prec \cdots \prec \gamma_n$, then, for any α, $Q()^\alpha$ must be in $\mathrm{ran}[Q()^{\gamma_j} : \gamma_j \preceq \alpha]$. This implies, for any α not equal to one of the γ_j, that $Q()^\alpha$ is a free column of QV, hence the only columns that can be bound are those n columns $Q()^\alpha$ with $\alpha \in \{\gamma_1, \ldots, \gamma_n\}$, and these must all be bound since we know there to be n bound columns. In other words, $F_1 = F$. □

For the special case that $\prec$ is a monomial ordering (see below for the definition) and $\ker Q$ is an ideal, this proposition is Theorem 4 of [S], and the earlier assertion concerning the existence of other $\prec$-reducing spaces is, essentially, Proposition 2 of [S] in that setting.

It is also possible to prove the following emended and extended version of Theorem 3 of [S], in which

$$\Lambda(p) \;:=\; ()^{\delta(p)}\widehat{p}(\delta(p))$$

denotes the **leading term** of $p \in \Pi$.

Proposition 3. *The n-dimensional linear subspace F_1 of Π is $\prec$-reducing for Q if and only if it has a spanning sequence $p_1 \prec \cdots \prec p_n$ so that (a)*

$$\eta_i p_j \;=\; \delta_{ij}, \quad 1 \le i \le j \le n,$$

for some suitable ordering $\eta_1, \ldots, \eta_n$ of the rows of Q; and, (b) for some elements $q_1, \ldots$ of $\ker Q$,

$$[\Lambda(p_1), \ldots, \Lambda(p_n), \Lambda(q_1), \ldots] \tag{4}$$

is a basis for $\Pi_{\preceq\delta(p_n)}$, with

$$\Pi_{\preceq\gamma} := \mathrm{ran}[()^\alpha : \alpha \preceq \gamma].$$

Proof: Assume that F_1 is $\prec$-reducing and let P_1 be the corresponding linear projector. As before, let $\beta_1 \prec \cdots \prec \beta_n$ be the indices of the bound columns of QV and set

$$r_j := P_1()^{\beta_j}, \quad j = 1{:}n.$$

Then, necessarily, $\delta(r_j) = \beta_j$, all j, since, otherwise, $\delta(r_j) \prec \beta_j$ and, since $Qr_j = Q()^{\beta_j}$, this would imply that column β_j were free. Since the r_j are in F_1 and satisfy $\delta(r_1) \prec \cdots \prec \delta(r_n)$, it follows that $[r_1, \ldots, r_n]$ is a basis for F_1. Then, for some ordering $\eta_1, \ldots, \eta_n$ (determinable by applying Gauss elimination with row interchanges to the invertible matrix $Q[r_1, \ldots, r_n]$), there is an upper triangular matrix U such that, for that ordering of the rows of Q, $Q[r_1, \ldots, r_n]U^{-1}$ is unit lower triangular, hence $[p_1, \ldots, p_n] := [r_1, \ldots, r_n]U^{-1}$ is a basis for F_1 with $p_1 \prec \cdots \prec p_n$ that satisfies (a). Now, for any $\alpha \in \{\gamma \preceq \beta_n\}\backslash\{\beta_1, \ldots, \beta_n\}$, set

$$q_\alpha \;:=\; ()^\alpha - P_1()^\alpha.$$

Then $q_\alpha \in \ker Q$, and $\delta(q_\alpha) = \alpha$ since the fact that F_1 is $\prec$-reducing implies that $\delta(P_1()^\alpha) \preceq \alpha$, hence $P_1()^\alpha \in \mathrm{ran}[p_j : \delta(p_j) = \beta_j \preceq \alpha] \subset \Pi_{\prec\alpha}$. Thus this choice of the q_α satisfies (b).

Conversely, assume that we have in hand a spanning sequence $p_1 \prec \cdots \prec p_n$ for F_1 satisfying (a) and (b). Then, by (a), the matrix $Q[p_1, \ldots, p_n]$ is unit lower triangular for some ordering of the rows of Q, hence, since $(p_1, \ldots, p_n)$ spans F_1, it must be a basis for F_1. Also, by (b), all columns $\alpha \preceq \delta(p_n)$ with $\alpha \notin \{\delta(p_1), \ldots, \delta(p_n)\}$ are free (since each such $()^\alpha$ is the leading term of an element of $\ker Q$). Therefore, necessarily, $(\delta(p_1), \ldots, \delta(p_n)) = (\beta_1, \ldots, \beta_n)$, the indices of the bound columns of QV. We will continue to use the notation

$$F = \mathrm{ran}[()^{\beta_j} : j = 1{:}n]$$

for the space spanned by the corresponding monomials, i.e., by the leading terms of the p_j, and use P for the corresponding projector. Since each of the monomials not in F corresponds to a free column of QV, we can write each p_j as

$$p_j \;=:\; f_j + q_j,$$

with

$$f_j \in F, \quad \delta(f_j) = \beta_j, \quad q_j \in \ker Q, \quad \delta(q_j) \prec \beta_j.$$

Then

$$Q[p_1, \ldots, p_n] = Q[f_1, \ldots, f_n],$$

hence, for any $p \in \Pi$,

$$P_1 p = \sum_j p_j a(j) \quad \Longleftrightarrow \quad Pp = \sum_j f_j a(j),$$

and therefore, in particular,

$$\delta(P_1 p) = \max\{\beta_j : a(j) \neq 0\} = \delta(Pp) \preceq \delta(p),$$

the inequality since, as we saw earlier, F is $\prec$-reducing. In other words, F_1 is $\prec$-reducing. □

We note that Theorem 3 of [S] has, in the equation corresponding to (4) here, the polynomials p_j and q_j rather than their leading terms. Also, with the assumption that $p_1 \prec \cdots \prec p_n$ spans F_1, the assumption (a) really plays no role in the above proof other than to ensure that $Q[p_1, \ldots, p_n]$ is invertible, hence there is a linear projector P_1 with $\operatorname{ran} P_1 = F_1$ and $\ker P_1 = \ker Q$. This is not surprising in view of the fact that every n-dimensional subspace of Π has a graded basis, i.e., a basis $[r_1, \ldots, r_n]$ with $\delta(r_1) \prec \cdots \prec \delta(r_n)$, and that, as we saw in the first part of the above proof, Gauss elimination derives from this a basis $[p_1, \ldots, p_n] = [r_1, \ldots, r_n]U^{-1}$ with U some upper triangular matrix, hence $\delta(p_j) = \delta(r_j)$, all j, for which $Q[p_1, \ldots, p_n]$ is unit lower triangular with respect to some ordering of the rows of Q. In fact, the proof of Proposition 3 also establishes the following simpler characterization.

Corollary. *A linear subspace of Π is $\prec$-reducing for Q if and only if it has a basis whose leading terms correspond to the bound columns of QV.*

Now we raise the stakes by assuming, in addition, that P is an **ideal projector** in the sense of [B], i.e., that $\ker Q$ is a (polynomial) ideal. This means that $\ker Q$ contains, for any $q \in Q$ and any $p \in \Pi$, also their (pointwise) product

$$qp : x \mapsto q(x)p(x),$$

and the ordering of the columns of V should be sensitive to that. Specifically, we assume from now on that the ordering is also **monomial**, meaning that it is consistent with addition on $\mathbb{Z}_+^d$, i.e.,

$$\alpha \preceq \beta \quad \Longrightarrow \quad \alpha + \gamma \preceq \beta + \gamma, \quad \text{all } \alpha, \beta, \gamma \in \mathbb{Z}_+^d,$$

and

$$0 \preceq \alpha, \quad \text{all } \alpha \in \mathbb{Z}_+^d.$$

Any such ordering refines the partial order given by divisibility, i.e., $()^\alpha$ dividing $()^\beta$ implies that $\alpha \preceq \beta$ since it implies that $\beta - \alpha \in \mathbb{Z}_+^d$, hence $0 \preceq \beta - \alpha$ and therefore $\alpha \preceq \beta$.

Proposition 5. *Under the given assumptions, F is D-**invariant**, i.e., closed under differentiation.*

Proof: $()^\alpha$ is free iff

$$Q()^\alpha = Qp$$

for some

$$p \in \Pi_{\prec\alpha}.$$

Hence, if $()^\alpha$ is free, and therefore $()^\alpha - p \in \ker Q$ for some $p \in \Pi_{\prec\alpha}$, then also, for any $\gamma \in \mathbb{Z}_+^d$, $()^\gamma(()^\alpha - p) \in \ker Q$, i.e.,

$$Q()^{\gamma+\alpha} = Q(()^\gamma p),$$

with $()^\gamma p \in \Pi_{\prec\gamma+\alpha}$, hence also $()^{\gamma+\alpha}$ is free. In other words, any monomial that divides a bound monomial must itself be bound, i.e., must lie in F. □

More than that and as pointed out in [S] and certainly already used in [MB], we also get immediately a reduced Gröbner basis for the ideal $\ker Q$, namely the set

$$G := \{()^\alpha - P()^\alpha : \alpha \in \mathrm{A}\},$$

with A the indices of all the free monomials not divisible by some other free monomial.

Indeed, G is **reduced**, i.e., no term of an element of G is divisible by the leading term of any other element of G, as is clear from the definition of A and from the fact that any monomial having a free monomial as a factor must be free while $\operatorname{ran} P$ is spanned by bound monomials. To show that every $p \in \ker Q \backslash 0$ is in ideal(G) (the ideal generated by G), proceed by induction on $\delta(p)$, assuming without loss of generality that $p \in ()^{\delta(p)} + \Pi_{\prec\delta(p)}$. If $p \in \ker Q$, then, necessarily, $()^{\delta(p)}$ is free. If it is not divisible by any other free monomial, then, for some $g \in G$, $p - g \in \Pi_{\prec\delta(p)}$ and in $\ker Q$, hence in ideal(G) by induction. Otherwise, $\delta(p) = \gamma + \beta$ for some some free $()^\beta$, hence, by induction, $p - ()^\gamma q \in \Pi_{\prec\delta(p)}$ for some $q \in \text{ideal}(G) \subseteq \ker Q$, hence that difference is in $\ker Q$ and so, by induction, in ideal(G). Hence, either way, $p \in \text{ideal}(G)$. More than that, it shows that p is writable as $\sum_{g\in G} g q_g$ with $\delta(g q_g) \le \delta(p)$ for all $g \in G$, hence G is a Gröbner basis for $\ker Q$.

Finally, both [MB] and [S] provide an algorithm for the construction of F and G, and both choose to do, in effect, Gauss elimination by columns, but working with polynomials rather than with the Gram matrix QV. But there seems to be no reason to deviate from the standard approach, of applying Gauss elimination by rows to QV, since it is just as easy there to introduce columns one at a time, hence to ignore any column known *a priori* to be free since its monomial is divisible by some monomial known to be free. In particular, as already pointed out in [S], it is sufficient to consider only columns α with $|\alpha| := \sum_j \alpha(j) \le n$, i.e., to consider the *finite* matrix QV_n with $V_n := [()^\alpha : |\alpha| \le n]$. The resulting factorization of QV_n, as CLU with C a permutation matrix, L unit lower triangular, and U in row echelon form, provides just as readily the set G and even a Newton basis for F, as follows.

Assume without loss that the rows of Q are so ordered that no row interchanges were necessary, hence $C = \text{id}$, and that, as before, the bound columns are $\beta_1 \prec \cdots \prec \beta_n$. Then, as already used in the proof of Proposition 3,

$$[p_j : j = 1{:}n] := [()^{\beta_j} : j = 1{:}n] U(:, (\beta_1, \ldots, \beta_n))^{-1}$$

is a **Newton basis for** F in the sense that $Q[p_1, \ldots, p_n]$ is unit lower triangular. Also, for each free $()^\alpha$ not divisible by some other free monomial,

$$q_\alpha := ()^\alpha - [p_1, \ldots, p_n] U(:, \alpha)$$

is an element of G, and G has no other elements than these.

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