SCATTERED DATA INTERPOLATION FROM PRINCIPAL SHIFT-INVARIANT SPACES

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Abstract

Under certain assumptions on the compactly supported function $\phi \in C(\mathbb{R}^d)$, we propose two methods of selecting a function s from the scaled principal shift-invariant space $S^h(\phi)$ such that s interpolates a given function f at a scattered set of data locations. For both methods, the selection scheme amounts to solving a quadratic programming problem and we are able to prove errror estimates similar to those obtained by Duchon for surface spline interpolation.

1. Introduction

The scattered data interpolation problem in $\mathbb{R}^d$ is the following: Given a set of scattered points $\Xi \subset \mathbb{R}^d$ and a function f defined at least on Ξ, one seeks a 'nice' function s which interpolates the data $f_{|\Xi}$; that is, which satisfies $s(\xi) = f(\xi)\ \forall \xi \in \Xi$. The reader is referred to the surveys [3] and [5] for descriptions of a variety of interpolation methods. One such method is that of surface spline interpolation (see [4]) which we now describe.

Let $m \in \mathbb{N} := \{1, 2, 3, \dots\}$ be such that $m > d/2$, and let H^m denote the set of all tempered distributions f for which $D^\alpha f \in L_2 := L_2(\mathbb{R}^d)$ for all $|\alpha| = m$. For measurable $A \subset \mathbb{R}^d$ and $f \in H^m$, we define the seminorm

$$|||f|||_{H^m(A)} := (2\pi)^{d/2} \sqrt{\sum_{|\alpha|=m} \tau_\alpha \left\| D^\alpha f \right\|^2_{L_2(A)}},$$

where the τ_α's are the positive integers determined by the equation $|x|^{2m} = \sum_{|\alpha|=m} \tau_\alpha x^{2\alpha}$, $x \in \mathbb{R}^d$. In case $A = \mathbb{R}^d$, we write simply $|||f|||_{H^m}$. The surface spline interpolation method dictates that $s \in H^m$ be chosen to minimize $|||s|||_{H^m}$ subject to the interpolation conditions $s_{|\Xi} = f_{|\Xi}$. If Ξ is finite and not contained in the zero-set of any nontrivial polynomial in $\Pi_{m-1} :=$ {polynomials of degree $\leq m-1$}, then the surface spline interpolant s can be realized as the unique function which interpolates the data $f_{|\Xi}$ and has the form $s = q + \sum_{\xi\in\Xi} \lambda_\xi \zeta(\cdot - \xi)$, where $q \in \Pi_{m-1}$, the λ_ξ's satisfy $\sum_{\xi\in\Xi} \lambda_\xi r(\xi) = 0\ \forall r \in \Pi_{m-1}$, and ζ is the radially symmetric function

$$\zeta(x) = \begin{cases} |x|^{2m-d} & \text{if } d \text{ is odd,} \\ |x|^{2m-d} \log|x| & \text{if } d \text{ is even,} \end{cases} \qquad x \in \mathbb{R}^d.$$

In order to discuss the error between f and s, let us assume that Ω is open, bounded, and has the cone property, and assume also that $\Xi \subset \overline{\Omega} := \text{closure}(\Omega)$. The 'fill distance' from Ξ to Ω is the quantity $\delta := \delta(\Xi, \Omega) := \sup_{x\in\Omega} \inf_{\xi\in\Xi} |x - \xi|$. Duchon [4] has shown that if s is the surface spline interpolant to f at Ξ, then

$$(1.1) \qquad \|f - s\|_{L_p(\Omega)} \leq const\, \delta^{m-d/2+d/p} |||f|||_{H^m} \quad \forall f \in H^m$$

for $2 \leq p \leq \infty$ and δ sufficiently small. What is interesting about the proof of (1.1) is that it hinges not on the fact that s minimizes $|||s|||_{H^m}$, but rather on the fact that $|||s|||_{H^m}$

is bounded by $const|||f|||_{H^m}$. The point being that the form of s is irrelevant. To obtain (1.1), all that is needed is that s interpolate $f_{|\Xi}$ while maintaining $|||s|||_{H^m} \le const|||f|||_{H^m}$. With this in mind we consider interpolation from principal shift-invariant spaces.

Let $\phi : \mathbb{R}^d \to \mathbb{C}$ be continuous and compactly supported. The semi-discrete convolution $\phi *' c$ between ϕ and a function c (defined at least on $\mathbb{Z}^d$) is defined by

$$\phi *' c := \sum_{j\in\mathbb{Z}^d} c(j)\phi(\cdot - j),$$

with convergence taken uniformly on compact sets. For $A \subset \mathbb{R}^d$ let

$$S(\phi, A) := \{\phi *' c : c(j) = 0 \text{ whenever } supp\,\phi(\cdot - j) \cap A = \emptyset\}.$$

The space $S(\phi, \mathbb{R}^d)$ is a *shift-invariant* space because $s(\cdot - j) \in S(\phi, \mathbb{R}^d)$ whenever $s \in S(\phi, \mathbb{R}^d)$ and $j \in \mathbb{Z}^d$. It is called a *principal* shift-invariant space because it is generated by the single function ϕ. The space $S(\phi, A)$ is refined by dilation for which we employ the dilation operator σ_h defined by

$$\sigma_h f := f(h\cdot).$$

For $h > 0$ and $A \subset \mathbb{R}^d$ let

$$S^h(\phi, A) := \{\sigma_{1/h} s : s \in S(\phi, h^{-1}A)\}.$$

In other words, $S^h(\phi, A)$ is the closure, in the topology of uniform convergence on compact sets, of $span\{\phi(\cdot/h - j) : j \in \mathbb{Z}^d,\ supp\,\phi(\cdot/h - j) \cap A \neq \emptyset\}$. Let $\Omega \subset \mathbb{R}^d$ be open and bounded. The approximation order of the scale of spaces $\{S^h(\phi, \Omega)\}_{h>0}$ can be characterized in terms of the Strang-Fix conditions:

Definition 1.2. ϕ is said to satisfy the *Strang-Fix conditions of order* m ($m \in \mathbb{N}$) if $\widehat{\phi}(0) \neq 0$ and $D^\alpha \widehat{\phi}(2\pi j) = 0\ \forall j \in \mathbb{Z}^d \backslash 0, |\alpha| < m$.

Here $\widehat{\phi}$ denotes the Fourier transform of ϕ. It is known (see [7]) that ϕ satisfies the Strang-Fix conditions of order m if and only if

$$\inf_{s\in S^h(\phi,\mathbb{R}^d)} \|f - s\|_{L_p} = O(h^m) \text{ as } h \to 0 \quad \forall f \in W_p^m,\ 1 \le p \le \infty,$$

where W_p^m denotes the Sobolev space (see [1]) of all tempered distributions f for which $D^\alpha f \in L_p := L_p(\mathbb{R}^d)\ \forall\, |\alpha| \le m$.

Assume that ϕ satisfies the Strang-Fix conditions of order m for some $m \in \mathbb{N}$ with $m > d/2$. We show in Section 2 that if Ξ is a finite subset of $\overline{\Omega}$, then $S^h(\phi, \Omega)$ contains functions which interpolate $f_{|\Xi}$ whenever h is sufficiently small; precisely, whenever $0 < h \le sep(\Xi)/\varepsilon_\phi$, where ε_ϕ is a positive constant depending only on ϕ and where

$$sep(\Xi) := \inf\{|\xi - \xi'| : \xi, \xi' \in \Xi, \xi \neq \xi'\}$$

denotes the separation distance in Ξ. Of course, in this case, there are infinitely many functions in $S^h(\phi, \Omega)$ which interpolate $f_{|\Xi}$. In light of the discussion surrounding (1.1), a sensible way of selecting a particular interpolant $s \in S^h(\phi,\Omega)$ is to choose one which minimizes $|||s|||_{H^m(\Omega)}$. In Section 7, under the additional assumptions that $\phi \in W_2^m$ and that Ω is connected and has a Lipschitz boundary, we show that if s is chosen in this manner then (1.1) holds whenever δ is sufficiently small and $0 < h \le sep(\Xi)/\varepsilon_\phi$.

The additional assumption that $\phi \in W_2^m$ is very strong, and a quick survey of 'distinguished' box-splines (see [2]) or B-splines reveals numerous examples where ϕ satisfies the Strang-Fix conditions of order m but $\phi \notin W_2^m$. For example, in the univariate case ($d = 1$), the function $\phi := (1-|\cdot|)\chi_{[-1,1]}$ satisfies the Strang-Fix conditions of order 2, but does not belong to W_2^2 because ϕ'' is not a function. A great share of the effort in the present work is devoted to replacing this assumption with the weaker assumption that $\phi \in W_2^\kappa$ where $\kappa \in \mathbb{N}$ is such that $d/2 < \kappa \le m$. Note that this supports the abovementioned univariate example $\phi := (1-|\cdot|)\chi_{[-1,1]}$ if we take $\kappa = 1$.

Unfortunately, the 'cost' functional $|||s|||_{H^m(\Omega)}$ is no longer meaningful when $\kappa < m$ for the simple reason that the functions in $S^h(\phi,\Omega)$ are not assumed to lie in H^m. This is very similar to the situation encountered in [9]. There, the natural choice of the cost functional was $|||s|||_{H^{2m}}$ but the functions s under consideration were spanned by translates of the function ζ (mentioned above) which does not locally belong to H^{2m}. This difficulty was overcome in [9] by using a cost functional of the form $|||\delta^{-d}\eta(\cdot/\delta) * s|||_{H^{2m}}$ where η is a well chosen exponentially decaying function. We employ a similar cure. In Section 3 we show that there exists a compactly supported distribution η such that $\widehat{\eta} \sim (1+|\cdot|^2)^{(\kappa-m)/2}$. The cost functional

$$|||h^{-d}\eta(\cdot/h) * s|||_{H^m} \tag{1.3}$$

is now well defined because $\eta * \phi \in W_2^m$ (by Proposition 3.3). In order to obtain something like (1.1) we have to slightly adjust our approach. We assume only that Ω is open, bounded and has the cone property, and we let Ω_0 be any open, bounded set which contains $\overline{\Omega}$. With $0 < h \le sep(\Xi)/\varepsilon_\phi$, we choose $s \in S^h(\phi,\Omega_0)$ to minimize (1.3) subject to the interpolation conditions $s_{|\Xi} = f_{|\Xi}$. In Section 6 we show that if δ is sufficiently small, then

$$\|f-s\|_{L_p(\Omega)} \le const\,\delta^{m-d/2+d/p}\,\|f\|_{W_2^m} \quad \forall 2 \le p \le \infty, f \in W_2^m,$$

where

$$\|f\|_{W_2^m} := \left\|(1+|\cdot|^2)^{m/2}\widehat{f}\right\|_{L_2}.$$

An outline of the paper is as follows: In Section 2 we prove that interpolants from $S^h(\phi,\mathcal{A})$ exist whenever $0 < h \le sep(\mathcal{A})/\varepsilon_\phi$, while in Section 3 we settle some technical issues relating to the convolution $\eta * f$ when f is a tempered distribution. We show in Section 4 that the error is controlled by the cost functional (1.3). The operator norm of the operator $\phi *'$ is analyzed in various settings in Section 5. Finally, in Section 6 and Section 7, the two abovementioned interpolation schemes are described and analyzed.

Throughout this paper we use standard multi-index notation: $D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\cdots\frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$. For multi-indices α, we define $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_d$, while for $x \in \mathbb{R}^d$, we define $|x| := \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$. The Fourier transform of a function f is defined formally by $\widehat{f}(w) := \int_{\mathbb{R}}^{d} e^{-iw\cdot x} f(x)\,dx$ and plays an important role in the sequel. One related fact which follows from the Plancherel Theorem is that $|||f|||_{H^m}$ has the representation in the Fourier domain as $\left\| |\cdot|^m \widehat{f} \right\|_{L_2(\mathbb{R}^d \setminus 0)}$ for all $f \in H^m$. It follows form this that $|||\sigma_h f|||_{H^m} = h^{m-d/2} |||f|||_{H^m}$ and $|||f|||_{H^m} \le \|f\|_{W_2^m}$. The space of compactly supported C^∞ functions is denoted $C_c^\infty(\mathbb{R}^d)$. The space $C_c(A)$ is the set of all continuous functions having compact support contained in A. If μ is a distribution and g is a test function, then the application of μ to g is denoted $\langle g, \mu \rangle$. We employ the notation $const$ to denote a generic constant in the range $(0\,..\,\infty)$ whose value may change with each occurence. In the statement of results we specify the dependencies of any $const$ while in proofs we omit the dependencies for the sake of brevity. Two oft employed subsets of $\mathbb{R}^d$ are the open unit ball $B := \{x \in \mathbb{R}^d : |x| < 1\}$ and the unit cube $C := [1/2..1/2)^d$.

2. Existence of Interpolants from $S^h(\phi, \mathcal{A})$

The following lemma gives sufficient conditions for the existence of interpolants to f from $S^h(\phi, \mathcal{A})$.

Lemma 2.1. *Let $\phi \in C_c(\mathbb{R}^d)$ satisfy the Strang-Fix conditions of order $m \ge 1$. There exists $\varepsilon_\phi > 0$ (depending only on ϕ) such that if $0 < h \le sep(\mathcal{A})/\varepsilon_\phi$ and $f \in \ell_2(\mathcal{A})$, then there exists $s \in S^h(\phi, \mathcal{A})$, say $s = \sigma_{1/h}(\phi *' c)$, such that $s_{|\mathcal{A}} = f_{|\mathcal{A}}$ and $\|c\|_{\ell_2} \le const(\phi)\, \|f\|_{\ell_2(\mathcal{A})}$.*

Proof. It suffices to consider the case $h = 1$ since the general case can then be obtained by scaling. It is known [8] that $\phi *' 1 = \widehat{\phi}(0)$. Put $\mathcal{N} := \{j \in \mathbb{Z}^d : supp\,\phi(\cdot - j) \cap C \ne \emptyset\}$. Let $b : \mathbb{Z}^d \to \mathbb{C}$ be given by $b := \chi_{\mathcal{N}} / \widehat{\phi}(0)$, and put $\psi := \phi *' b$. Note that $\psi = 1$ on C. Put $r := \max\{|x| : x \in \mathcal{N} \cup supp\,\psi\}$ and $\varepsilon_\phi := 2r + \sqrt{d}$. Assume $sep(\mathcal{A}) \ge \varepsilon_\phi$. For $x \in \mathbb{R}^d$, let $[x] \in \mathbb{Z}^d$ be defined by $x \in [x] + C$. Put $\widetilde{c} := \sum_{a \in \mathcal{A}} f(a) b(\cdot - [a])$ and $\widetilde{s} := \phi *' \widetilde{c}$. The choice of ε_ϕ ensures that the supports of the sequences $\{b(\cdot - [a])\}_{a \in \mathcal{A}}$ are pairwise disjoint. Consequently, $\|\widetilde{c}\|_{\ell_2}^2 = \sum_{a \in \mathcal{A}} |f(a)|^2 \|b(\cdot - [a])\|_{\ell_2}^2 = \|b\|_{\ell_2}^2 \|f\|_{\ell_2(\mathcal{A})}^2$. The choice of ε_ϕ also ensures that the supports of the functions $\{\psi(\cdot - [a])\}_{a \in \mathcal{A}}$ are pairwise disjoint. Hence, if $a \in \mathcal{A}$, then $\widetilde{s}(a) = \sum_{a' \in \mathcal{A}} f(a')\psi(a - [a']) = f(a)\psi(a - [a]) = f(a)$. It may be the case that $\widetilde{s} \notin S(\phi, \mathcal{A})$, so define $c : \mathbb{Z}^d \to \mathbb{C}$ by $c(j) := \widetilde{c}(j)$, if $supp\,\phi(\cdot - j) \cap \mathcal{A} \ne \emptyset$, and $c(j) = 0$ otherwise. Put $s := \phi *' c$. Then $s(a) = \widetilde{s}(a) = f(a)$ for all $a \in \mathcal{A}$ and $\|c\|_{\ell_2} \le \|\widetilde{c}\|_{\ell_2} = const\, \|f\|_{\ell_2}$. $\square$

3. Convolution with the distribution η

In this section we settle some technical issues related to our cost funcitonal (1.3). We begin by proving the existence of the compactly supported distribution η mentioned in the introduction.

Lemma 3.1. *Let $\kappa, m \in \mathbb{N}$ be such that $d/2 < \kappa \le m$. There exists a compactly supported distribution η such that*

$$const(m,d)(1+|w|^2)^{(\kappa-m)/2} \le \widehat{\eta}(w) \le const(m,d)(1+|w|^2)^{(\kappa-m)/2} \quad \forall w \in \mathbb{R}^d. \tag{3.2}$$

Proof. Define the tempered distribution η_1 by $\widehat{\eta}_1 := (1+|\cdot|^2)^{(\kappa-m)/2} \in \mathcal{M}$, and let $\zeta \in C_c^\infty(\mathbb{R}^d)$ be such that $\zeta(0)=1$ and $\widehat{\zeta} \ge 0$. Put $\eta := \eta_1\zeta$. Then η is compactly supported and $\widehat{\eta} = (2\pi)^{-d}\widehat{\zeta} * \widehat{\eta}_1$. Since $\widehat{\zeta}$ decays rapidly and is non-negative (and not identically 0), we obtain (3.2). $\square$

With the existence of η settled, we turn now to the issue of defining the convolution $\eta * f$ assuming only that f is a tempered distribution. Our definition is valid not just for η, but for any tempered distribution whose Fourier transform lies in the space $\mathcal{M}$ defined below.

Let $\mathcal{S}$ denote the 'rapidly decreasing functions' (the test functions associated with tempered distributions) topologized (as usual) by the seminorms $\{\rho_n\}_{n\in\mathbb{N}}$, where

$$\rho_n(g) := \max_{|\alpha|\le n} \|(1+|\cdot|)^n D^\alpha g\|_{L_\infty}.$$

Let $\mathcal{M}$ denote the set of all $g \in C^\infty(\mathbb{R}^d)$ which satisfy

$$\forall N \in \mathbb{N}\ \exists n \in \mathbb{N} \quad \max_{|\alpha|\le N} \left\|(1+|\cdot|)^{-n} D^\alpha g\right\|_{L_\infty} < \infty.$$

For example, if u is a compactly supported distribution, then it follows from a theorem of Paley-Wiener that $\widehat{u} \in \mathcal{M}$. If $g \in \mathcal{M}$, then it is a consequence of Leibniz' formula that $g\zeta \in \mathcal{S}\ \forall \zeta \in \mathcal{S}$, and it is a consequence of the closed graph theorem that the mapping $\zeta \mapsto g\zeta$ is a continuous operator on $\mathcal{S}$. Consequently, the mapping $f \mapsto gf$ is a continuous operator on $\mathcal{S}'$ (the space of tempered distributions) whenever $g \in \mathcal{M}$.

Definition. Let u and v be tempered distributions with $\widehat{u} \in \mathcal{M}$ or $\widehat{v} \in \mathcal{M}$. The convolution $u * v$ is defined as the inverse Fourier transform of the tempered distribution $\widehat{u}\widehat{v}$:

$$u * v := (\widehat{u}\widehat{v})^\vee.$$

If $\widehat{u} \in \mathcal{M}$, then it follows that $u*$ is a continuous operator on $\mathcal{S}'$. We collect in the following proposition several properties of the convolution operator $\eta*$ which will be used in the sequel.

Proposition 3.3. *Let κ, m, η be as in Lemma 3.1, and let $\phi \in W_2^\kappa$ be compactly supported. Put $\psi := \eta * \phi$. Then $\psi \in W_2^m$ and $\operatorname{supp}\psi \subset \operatorname{supp}\eta + \operatorname{supp}\phi$. Let $c : \mathbb{Z}^d \to \mathbb{C}$ have at most polynomial growth and for $n \in \mathbb{N}$ define $c_n \in \ell_0$ by $c_n(j) := \begin{cases} c(j) & \text{if } |j| \le n, \\ 0 & \text{else} \end{cases}$. Then*

$$\begin{aligned} &(i) \qquad \phi *' c_n \to \phi *' c \text{ in } \mathcal{S}' \quad \text{and} \\ &(ii) \qquad \eta * (\phi *' c) = \psi *' c. \end{aligned}$$

Proof. To see that $\psi \in W_2^m$ note that by (3.2)

$$\|\psi\|_{W_2^m} = \left\|(1+|\cdot|^2)^{m/2}\widehat{\eta}\widehat{\phi}\right\|_{L_2} \le const \left\|(1+|\cdot|^2)^{\kappa/2}\widehat{\phi}\right\|_{L_2} = const\, \|\phi\|_{W_2^\kappa} < \infty.$$

That $supp\,\psi \subset supp\,\eta + supp\,\phi$ is proved in [6, Th. 4.9 and p. 87]. Let r be the smallest positive real number for which $supp\,\phi \subset r\overline{B}$. There exists a polynomial q, say of degree k, such that $|c(j)| \le q(j)$ $\forall j \in \mathbb{Z}^d$. If $g \in \mathcal{S}$, then

$$|\langle g, \phi *' c\rangle - \langle g, \phi *' c_n\rangle| = |\langle g, \phi *' (c - c_n)\rangle| \le \sum_{|j|>n} |c(j)|\, |\langle g, \phi(\cdot - j)\rangle|$$

$$\le \sum_{|j|>n} q(j)\, \|\phi\|_{L_1}\, \|g\|_{L_\infty(j+rB)} \le const \left(\sum_{|j|>n} q(j)(1+|j|)^{-k-d-1}\right) \rho_{k+d+1}(g).$$

Since $\sum_{|j|>n} q(j)(1+|j|)^{-k-d-1} \to 0$ as $n \to \infty$, we obtain (i). Since $\psi \in W_2^m$ has compact support, we have by (i) that $\psi *' c_n \to \psi *' c$ in $\mathcal{S}'$. Since $\eta *$ is a continuous operator on $\mathcal{S}'$, it follows from (i) that $\eta * (\phi *' c_n) \to \eta * (\phi *' c)$ in $\mathcal{S}'$. Noting that $\eta * (\phi *' c_n) = \psi *' c_n$ $\forall n \in \mathbb{N}$, we obtain (ii). $\square$

4. An Error Estimate

The following theorem contains our basic error estimate. In practice, the function g will be the error $f - s$. Of course, if s interpolates f at Ξ, then $f - s$ will vanish on Ξ.

Theorem 4.1. *Let κ, m, η be as in Lemma 3.1. Let Ω be an open, bounded subset of $\mathbb{R}^d$ having the cone property and let $\Xi \subset \overline{\Omega}$. There exists $\delta_0 > 0$ such that if $\delta := \delta(\Xi;\Omega) \le \delta_0$, then for $2 \le p \le \infty$*

$$\|g\|_{L_p(\Omega)} \le const(\eta, m, \Omega)\delta^{m-d/2+d/p}|||\delta^{-d}\eta(\cdot/\delta) * g|||_{H^m} \quad \forall g \in H^m + H^\kappa \textit{ satisfying } g_{|\Xi} = 0.$$

We mention that in the case $\kappa = m$, the above conclusion reduces to

$$\|g\|_{L_p(\Omega)} \le const(m, \Omega)\delta^{m-d/2+d/p}|||g|||_{H^m} \quad \forall g \in H^m \text{ which vanish on } \Xi,$$

which is known [4]. Our proof of this theorem requires two supporting lemmas. The proof of the first is essentially the same as the proof of [9, Prop. 3.1] if one replaces $|||f|||_{H^m}$ with $|||f|||_{H^\kappa}$, $|||f|||_{H^{2m}}$ with $|||f|||_{H^m}$, and $|||f|||_*$ with $|||\eta * f|||_{H^m}$.

Lemma 4.2. *Let κ, m, η be as in Lemma 3.1, and let $r > 0$. For each $j \in \mathbb{Z}^d$, let $\mathcal{N}_j$ be a finite subset of $j + rB$. If $\{b_{j,\xi}\}_{j\in\mathbb{Z}^d, \xi\in\mathcal{N}_j}$ is such that*

$$\sum_{\xi\in\mathcal{N}_j} b_{j,\xi}q(\xi) = 0 \quad \forall q \in \Pi_{m-1}, j \in \mathbb{Z}^d \quad \textit{and}$$

$$M := \sup_{j\in\mathbb{Z}^d} \sum_{\xi\in\mathcal{N}_j} |b_{j,\xi}| < \infty,$$

then

$$\sum_{j\in\mathbb{Z}^d} \left|\sum_{\xi\in\mathcal{N}_j} b_{j,\xi} f(\xi)\right|^2 \le const(\eta, m, r)M^2|||\eta * f|||^2_{H^m} \quad \forall f \in H^m + H^\kappa.$$

The following lemma is taken from [9, Lemma 4.2].

Lemma 4.3. *Let $n \ge 0$. If $\Omega \subset \mathbb{R}^d$ is bounded, open, and has the cone property, then there exists $\delta_0, r_0 \in (0\,..\,\infty)$ (depending only on n and Ω) such that if Ξ is a finite subset of $\overline{\Omega}$ with $\delta := \delta(\Xi;\Omega) \le \delta_0$, then for all $x \in \Omega/\delta$ there exists a finite $\mathcal{N} \subset (\Xi/\delta) \cap (x + r_0 B)$ and $\{b_\xi\}_{\xi\in\mathcal{N}}$ such that*

$$q(x) + \sum_{\xi\in\mathcal{N}} b_\xi q(\xi) = 0 \quad \forall q \in \Pi_n \quad \textit{and}$$
$$\sum_{\xi\in\mathcal{N}} |b_\xi| \le const(n,\Omega).$$

Proof of Theorem 4.1. Let δ_0, r_0 be as in Lemma 4.3 with $n = m-1$. Put $\mathcal{A} := \{j \in \mathbb{Z}^d : (j + C) \cap (\Omega/\delta) \ne \emptyset\}$. For each $j \in \mathcal{A}$, let $x_j \in (j + C) \cap (\Omega/\delta)$ be such that $\|\sigma_\delta g\|_{L_\infty((j+C)\cap(\Omega/\delta))} \le 2\,|g(\delta x_j)|$. By Lemma 4.3, for each $j \in \mathcal{A}$, there exists $\mathcal{N}_j \subset (\Xi/\delta) \cap (x_j + r_0 B)$ and $\{b_{j,\xi}\}_{\xi\in\mathcal{N}_j}$ such that

$$q(x_j) + \sum_{\xi\in\mathcal{N}_j} b_{j,\xi} q(\xi) = 0 \quad \forall q \in \Pi_{m-1} \text{ and}$$
$$\sum_{\xi\in\mathcal{N}_j} |b_{j,\xi}| \le const(m,\Omega).$$

Put $r := r_0 + \sqrt{d}/2$ and note that $\{x_j\} \cup \mathcal{N}_j \subset j + rB$ for all $j \in \mathcal{A}$. Now,

$$\begin{aligned}
\|g\|_{L_p(\Omega)} &= \delta^{d/p}\,\|\sigma_\delta g\|_{L_p(\Omega/\delta)} \\
&\le \delta^{d/p}\,\Big\| j \mapsto \|\sigma_\delta g\|_{L_\infty((j+C)\cap(\Omega/\delta))} \Big\|_{\ell_p(\mathcal{A})} \\
&\le 2\delta^{d/p}\,\|j \mapsto g(\delta x_j)\|_{\ell_p(\mathcal{A})} \le 2\delta^{d/p}\,\|j \mapsto g(\delta x_j)\|_{\ell_2(\mathcal{A})}, \quad \text{since } 2 \le p, \\
&= 2\delta^{d/p}\sqrt{\sum_{j\in\mathcal{A}} |g(\delta x_j)|^2}.
\end{aligned} \tag{4.4}$$

Since $g(\delta\xi) = 0$ for all $\xi \in \Xi/\delta$, we have

$$|g(\delta x_j)| = \left| g(\delta x_j) + \sum_{\xi\in\mathcal{N}_j} b_{j,\xi}\, g(\delta\xi) \right|, \quad \forall j \in \mathcal{A}.$$

We thus obtain from (4.4) and Lemma 4.2 that

$$\|g\|_{L_p(\Omega)} \le const\,\delta^{d/p}|||\eta * (\sigma_\delta g)|||_{H^m} = const\,\delta^{m-d/2+d/p}|||\delta^{-d}\eta(\cdot/\delta) * g|||_{H^m}.$$

□

5. An analysis of $\phi *'$

As mentioned just prior to the statement of Theorem 4.1, our error estimates will employ Theorem 4.1 with $g = f - s$. Roughly speaking, the factor $|||\delta^{-d}\eta(\cdot/\delta) * (f-s)|||_{H^m}$ will be estimated by $|||\delta^{-d}\eta(\cdot/\delta) * f|||_{H^m} + |||\delta^{-d}\eta(\cdot/\delta) * s|||_{H^m}$, where the first term will be shown to be bounded by a constant times $|||f|||_{H^m}$. The second term is our cost functional (1.3) with δ in place of h. Although this second term involves the parameter δ, its action on any $s \in S^\delta(\phi, \mathbb{R}^d)$ exhibits a certain stationarity. Namely, if $s = \sigma_{1/\delta}(\phi *' c)$, then

$$\delta^{m-d/2}|||\delta^{-d}\eta(\cdot/\delta) * s|||_{H^m} = |||\eta * (\phi *' c)|||_{H^m}. \tag{5.1}$$

Thus, the right side of (5.1) is an important quantity. Two estimates of this quantity are given in the following proposition.

Proposition 5.2. *Let κ, m, η be as in Lemma 3.1, and let $\phi \in W_2^\kappa$ be compactly supported. Then*

$$|||\eta * (\phi *' f)|||_{H^m} \le const(\eta, m, \phi)\,\|f\|_{\ell_2} \quad \forall f \in \ell_2. \tag{5.2}$$

If, in addition, ϕ satisfies the Strang-Fix conditions of order m, then

$$|||\eta * (\phi *' f)|||_{H^m} \le const(\eta, m, \phi)|||f|||_{H^m} \quad \forall f \in H^m. \tag{5.3}$$

Our proof of this proposition requires the following lemma which is a consequence of [10, Théorème 1.6] and the Sobolev embedding theorem [1, p. 97].

Lemma 5.4. *Let $y \in \mathbb{R}^d$, $r > 0$, and $m \in \mathbb{N}$ with $m > d/2$. For all $f \in H^m$ there exists $q \in \Pi_{m-1}$ such that*

$$\|f - q\|_{L_\infty(y+rB)} \le const(d, m, r)|||f|||_{H^m(y+rB)}.$$

Proof of Propositon 5.2. Put $\psi := \eta * \phi$. By Proposition 3.3, $\psi \in W_2^m$ is compactly supported and

$$\eta * (\phi *' f) = \psi *' f. \tag{5.5}$$

Put $\mathcal{N} := \{j \in \mathbb{Z}^d : supp\,\psi(\cdot - j) \cap C \ne \emptyset\}$, and note that $\#\mathcal{N} < \infty$. In consideration of (5.2), assume $f \in \ell_2$. Then

$$\begin{aligned}|||\psi *' f|||^2_{H^m} &= \sum_{\ell\in\mathbb{Z}^d} |||\psi *' f|||^2_{H^m(\ell+C)} = \sum_{\ell\in\mathbb{Z}^d} \Big|\Big|\Big| \sum_{j\in\mathcal{N}} f(\ell+j)\psi(\cdot - j)\Big|\Big|\Big|^2_{H^m(C)} \\ &\le const \sum_{\ell\in\mathbb{Z}^d} |||\psi|||^2_{H^m} \sum_{j\in\mathcal{N}} |f(\ell+j)|^2 \le const\,\|f\|^2_{\ell_2}\end{aligned}$$

which, in view of (5.5), proves (5.2). In consideration of (5.3), assume ϕ satisfies the Strang-Fix conditions of order m, and let $f \in H^m$. Since $\widehat{\psi} = \widehat{\eta}\widehat{\phi}$, it follows that ψ

also satisfies the Strang-Fix conditions of order m. Consequently, $\psi *' q \in \Pi_{m-1}$ for all $q \in \Pi_{m-1}$ (see [8]). Let r be the least positive real number such that $\mathcal{N} \subset r\overline{B}$. By Lemma 5.4, for each $\ell \in \mathbb{Z}^d$ there exists $q_\ell \in \Pi_{m-1}$ such that

$$\|f - q_\ell\|_{L_\infty(\ell+rB)} \le const|||f|||_{H^m(\ell+rB)}.$$

This yields the estimate

$$|||\psi *' f|||_{H^m(\ell+C)} = |||\psi *' (f - q_\ell)|||_{H^m(\ell+C)} = \Big|\Big|\Big| \sum_{j\in\mathcal{N}} (f(\ell+j) - q_\ell(\ell+j))\psi(\cdot - j)\Big|\Big|\Big|_{H^m(\ell+C)}$$
$$\le \#\mathcal{N}\, \|f - q_\ell\|_{L_\infty(\ell+rB)}\, |||\psi|||_{H^m} \le const|||f|||_{H^m(\ell+rB)}.$$

Therefore,

$$|||\psi *' f|||^2_{H^m} = \sum_{\ell\in\mathbb{Z}^d} |||\psi *' f|||^2_{H^m(\ell+C)} \le const \sum_{\ell\in\mathbb{Z}^d} |||f|||^2_{H^m(\ell+rB)} \le const|||f|||^2_{H^m}$$

which, in view of (5.5), proves (5.3). □

Our proof of the following result uses the standard quasi-interpolation argument (see [2, Ch. III]) which greatly simplifies when $m > d/2$.

Proposition 5.6. *Let $\psi \in C_c(\mathbb{R}^d)$ and $m \in \mathbb{N}$ with $m > d/2$ be such that $\psi *' q = q$ $\forall q \in \Pi_{m-1}$. If $sep(\mathcal{A}) \ge const$, then*

$$\|f - \psi *' f\|_{\ell_2(\mathcal{A})} \le const(m,\psi)|||f|||_{H^m} \quad \forall f \in H^m.$$

Proof. Let $\mathcal{N}$, r, and $\{q_\ell\}_{\ell\in\mathbb{Z}^d}$ be as defined in the proof of Proposition 5.2. Then for $\ell \in \mathbb{Z}^d$

$$\|f - \psi *' f\|_{L_\infty(\ell+C)} = \|f - q_\ell - \psi *' (f - q_\ell)\|_{L_\infty(\ell+C)}$$
$$\le \|f - q_\ell\|_{L_\infty(\ell+C)} + \left\| \sum_{j\in\mathcal{N}} (f(\ell+j) - q_\ell(\ell+j))\psi(\cdot - j)\right\|_{L_\infty(C)}$$
$$\le (1 + \#\mathcal{N}\, \|\psi\|_{L_\infty})\, \|f - q_\ell\|_{L_\infty(\ell+(r+1)B)} \le const|||f|||_{H^m(\ell+(r+1)B)}.$$

Since $sep(\mathcal{A}) \ge const$, we have

$$\|f - \psi *' f\|^2_{\ell_2(\mathcal{A})} \le const \sum_{\ell\in\mathbb{Z}^d} \|f - \psi *' f\|^2_{L_\infty(\ell+C)} \le const \sum_{\ell\in\mathbb{Z}^d} |||f|||^2_{H^m(\ell+(r+1)B)} \le const|||f|||^2_{H^m}.$$

□

6. An Interpolation Method for the Case $\kappa \le m$

In the following, the phrase *nearly minimize* means to bring to within a constant of its minimal value.

Interpolation Method 6.1. *Let κ, m, η be as in Lemma 3.1. Let $\phi \in W_2^\kappa$ be compactly supported and satisfy the Strang-Fix conditions of order m, and let ε_ϕ be as in Lemma 2.1. Let Ω be an open, bounded subset of $\mathbb{R}^d$ having the cone property, and let Ω_0 be an open, bounded set which contains $\overline{\Omega}$. Let Ξ be a finite subset of $\overline{\Omega}$ and let $0 < h \le sep(\Xi)/\varepsilon_\phi$. Choose $s \in S^h(\phi, \Omega_0)$ to nearly minimize $|||h^{-d}\eta(\cdot/h) * s|||_{H^m}$ subject to the interpolation conditions $s_{|_\Xi} = f_{|_\Xi}$. There exists $\delta_1 > 0$ such that if $\delta := \delta(\Xi, \Omega) \le \delta_1$, then for all $f \in W_2^m$*

$$(i) \qquad |||h^{-d}\eta(\cdot/h) * s|||_{H^m} \le const(\eta, m, \Omega, \Omega_0, \phi) \, \|f\|_{W_2^m} \qquad and$$

$$(ii) \qquad \|f - s\|_{L_p(\Omega)} \le const(\eta, m, \Omega, \Omega_0, \phi)\delta^{m-d/2+d/p} \, \|f\|_{W_2^m} \quad \forall 2 \le p \le \infty.$$

Remark 6.2. The interpolant s can be found by *nearly* solving a quadratic programming problem. To see this, let s be written as $s = \sum_{j=1}^{M} c_j \phi(\cdot/h - k_j)$, where $\{k_1, k_2, \ldots, k_M\} := \{k \in \mathbb{Z}^d : supp\, \phi(\cdot/h - k) \cap \Omega_0 \ne \emptyset\}$, and put $\{\xi_1, \xi_2, \ldots, \xi_N\} := \Xi$. The interpolation conditions become $Ac = F$ where A is the $N \times M$ matrix having (i,j)-entry $\phi(\xi_i/h - k_j)$ and $F = [f(\xi_i)]_{1 \le i \le N}$. Put $\psi := \eta * \phi$ and let G be the $M \times M$ matrix having (i,j)-entry $(\psi, \psi(\cdot + k_j - k_i))_{H^m}$, where $(\ ,\ )_{H^m}$ denotes the semi-inner product associated with $|||\cdot|||_{H^m}$. The cost functional can then be written as

$$|||h^{-d}\eta(\cdot/h) * s|||_{H^m} = h^{-m+d/2}\sqrt{c^* G c},$$

where c^* denotes the complex conjugate of the transpose of c. Thus c is any near solution of the quadratic programming problem

$$\begin{aligned} &\text{minimize } c^* G c \\ &\text{subject to } Ac = F. \end{aligned}$$

We mention that the matrices A and G are sparse in the sense that the number of nonzero entries in each row or column is bounded independently of M and N.

Proof of 6.1. Let $\varepsilon > 0$ be the largest positive real number for which $\Omega + \varepsilon B \subset \Omega_0$, and let $\zeta \in C_c^\infty(\Omega + (\varepsilon/2)B)$ be such that $\zeta = 1$ on Ω. The assumptions on ϕ ensure (see [8]) that there exists a finitely supported sequence $a : \mathbb{Z}^d \to \mathbb{C}$ such that $\psi := \phi *' a$ satisfies the Strang-Fix conditions of order m and the condition $\psi *' q = q$ for all $q \in \Pi_{m-1}$. Let δ_0 be as in Theorem 4.1, and let $\delta_1 \in (0, \delta_0]$ be sufficiently small to ensure that

$$\delta(\Xi, \Omega) \le \delta_1,\ 0 < h \le sep(\Xi)/\varepsilon_\phi \text{ and } supp\, g \subset h^{-1}(\Omega + (\varepsilon/2)B) \Rightarrow \psi *' g \in S(\phi, h^{-1}\Omega_0).$$

Let $f \in W_2^m$ and put $\widetilde{f} := \zeta f$. Then $\left\|\widetilde{f}\right\|_{W_2^m} \le const \|f\|_{W_2^m}$. Assume $\delta := \delta(\Xi, \Omega) \le \delta_1$. Put $s_1 := \sigma_{1/h}(\psi *' \sigma_h \widetilde{f}) \in S^h(\phi, \Omega_0)$. Since $0 < h \le sep(\Xi)/\varepsilon_\phi$, it follows by Lemma 2.1 that there exists $s_2 \in S^h(\phi, \Xi)$, say $s_2 = \sigma_{1/h}(\phi *' c)$, such that $\|c\|_{\ell_2} \le const \left\|\widetilde{f} - s_1\right\|_{\ell_2(\Xi)}$ and $s_2(\xi) = \widetilde{f}(\xi) - s_1(\xi)$ for all $\xi \in \Xi$. Put $\widetilde{s} := s_1 + s_2 \in S^h(\phi, \Omega_0)$, and note that $\widetilde{s}(\xi) = s_1(\xi) + \widetilde{f}(\xi) - s_1(\xi) = f(\xi)$ for all $\xi \in \Xi$. Consequently,

$$
\begin{aligned}
|||h^{-d}\eta(\cdot/h) * s|||_{H^m} &\le const|||h^{-d}\eta(\cdot/h) * \widetilde{s}|||_{H^m} = const h^{-m+d/2}|||\eta * \sigma_h \widetilde{s}|||_{H^m} \\
&\le const h^{-m+d/2}(|||\eta * \sigma_h s_1|||_{H^m} + |||\eta * \sigma_h s_2|||_{H^m}).
\end{aligned}
\tag{6.2}
$$

By Proposition 5.2, we have

$$
|||\eta * \sigma_h s_1|||_{H^m} = |||\eta * (\psi *' \sigma_h \widetilde{f})|||_{H^m} \le const|||\sigma_h \widetilde{f}|||_{H^m}
$$

and

$$
\begin{aligned}
&|||\eta * \sigma_h s_2|||_{H^m} = |||\eta * (\phi *' c)|||_{H^m} \le const \|c\|_{\ell_2} \\
&\le const \left\|\widetilde{f} - s_1\right\|_{\ell_2(\Xi)} = const \left\|\sigma_h \widetilde{f} - \psi *' \sigma_h \widetilde{f}\right\|_{\ell_2(\Xi/h)} \le const|||\sigma_h \widetilde{f}|||_{H^m}
\end{aligned}
$$

by Proposition 5.6. Therefore, by (6.2),

$$
\begin{aligned}
&|||h^{-d}\eta(\cdot/h) * s|||_{H^m} \le const h^{-m+d/2}|||\sigma_h \widetilde{f}|||_{H^m} = const|||\widetilde{f}|||_{H^m} \\
&\le const \left\|\widetilde{f}\right\|_{W_2^m} \le const \|f\|_{W_2^m}
\end{aligned}
$$

which proves (i). Since $h \le const\delta$, it follows that $|||\delta^{-d}\eta(\cdot/\delta) * s|||_{H^m} \le const|||h^{-d}\eta(\cdot/h) * s|||_{H^m}$. Hence, by Theorem 4.1,

$$
\begin{aligned}
&\|f - s\|_{L_p(\Omega)} \le const\delta^{m-d/2+d/p}|||\delta^{-d}\eta(\cdot/\delta) * (f - s)|||_{H^m} \\
&\le const\delta^{m-d/2+d/p}(|||\delta^{-d}\eta(\cdot/\delta) * f|||_{H^m} + |||\delta^{-d}\eta(\cdot/\delta) * s|||_{H^m}) \le const\delta^{m-d/2+d/p} \|f\|_{W_2^m}
\end{aligned}
$$

which proves (ii). □

7. An Interpolation Method for the case when $\phi \in W_2^m$

The conclusion of the following result is an improvement over that of 6.1 as $|||f|||_{H^m(\Omega)}$ has taken the place of $\|f\|_{W_2^m}$ in (i) and (ii). To obtain this improvement, we have assumed further that $\phi \in W_2^m$ and that Ω is connected and has a Lipschitz boundary.

Interpolation Method 7.1. *Let $m \in \mathbb{N}$ with $d/2 < m$, and let $\phi \in W_2^m$ be compactly supported. Let Ω be an open, bounded, connected subset of $\mathbb{R}^d$ having the cone property and a Lipschitz boundary (in the sense of* [10]*), and let δ_0 and ε_ϕ be as in Theorem 4.1 and Lemma 2.1, respectively. Let Ξ be a finite subset of $\overline{\Omega}$ and let $0 < h \le sep(\Xi)/\varepsilon_\phi$. Let Ω_h be any measurable set which contains Ω, and let $s \in S^h(\phi, \Omega_h)$ be chosen to nearly minimize $|||s|||_{H^m(\Omega_h)}$ subject to the interpolation conditions $s_{|\Xi} = f_{|\Xi}$. If $\delta := \delta(\Xi, \Omega) \le \delta_0$, then for all $f \in H^m$*

$$\begin{array}{ll} (i) & |||s|||_{H^m(\Omega_h)} \le const(m, \Omega, \phi)|||f|||_{H^m(\Omega)} \quad \text{and} \\ (ii) & \|f - s\|_{L_p(\Omega)} \le const(m, \Omega, \phi)\delta^{m-d/2+d/p}|||f|||_{H^m(\Omega)} \quad \forall 2 \le p \le \infty. \end{array}$$

Remark. The interpolant s can be found by *nearly* solving the same quadratic programming problem described in Remark 6.2 excepting that $\{k_1, k_2, \ldots, k_M\} := \{k \in \mathbb{Z}^d : supp\,\phi(\cdot/h - k) \cap \Omega_h \ne \emptyset\}$ and $G(i,j) := (\phi, \phi(\cdot + k_j - k_i))_{H^m(h^{-1}\Omega_h)}$. If Ω_h is a complicated set, then the computation of G will likely be difficult. One way to ease this task is to choose Ω_h as

$$\Omega_h := \cup_{\ell \in \mathcal{A}_h} h(\ell + C),$$

where $\mathcal{A}_h := \{\ell \in \mathbb{Z}^d : \Omega \cap h(\ell + C) \ne \emptyset\}$. Using the auxillary function $u : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{C}$ given by $u(k, \ell) := (\phi, \phi(\cdot - k))_{H^m(\ell + C)}$ (which has a fixed number of nonzero entries), we can compute $G(i,j)$ as

$$G(i,j) = \sum_{\ell \in \mathcal{A}_h} u(k_i - k_j, \ell).$$

Our proof of 7.1 requires the following result which comes out of [4, p. 331].

Theorem 7.2. *Let $m \in \mathbb{N}$ with $m > d/2$. If Ω is an open, bounded, connected subset of $\mathbb{R}^d$ having the cone property and a Lipschitz boundary (in the sense of* [10]*), then for all $f \in H^m$ there exists $f_\Omega \in H^m$ such that*

$$\begin{array}{ll} (i) & f_\Omega = f \text{ on } \Omega \text{ and} \\ (ii) & |||f_\Omega|||_{H^m} \le const(m, \Omega)|||f|||_{H^m(\Omega)}. \end{array}$$

Proof of 7.1. Let a, ψ be as in the proof of 6.1. Let $f \in H^m$ and let f_Ω be as in Theorem 7.2. Put $s_1 := \sigma_{1/h}(\psi *' \sigma_h f_\Omega)$. By Proposition 5.2, $s_1 \in H^m$ and

$$|||\sigma_h s_1|||_{H^m} \le const|||\sigma_h f_\Omega|||_{H^m}. \tag{7.3}$$

Since $0 < h \le sep(\Xi)/\varepsilon_\phi$, it follows by Lemma 2.1 that there exists $s_2 \in S^h(\phi, \Xi)$, say $s_2 = \sigma_{1/h}(\phi *' c)$, such that $\|c\|_{\ell_2} \le const\,\|f_\Omega - s_1\|_{\ell_2(\Xi)}$ and $s_2(\xi) = f_\Omega(\xi) - s_1(\xi)$ for all $\xi \in \Xi$. Put $s_3 := s_1 + s_2 \in S^h(\phi, \mathbb{R}^d)$. Then $s_{3|\Xi} = f_{\Omega|\Xi} = f_{|\Xi}$, and

$$\begin{aligned} |||s_3|||_{H^m} &\le |||s_1|||_{H^m} + |||s_2|||_{H^m} = h^{-m+d/2}\left(|||\sigma_h s_1|||_{H^m} + |||\sigma_h s_2|||_{H^m}\right) \\ &= h^{-m+d/2}\left(|||\psi *' \sigma_h f_\Omega|||_{H^m} + |||\phi *' c|||_{H^m}\right) \le const h^{-m+d/2}\left(|||\sigma_h f_\Omega|||_{H^m} + \|c\|_{\ell_2}\right) \end{aligned}$$

by Proposition 5.2. Since $\|c\|_{\ell_2} \le const\, \|f_\Omega - s_1\|_{\ell_2(\Xi)} = const\, \|\sigma_h f_\Omega - \psi *' \sigma_h f_\Omega\|_{\ell_2(\Xi/h)}$, we have by Proposition 5.6, that $\|c\|_{\ell_2} \le const|||\sigma_h f_\Omega|||_{H^m}$. Therefore,

$$(7.4) \qquad |||s_3|||_{H^m} \le const h^{-m+d/2} |||\sigma_h f_\Omega|||_{H^m} = const|||f_\Omega|||_{H^m} \le const|||f|||_{H^m(\Omega)}$$

by Theorem 7.2. Let $s_4 \in S^h(\phi, \Omega_h)$ be such that $s_4 = s_3$ on Ω_h. Then ${s_4}_{|\Xi} = {s_3}_{|\Xi} = f_{|\Xi}$. Hence,

$$|||s|||_{H^m(\Omega_h)} \le const|||s_4|||_{H^m(\Omega_h)} = const|||s_3|||_{H^m(\Omega_h)} \le const|||s_3|||_{H^m} \le const|||f|||_{H^m(\Omega)}$$

which proves (i). By Theorem 7.2, there exists $s_\Omega \in H^m$ such that $s_\Omega = s$ on Ω and $|||s_\Omega|||_{H^m} \le const|||s|||_{H^m(\Omega)}$. Hence, by Theorem 4.1,

$$\begin{aligned}
&\|f - s\|_{L_p(\Omega)} = \|f_\Omega - s_\Omega\|_{L_p(\Omega)} \le const\delta^{m-d/2+d/p} |||f_\Omega - s_\Omega|||_{H^m} \\
&\le const\delta^{m-d/2+d/p} \left(|||f_\Omega|||_{H^m} + |||s_\Omega|||_{H^m}\right) \le const\delta^{m-d/2+d/p} |||f|||_{H^m(\Omega)}
\end{aligned}$$

which proves (ii). □

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