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On Cubic Spline Functions that Vanish at All Knots*

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INTRODUCTION

In [4], Birkhoff and de Boor improved on earlier results by Ahlberg and Nilson [1] concerning the convergence of cubic spline interpolants to a smooth interpoland. Shortly thereafter, Sharma and Meir [20] gave much more general results using much simpler means of proof and thus made [4] seemingly obsolete. Yet, the basic idea of [4] has been of help recently in illuminating certain problems, as recounted below, and seems at present to be the one most likely to provide the right insight into general odd-degree spline interpolation at knots. This note is therefore intended to give [4] a second chance.

1. NULLSPLINES AND FUNDAMENTAL SPLINES

The basic idea of [4] was Birkhoff's observation that the first and second derivative of a nonzero cubic spline C vanishing at its (simple) knots

$$\cdots < x_{i-1} < x_i < x_{i+1} < \cdots$$

must increase exponentially either for increasing or for decreasing argument. Explicitly, for $r = 1, 2$, either

$$-C^{(r)}(x_{j+1})/C^{(r)}(x_j) > 2, \qquad j = i+1, i+2,\ldots$$

or

$$-C^{(r)}(x_{j-1})/C^{(r)}(x_j) > 2, \qquad j = i-1, i-2,\ldots$$

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This follows at once from the fact that, for a cubic polynomial p vanishing at a and b (with $a \neq b$),

$$\begin{pmatrix} p'(b) \\ p''(b)/2 \end{pmatrix} = -A(b-a) \begin{pmatrix} p'(a) \\ p''(a)/2 \end{pmatrix} \tag{1a}$$

with

$$A(h) := \begin{pmatrix} 2 & h \\ 3/h & 2 \end{pmatrix}. \tag{1b}$$

Hence, if $(b-a)\,p'(a)\,p''(a) \geqslant 0$, then also $(b-a)\,p'(b)\,p''(b) \geqslant 0$ and $|\,p^{(r)}(b)| \geqslant 2\,|\,p^{(r)}(a)|$ with equality only if $p^{(3-r)}(a) = 0$, $r = 1, 2$. Now, with $a = x_i$, this situation must exist either for $b = x_{i+1}$ and $p = C\,|_{(x_i, x_{i+1})}$, or else for $b = x_{i-1}$ and $p = C\,|_{(x_{i-1}, x_i)}$.

This observation implies the exponential decay of the fundamental functions of cubic spline interpolation at knots.

THEOREM 1 [4]. *With* $\Delta := (x_i)_0^{N+1}$ *so that* $0 = x_0 < \cdots < x_{N+1} = 1$, *let* $C_i = C_{i,\Delta}$ *be the cubic spline on* $[0, 1]$ *with simple interior knots* $x_1, \ldots, x_N$ *that satisfies*

$$C_i(x_j) = \delta_{i,j}, \qquad j = 1,\ldots, N \tag{2}$$

together with the homogeneous end conditions

$$C_i(0) = C_i'(0) = C_i(1) = C_i'(1) = 0, \tag{3a}$$

$i = 1,\ldots, N$. *Then*

$$\max_{x \notin (x_{i-j}, x_{i+j})} |\,C_i(x)| \leqslant K(m_\Delta/2)^j$$

with K *some absolute constant and*

$$m_\Delta := \max_{|r-s|=1} \Delta x_r/\Delta x_s \tag{4}$$

the local mesh ratio. Also

$$\max_{x \notin (x_{i-j}, x_{i+j})} |\,C_i(x)| \leqslant K' 2^{-j}$$

with K' *a constant which can be bounded in terms of the global mesh ratio*

$$M_\Delta := \max_{r,s} \Delta x_r/\Delta x_s\,. \tag{5}$$

Without going into details (see [4]), note that the boundary conditions (3a) insure that

$$C_i'(x_1)\, C_i''(x_1) > 0, \qquad i = 2,\ldots, N$$
$$C_i'(x_N)\, C_i''(x_N) < 0, \qquad i = 1,\ldots, N-1. \tag{6}$$

Hence $C_i'(x_j)$ grows exponentially from the boundary toward x_i, the only knot at which C_i does not vanish but at which these two *nullsplines* $C_i\,|_{x<x_i}$ and $C_i\,|_{x>x_i}$ must join smoothly. Since $C_i(x_i) = 1$, this implies the bounds

$$2^j\,|\,C_i'(x_{i\pm j})| < |\,C_i'(x_{i\pm 1})| \leqslant K_{m_\Delta}/|\,x_i - x_{i\pm 1}\,|$$

hence

$$\max_{x_{i+j}\leqslant x\leqslant x_{i+j+1}} |\,C_i(x)| \leqslant (\Delta x_{i+j}/\Delta x_i)\, K_{m_\Delta} 2^{-j}$$

and a corresponding bound for $|\,C_i(x)|$ on (x_{i-j-1}, x_{i-j}).

Remark. The term *cardinal spline* was introduced in [4] to denote these functions C_i, thus stressing their kinship to Whittaker's Cardinal Function $(\sin nx)/nx$ to which these functions converge as the degree is increased to infinity, provided Δ is chosen appropriately uniform. Since the publication of [4], Schoenberg chose to call cardinal spline any spline function defined on the real line with knots at the (half) integers. For this reason, I refrain here from using the term "cardinal", and use the term "fundamental" instead (but retain the letter C).

Note that Theorem 1 is easily extended to end conditions other than (3a). Thus, (6) is implied by

$$C_i(0) = C_i''(0) = C_i(1) = C_i''(1) = 0 \tag{3b}$$

important when second derivatives are prescribed or for *natural* cubic spline interpolation, or by

$$C_i(0) = C_i(\Delta x_0/2) = C_i(1 - \Delta x_N/2) = C_i(1) = 0 \tag{3c}$$

important for cubic spline interpolation without derivatives. The case of periodic boundary conditions

$$C_i(0) = C_i(1) = C_i'(1) - C_i'(0) = C_i''(1) - C_i''(0) = 0 \tag{3d}$$

is dealt with in Section 3.

2. The Question of the Largest Allowable Local Mesh Ratio

Take again $\Delta = (x_i)_0^{N+1}$ with

$$0 = x_0 < \cdots < x_{N+1} = 1$$

and define

$$P_\Delta f := \sum_{i=1}^{N} f(x_i)\, C_i\,,$$

the cubic spline that agrees with f at its knots $x_1, \ldots, x_N$ and satisfies appropriate end conditions, e.g., one of the four conditions (3a)–(3d) (with $P_\Delta f$ replacing C_i). P_Δ is a bounded linear projector on $C[0, 1]$ (and on even larger spaces). In considering in what sense (if at all) $P_\Delta f$ converges to a given $f \in C[0, 1]$ as

$$|\,\Delta\,| := \max_i \Delta x_i$$

goes to zero, it becomes important to bound

$$\|\,P_\Delta\,\| := \sup_{f \in C} \|\,P_\Delta f\,\|_\infty / \|\,f\,\|_\infty\,.$$

It is fairly easy to see that, even for fixed N, $\|\,P_\Delta\,\|$ may become arbitrarily large [7] unless Δ is restricted to be more or less uniform. In [17] it is proven that

$$\sup\{\|\,P_\Delta\,\| \mid N \text{ arbitrary};\ M_\Delta = \max_{i,j} \Delta x_i/\Delta x_j \leqslant M\} < \infty.$$

Further, it was shown that

$$\sup\{\|\,P_\Delta\,\| \mid N \text{ arb.},\ m_\Delta = \max_{|i-j|=1} \Delta x_i/\Delta x_j \leqslant m\} < \infty$$

provided

$$\begin{aligned} m &< \sqrt{2} && [17] \\ &< 2 && [8] \\ &< 1 + \sqrt{2} && [9] \\ &< 2.439+ && [16]. \end{aligned} \tag{7}$$

All these results assumed the periodic end conditions (3d). Similar results for end conditions (3a)–(3b) can be found in [15].

Since

$$\| P_\Delta \| = \left\| \sum_{i=1}^{N} | C_i | \right\|_\infty , \tag{8}$$

all of these results except those of [9] and of [15, 16] could have been obtained directly from the exponential decay of the C_i's as proven in [4].

Marsden [16] also shows that [for conditions (3a, b, d)],

$$\sup\{\| P_\Delta \| \mid N \text{ arb.}, m_\Delta \leqslant m\} = \infty$$

if $m > (3 + \sqrt{5})/2 = 2.618...$. Since $\| P_\Delta \|$ is clearly a continuous function of $x_1 ,..., x_N$, the supremum must also be infinite when $m = (3 + \sqrt{5})/2$. As it turns out, the existing gap between 2.439+ and 2.618+ can be filled by a careful consideration of cubic nullsplines in the manner of [4], thus terminating the iteration (7).

THEOREM 2. *For every $m < m^* := (3 + \sqrt{5})/2$, there exists $\alpha = \alpha_m \in [0, 1)$ and a constant $K = K_m$ so that for every $\Delta = (x_i)_0^{N+1}$ with*

$$0 = x_0 < \cdots < x_{N+1} = 1 \qquad \textit{and} \qquad m_\Delta \leqslant m$$

the fundamental cubic splines $C_i = C_{i,\Delta}$ satisfy

$$\max_{x \notin (x_{i-j}, x_{i+j})} | C_i(x)| \leqslant K_m(\alpha_m)^j. \tag{9}$$

Hence

$$\sup\{\| P_\Delta \| \mid N \text{ arb.}, m_\Delta \leqslant m\} < \infty \qquad \textit{iff} \quad m < m^* = (3 + \sqrt{5})/2.$$

Proof. If the cubic polynomial p vanishes at 0 and at $h > 0$, and if $p'p'' \geqslant 0$ at 0, then

$$r := hp''(0)/(2p'(0)) \geqslant 0$$

and

$$\max_{0 \leqslant x \leqslant h} | p(x)| = h \mid p'(0)| \, F(r)$$

with

$$F(r) := \left(3r + 2 \, \frac{3 + 3r + r^2}{1 + r} \, [r + (3 + 3r + r^2)^{1/2}]\right) \Big/ (27(1 + r)), \tag{10}$$

as one verifies. One checks that $F(r)$ strictly increases with r. Further, by (1),

$$p'(h) = 2p'(0) + hp''(0)/2 = p'(0)(2 + r) \tag{11}$$

and

$$hp''(h)/2 = 3p'(0) + 2hp''(0)/2 = p'(0)(3 + 2r),$$

hence

$$hp''(h)/(2p'(h)) = (3 + 2r)/(2 + r),$$

which strictly increases from 3/2 to 2 as r goes from 0 to ∞.

If now C is a cubic spline which vanishes at its simple knots

$$\cdots < x_i < x_{i+1} < x_{i+2} < \cdots$$

and if

$$r_i := \Delta x_i \, C''(x_i)/(2C'(x_i))$$

is non-negative, then it follows that

$$r_{i+1} = \Delta x_{i+1} \frac{C''(x_{i+1})}{2C'(x_{i+1})} = \frac{(3 + 2r_i)/(2 + r_i)}{m_{i+1}}$$

is positive, with

$$m_{i+1} := \Delta x_i / \Delta x_{i+1} .$$

Hence, for arbitrary $r_i \in [0, \infty]$, $r_{i+1} \geqslant 3/(2m_\Delta)$, and, in general, with $\rho_0 = 0$, we find that for $j = 1, 2,...$

$$r_{i+j} \geqslant \rho_j := \frac{(3 + 2\rho_{j-1})/(2 + \rho_{j-1})}{m_\Delta} .$$

The sequence $(\rho_j)_0^\infty$ is strictly increasing and converges to

$$\rho = \rho(m_\Delta) := [1 - m_\Delta + (m_\Delta^2 + m_\Delta + 1)^{1/2}]/m_\Delta ,$$

a strictly decreasing function of m_Δ.

Further, from (10), (11), and (12),

$$\begin{aligned} \alpha_{i+1} &:= \max_{x_i \leqslant x \leqslant x_{i+1}} | C(x)| / \max_{x_{i+1} \leqslant x \leqslant x_{i+2}} | C(x)| \\ &= \alpha(r_i , m_{i+1}) \end{aligned}$$

with

$$\alpha(r, m) := mF(r)/\{(2 + r)\,F((3 + 2r)/[(2 + r)m])\}$$

positive on r, $m > 0$ and satisfying

$$\partial\alpha/\partial r < 0, \qquad \partial\alpha/\partial m > 0 \tag{13}$$

there. Since

$$\rho(m) = \{[3 + 2\rho(m)]/[2 + \rho(m)]\}/m, \tag{14}$$

the equation

$$\alpha[\rho(m), m] = 1$$

is equivalent to $m/[2 + \rho(m)] = 1$, or $\rho(m) = m - 2$. From this and (14),

$$m^3 - 2m^2 - 2m + 1 = 0$$

which has the solutions -1, m^*, and $1/m^*$, with

$$m^* = (3 + \sqrt{5})/2$$

i.e., the square of the golden ratio, as already remarked upon by Marsden [16].

If now $m_\Delta \leqslant m < m^*$, then there exists $\epsilon > 0$ and $j_0 = j_0(m)$ so that, for all $j \geqslant j_0$,

$$r_{i+j} \geqslant \rho(m) - \epsilon \geqslant \rho(m^*) + \epsilon.$$

Hence, for all $j \geqslant j_0$,

$$\begin{aligned}\alpha_{i+j+1} = \alpha(r_{i+j}, m_{i+j+1}) &\leqslant \alpha(\rho(m^*) + \epsilon, m)\\ &< \alpha(\rho(m^*), m^*) = 1,\end{aligned}$$

using (13) to establish the two inequalities.

The exponential decay of the $C_{i,\Delta}$ now follows as in the proof in [4] for Theorem 1 above. Specifically, one obtains (9) with $\alpha_m = \alpha(\rho(m^*) + \epsilon, m)$ and K_m adjusted to cover the possible (but bounded) increase in $|C_i(x)|$ in the first and last $j_0 = j_0(m)$ intervals.

There are certainly sequences (Δ) of meshes with $m_\Delta > 2.62...$, all Δ, but for which $(\| P_\Delta \|)$ is nevertheless bounded [9]. A more elaborate argument along the above lines but taking into consideration the relationship between three or more pieces of a nullspline would reveal such sequences and many others. A limit of sorts to further weakening of the assumptions on (Δ) is set by the observation in [7] that $\sup_\Delta m_\Delta$ has to be finite for $\sup_\Delta \| P_\Delta \|$ to be finite.

3. The Question of Local Convergence

I continue to denote by P_Δ the linear projector given by the rule

$$P_\Delta f = \sum_{i=1}^{N} f(x_i)\, C_{i,\Delta}$$

with $C_{i,\Delta}$ the fundamental splines associated with the sequence $\Delta = (x_i)_0^{N+1}$, and satisfying appropriate end conditions, e.g., one of (3a)–(3d). It is convenient to denote by $P_0 f$ the unique cubic polynomial for which $f - P_0 f$ satisfies these same end conditions.

$$Q_\Delta := P_0 + P_\Delta(1 - P_0)$$

is then the linear projector defined on sufficiently smooth f, with range the cubic splines on $[0, 1]$ with simple knots at $x_1, \dots, x_N$.

While it is well known (e.g., as a consequence of [20]) that, for $f \in L_\infty^{(4)}[0, 1] := \{f \in C^{(3)}[0, 1] \mid f^{(3)}$ abs. continuous, $f^{(4)} \in L_\infty[0, 1]\}$,

$$\| f - Q_\Delta f \|_\infty \leqslant \text{const} \mid \Delta \mid^4 \| f^{(4)} \|_\infty , \tag{15}$$

two questions of local convergence seem to continue to attract attention.

(i) If f is smooth enough so that $Q_\Delta f$ is defined but otherwise only $f \in L_\infty^{(4)}[\alpha, \beta]$ for some subinterval $[\alpha, \beta]$ of $(0, 1)$, is it still true that

$$|(f - Q_\Delta f)(t)| \leqslant \text{const} \mid \Delta \mid^4, \qquad \text{for} \quad t \in [\alpha, \beta]$$

with const depending on f and possibly on t [2, 10]?

(ii) Although $P_\Delta f$ may be bounded away from f at 0 and 1 independently of Δ, is it nevertheless true that

$$|(f - P_\Delta f)(t)| \leqslant \text{const} \mid \Delta \mid^4, \qquad \text{for} \quad t \in (0, 1)$$

if $f \in L_\infty^{(4)}[0, 1]$, with const depending on f and possibly on t [3, 11–14]?

Questions of this nature can all be answered in the affirmative using such results as Theorems 1 and 2, provided attention is restricted to partitions Δ for which the corresponding nullsplines, and therefore the $C_{i,\Delta}$, decay exponentially.

Take the second question first. Since Q_Δ can also be written as

$$Q_\Delta = P_\Delta + (1 - P_\Delta) P_0 ,$$

it is sufficient to show that, for any fixed cubic polynomial p (e.g., for $p = P_0 f$),

$$\max_{x \in (\epsilon, 1-\epsilon)} |(1 - P_\Delta) p(x)| \leqslant \text{const} \mid \Delta \mid^4$$

with const depending on $\epsilon > 0$ and on p. For this, observe that

$$s_\Delta := (1 - P_\Delta) p$$

is a cubic nullspline, i.e., a cubic spline which vanishes at its interior knots $x_1, \ldots, x_N$, and consider s_Δ at x_i for some $i \in (1, N)$. Take first the case that $s_\Delta' s_\Delta''$ is non-negative at x_i. Then

$$\max_{x_i \leqslant x \leqslant x_{i+1}} | s_\Delta(x)| \leqslant \text{const}\ \alpha^j \max_{x_{i+j} \leqslant x \leqslant x_{i+j+1}} | s_\Delta(x)|, \qquad j = 1, 2, \ldots$$

for some $\alpha \in [0, 1)$ with const bounded either in terms of M_Δ or else m_Δ (provided $m_\Delta < m^*$). Also,

$$\max_{x_{N-1} \leqslant x \leqslant x_N} | s_\Delta(x)| \leqslant \Delta x_{N-1} \mid s_\Delta'(x_N)|.$$

But $| s_\Delta'(x_N)|$ can be bounded in terms of p and $1/\Delta x_N$ since $s(x_N) = 0$ and $s_\Delta'(x_N)\, s_\Delta''(x_N) > 0$. To be specific, the end conditions (3a) imply that

$$s_\Delta(1) = p(1), \qquad s_\Delta'(1) = p'(1),$$

hence

$$| s_\Delta'(x_N)| \leqslant (3 \mid p(1)|/\Delta x_N + | p'(1)|)/4.$$

For the end conditions (3b), $s_\Delta(1) = p(1)$, $s_\Delta''(1) = p''(1)$, hence

$$| s_\Delta'(x_N)| \leqslant | p(1)|/\Delta x_N + \Delta x_N \mid p''(1)|/6.$$

For the end conditions (3c), $s_\Delta(1) = p(1)$, $s_\Delta(1 - \Delta x_N/2) = p(1 - \Delta x_N/2)$, so

$$| s_\Delta'(x_N)| \leqslant | p(1) - p(1 - \Delta x_N/2)|/(3 \Delta x_N).$$

Finally, for the periodic end conditions (3d), I have to assume, in addition, that p is 1-periodic. Then p is a constant and $s_\Delta = (1 - P_\Delta)p$ is simply the periodic cubic spline satisfying

$$s_\Delta(x_j) = p(0)\,\delta_{0j}, \qquad \text{all } j,$$

with $x_j = x_{N+1+j}$, all j; i.e., s_Δ is the multiple of a fundamental spline. Hence, if $s_\Delta' s_\Delta''$ is nonpositive at $x_1 = x_{N+2}$, [4] supplies the bound for $|s_\Delta'(x_N)|$ in terms of $|p(0)|$ and $1/\Delta x_N$ as it does for the nonperiodic fundamental spline. Otherwise, $s_\Delta' s_\Delta''$ is positive at $x_1 = x_{N+2}$. But then $s_\Delta' s_\Delta''$ is positive at $x_2, \ldots, x_N$ and

$$|s_\Delta'(x_N)| = |s_\Delta'(x_{-1})| > 2^{N-1}\,|s_\Delta'(x_1)|.$$

Further, on subtracting

$$p(0)\left\{\left(\frac{(x - x_{-1})_+}{x_0 - x_{-1}}\right)^3 - \left(\frac{(x_1 - x_{-1})(x - x_0)_+}{(x_0 - x_{-1})(x_1 - x_0)}\right)^3\right\}$$

from s_Δ, I obtain a cubic spline $\hat{s}$ which vanishes at x_{-1}, x_0, x_1 while $\hat{s}'\hat{s}'' = s_\Delta' s_\Delta'' > 0$ at x_{-1}. Hence

$$4\,|s_\Delta'(x_{-1})| < |\hat{s}'(x_1)| = |s_\Delta'(x_1) + 3p(0)(x_1 - x_{-1})^2/(\Delta x_0(\Delta x_{-1})^2)|$$

or

$$|s_\Delta'(x_N)| = |s_\Delta'(x_{-1})| < 3\,|p(0)|\,[(1 + m_\Delta)^2/(4 - 2^{1-N})]/\Delta x_N,$$

the required bound.

It follows that if $s_\Delta = (1 - P_\Delta)p$ is determined by one of the four side conditions (3a)–(3d), and if $s_\Delta' s_\Delta'' \geqslant 0$ at x_i, then

$$\max_{x_i \leqslant x \leqslant x_{i+1}} |s_\Delta(x)| \leqslant \text{const}\ \alpha^{N-i-1}$$

for some $\alpha \in [0, 1)$ and some const depending on m_Δ and p. Of course, if not $s_\Delta' s_\Delta'' \geqslant 0$ at x_i, then $s_\Delta' s_\Delta''$ must increase exponentially toward the left and the analogous argument now produces

$$\max_{x_i \leqslant x \leqslant x_{i+1}} |s_\Delta(x)| \leqslant \text{const}\ \alpha^{i-1}.$$

THEOREM 3. *For given $\Delta = (x_i)_0^{N+1}$ with $0 = x_0 < \cdots < x_{N+1} = 1$, let $P_\Delta f = \sum_{i=1}^{N} f(x_i) C_{i,\Delta}$ with $C_{i,\Delta}$ the fundamental cubic splines on Δ satisfying one of the end conditions* (3a)–(3d). *Then, for every cubic poly-*

nomial p [1-periodic in case of end conditions (3d)], $(1 - P_\Delta)p$ *converges to zero exponentially uniformly on every closed subinterval of* (0, 1) *as* $|\Delta| \to 0$ *provided* M_Δ *stays bounded or, at least,* m_Δ *stays below* $(3 + \sqrt{5})/2$.

The first question is essentially settled by the following lemma, for which I am unable to supply a reference, although it is part of the technical equipment of many an approximator.

LEMMA. *For given* $\Delta = (x_i)_0^{N+1}$, *let* R_Δ *be defined by the rule*

$$R_\Delta f := \sum_{i=1}^{N} f(x_i)\, c_i$$

and assume that (i) R_Δ *reproduces polynomials of degree* $< k$, *i.e.*, $R_\Delta p = p$ *for all polynomials of degree* $< k$, *and* (ii) *the* c_i*'s decay exponentially, i.e., for some* const_c *and some* $\alpha_c \in [0, 1)$,

$$\max_{x \notin (x_{i-j}, x_{i+j})} |c_i(x)| \leqslant \text{const}_c (\alpha_c)^j, \qquad \text{all} \quad i, j.$$

If f is bounded on [0, 1] *and k times continuously differentiable in a neighborhood of* $\hat{x} \in (0, 1)$, *then there exists a number* const_f *such that*

$$\Big| f(x) - \sum_{j<k} f^{(j)}(\hat{x})(x - \hat{x})^j / j! \Big| \leqslant \text{const}_f\, |x - \hat{x}|^k, \qquad \text{all} \quad x \in [0, 1],$$

hence then

$$|f(\hat{x}) - (R_\Delta f)(\hat{x})| \leqslant \Big(\text{const}_c\, \text{const}_f \sum_{r=-\infty}^{\infty} |r|^k\, \alpha_c^{|r|-1}\Big)\, |\Delta|^k.$$

Proof. Abbreviate $\sum_{j<k} f^{(j)}(\hat{x})(x - \hat{x})^j / j!$ to p. Then as $R_\Delta p = p$ and $p(\hat{x}) = f(\hat{x})$,

$$f(\hat{x}) - (R_\Delta f)(\hat{x}) = -(R_\Delta(f - p))(\hat{x}).$$

Hence, with $x_j \leqslant \hat{x} \leqslant x_{j+1}$,

$$\begin{aligned} |(f - R_\Delta f)(\hat{x})| &= \Big| \sum_i (f - p)(x_i)\, c_i(\hat{x}) \Big| \\ &\leqslant \text{const}_c\, \text{const}_f \Big(\sum_{i \leqslant j} (\hat{x} - x_i)^k\, \alpha_c^{j-i} + \sum_{i > j} (x_i - \hat{x})^k\, \alpha_c^{i-j-1} \Big) \\ &\leqslant \text{const}_c\, \text{const}_f \sum_r |r|^k\, \alpha_c^{|r|-1}\, |\Delta|^k. \end{aligned}$$

Remark. The lemma can be improved in various ways.

(i) If f has a bounded kth derivative in the interval $[\hat{x} - \epsilon, \hat{x} + \epsilon]$ for some positive ϵ, then it is possible to replace the global mesh length $| \Delta |$ by the local mesh length

$$h := \max\{\Delta x_i \mid | x_i - \hat{x} | \leqslant \epsilon\}$$

and still get

$$| f(\hat{x}) - (R_\Delta f)(\hat{x})| \leqslant \text{const } h^k + \text{const } \alpha_c^{\epsilon/h},$$

the last term being, of course, $o(h^n)$ for all n.

(ii) The c_i's need only decay polynomially of sufficiently high degree, i.e., it is sufficient to have

$$\max_{x \notin (x_{i-j}, x_{i+j})} | c_i(x)| \leqslant \text{const}_c \, | i - j |^{k+1+\epsilon}, \qquad \text{all} \quad i, j$$

for some positive ϵ.

(iii) The lemma applies verbatim to linear maps of the form

$$R_\Delta f := \sum_{i=1}^{N} (\lambda_i f) \, c_i$$

provided $\sup_i \| \lambda_i \|$ is bounded, say no bigger than 1, and the λ_i's are *local* linear functionals in the sense that, for some fixed r and all i, λ_i has its support in (x_{i-r}, x_{i+r}).

It follows that most of the local convergence results for cubic spline interpolation now in the literature could have been deduced directly from [4]. Here is a sample result.

THEOREM 4. *For* $\Delta = (x_i)_0^{N+1}$ *with* $0 = x_0 < \cdots < x_{N+1}$, *let*

$$P_\Delta f = \sum_{i=1}^{N} f(x_i) \, C_{i,\Delta}$$

denote the cubic spline interpolant, as at the beginning of this section, with $C_i = C_{i,\Delta}$ *satisfying one of the three end conditions* (3a)–(3c), *or the end condition* (3d) *if f is periodic. If f is bounded on* [0, 1] *and, for some integer* $k \in [1, 4]$, $f^{(k-1)}$ *exists and is continuous at* $\hat{x} \in (0, 1)$, *then*

$$| f(\hat{x}) - (P_\Delta f)(\hat{x})| \leqslant \text{const } | \Delta |^{k-1} \, \omega(| \Delta |)$$

with ω the modulus of continuity of $f^{(k-1)}$ at $\hat{x}$, and const *depending on the bound on f, and on M_Δ and/or m_Δ (provided $m_\Delta < 2.618...$).*

4. Cubic Spline Interpolation at Infinitely Many Knots

Recently, Schoenberg raised the following question:

Given a strictly increasing sequence $\Delta = (x_i)_{-\infty}^{\infty}$ and a corresponding bounded biinfinite sequence $(y_i)_{-\infty}^{\infty}$, does there exist a *bounded* cubic spline s with knot sequence Δ for which

$$s(x_i) = y_i, \qquad \text{all } i? \tag{16}$$

If so, how many?

The earlier considerations of nullsplines allow the following partial answers:

(i) If $\lim_{i\to\pm\infty} x_i = \pm\infty$, then there exists at most one solution to the interpolation problem. For, if both s and $\hat{s}$ are solutions, then their difference $d := s - \hat{s}$ is a bounded nullspline. If now $d \neq 0$, then we may assume without loss that d' and d'' are both positive at x_0, which then implies that $(-)^i\, d'(x_i) > 2^i d'(x_0)$, $i = 1, 2,...$. Since, for a cubic polynomial p vanishing at 0 and h,

$$p(h/2) = h(p'(0) - p'(h))/8,$$

it follows that

$$| d((x_i + x_{i+1})/2)| = (-)^i\, d((x_i + x_{i+1})/2) > 3\Delta x_i 2^i d'(x_0)/8, \qquad i = 1, 2,...$$

hence, the boundedness of d implies that the sequence $(2^i \Delta x_i)_0^\infty$ is bounded. But then

$$\lim_{i\to\infty} x_i = x_0 + \sum_{i=0}^{\infty} \Delta x_i \leqslant x_0 + M \sum_{i=0}^{\infty} 2^{-i} < \infty,$$

a contradiction.

(ii) Let $\lim_{i\to\pm\infty} x_i = \pm\infty$. If the interpolation problem has a solution $s_{\mathbf{y}}$ (necessarily unique) for every bounded sequence $\mathbf{y} = (y_i)$, then

$$\sup_{\mathbf{y}} \| s_{\mathbf{y}} \|_\infty / \| \mathbf{y} \|_\infty < \infty.$$

For, the linear space $S_{4,\Delta}$ of all bounded cubic splines with knot sequence Δ is known to be a Banach space with respect to the sup-norm [5], as is the space $l_\infty(\mathbb{Z})$ of bounded biinfinite sequences. We just proved that

$$R_\Delta\colon S_{4,\Delta} \to l_\infty(\mathbb{Z})\colon s \mapsto (s(x_i))$$

is one–one. If the interpolation problem has a solution for every $\mathbf{y} \in l_\infty(\mathbb{Z})$, then R_Δ is also onto. But then, its linear inverse, $\mathbf{y} \mapsto s_\mathbf{y}$, must be bounded, by the Open Mapping Theorem.

(iii) Let $\lim_{i\to\pm\infty} x_i = \pm\infty$. If $M_\Delta := \sup_{i,j} \Delta x_i/\Delta x_j < \infty$, or if $m_\Delta := \sup_{|i-j|=1} \Delta x_i/\Delta x_j < (3 + \sqrt{5})/2$, then there exists exactly one bounded cubic spline s in $S_{4,\Delta}$ satisfying (16) for given bounded (y_i). This spline can be written as

$$\sum_i y_i C_{i,\Delta}$$

with the sum converging uniformly on compact sets, where $C_{i,\Delta}$ are the unique bounded cubic fundamental splines on Δ, i.e., $C_{i,\Delta} \in S_{4,\Delta}$ and $C_{i,\Delta}(x_j) = \delta_{i,j}$, all i, j.

For this, it is certainly sufficient to ascertain that, under the given conditions, there exists a bounded cubic spline C_0 with knot sequence Δ so that $C_0(x_i) = \delta_{0i}$, all i, and that this C_0 decays exponentially, i.e.,

$$\sup_{x\notin(x_{-j},x_j)} |C_0(x)| \leqslant K\alpha^j, \qquad j = 1, 2, \ldots$$

for some $\alpha \in [0, 1)$ and some K, both depending only on M_Δ or m_Δ.

Let B_i denote a B-spline of order two with knots x_{i-1}, x_i, x_{i+1},

$$\begin{aligned} B_i(x) &:= (x - x_{i-1})/\Delta x_{i-1}, && x_{i-1} \leqslant x \leqslant x_i \\ &= (x_{i+1} - x)/\Delta x_i, && x_i \leqslant x \leqslant x_{i+1} \\ &:= 0, && x \notin (x_{i-1}, x_{i+1}). \end{aligned}$$

Every linear spline with knot sequence Δ can be written uniquely as $\sum \alpha_i B_i$, with α_i the value of the spline at x_i, all i, the sum being taken pointwise. According to [6], there exists a positive constant D_2 independent of Δ so that

$$D_2^{-1} \|(w_i\alpha_i)\|_2 \leqslant \Big|\sum_i \alpha_i B_i\Big|_2 \leqslant \|(w_i\alpha_i)\|_2 \tag{17}$$

with $w_i := ((x_{i+1} - x_{i-1})/2)^{1/2}$, all i. This implies that $(w_i^{-1}B_i)$ is a

Schauder basis for the closed linear subspace of $L_2(\mathbb{R})$ spanned by the B_i's. In particular, for every choice of α_{-1}, α_0, α_1, the function $\alpha_{-1}B_{-1} + \alpha_0 B_0 + \alpha_1 B_1$ has a best L_2-approximation in the span of $(B_i)_{i\neq -1,0,1}$. The error in this best approximation can be written

$$e = \sum_i \alpha_i B_i$$

for an appropriate (α_i) with

$$\|(w_i\alpha_i)\|_2 \leqslant D_2 \| e \|_2 \leqslant D_2 \| \alpha_{-1}B_{-1} + \alpha_0 B_0 + \alpha_1 B_1 \|_2 . \tag{18}$$

Let now C be the cubic spline with knot sequence Δ that vanishes at x_{-1} and x_1 and whose second derivative equals e. Then $C'' = e$ is orthogonal to B_i for all $i \neq -1, 0, 1$, therefore

$$\Delta C(x_i)\, \Delta x_i - \Delta C(x_{i-1})\, \Delta x_{i-1} = \int B_i(x)\, C''(x)\, dx/2 = 0, \qquad \text{all} \quad i \neq -1, 0, 1. \tag{19}$$

Choose α_{-1}, α_0, α_1 so that $C(x_0) = 1$, $C(x_{-2}) = C(x_2) = 0$ (as can be done in exactly one way). Since also $C(x_{-1}) = C(x_1) = 0$, (19) implies that then $C(x_i) = 0$ for all $i \neq 0$. C is therefore the desired fundamental cubic spline C_0. In particular, $C\,|_{x\geqslant x_1}$ is then a cubic nullspline. If now $C'C''$ were non-negative at some $x_i \geqslant x_1$, then it would follow that, for some positive i and all $j > i$,

$$| \alpha_j | = | C''(x_j)| > 2^{j-i}\, | C''(x_i)| > 0.$$

On the other hand, by (18), $\sum_j (x_{j+1} - x_{j-1})\, | \alpha_j |^2/2 = \|(w_j\alpha_j)\|_2^2 < \infty$, hence

$$0 \leqslant (2^{j-i}\, | C''(x_i))^2\, (x_{j+1} - x_{j-1})/2 < | \alpha_j |^2\, (x_{j+1} - x_{j-1})/2 \xrightarrow[j\to\infty]{} 0$$

which would imply that $\lim_{j\to\infty} x_j < \infty$, a contradiction. Consequently, $C'C''$ is negative at all $x_i > x_0$, and therefore, as in the arguments for Theorems 1 and 2, $\max_{x\geqslant x_j} | C(x)| \leqslant K\alpha^j$ for some $\alpha \in [0, 1)$. The exponential decay for $x \to -\infty$ is proved analogously.

5. Higher Order Nullsplines

The material presented in this paper indicates that higher order spline interpolation could be analyzed once the corresponding nullsplines are understood.

One establishes easily that a polynomial p of degree $< k$ which vanishes at 0 and at h satisfies

$$p^{(i)}(h)/i! = -\sum_{j=0}^{k-2} \left[\binom{k-1}{i} - \binom{j}{i}\right] h^{j-i} p^{(j)}(0)/j!.$$

Hence, if C is a spline of degree $< k$ that vanishes at its simple knots

$$\cdots < x_{i-1} < x_i < x_{i+1} < \cdots$$

then

$$\mathbf{C}_{i+1} = -A_k(\Delta x_i)\, \mathbf{C}_i$$

with

$$\mathbf{C}_j := (C'(x_j), C''(x_j)/2, \ldots, C^{(k-2)}(x_j)/(k-2)!),$$

$$A_k(h) := \operatorname{diag}(1, h^{-1}, \ldots, h^{-k+3})\, A_k(1)\, \operatorname{diag}(1, h, \ldots, h^{k-3}),$$

and

$$A_k(1) := \left(\binom{k-1}{i} - \binom{j}{i}\right)_{i,j=1}^{k-2}.$$

From Schoenberg's work (see, e.g., [19]) and earlier work going back to Collatz and Quade [18] and before, $-A_k(1)$ is known to be diagonalizable with its $k-2$ eigenvalues the roots of the appropriate Euler–Frobenius polynomial. In particular, these roots are simple and negative,

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{k-2} < 0$$

and paired so that $\lambda_i \lambda_{k-1-i} = 1$, all i. Further, $A_k^{-1}(h) = A_k(-h)$, and $A_k(1)$ is totally positive.

In fact, there seems to be enough structure here to allow the conclusion that, for such a nullspline C and for even k, $\mathbf{C}_j$ increases exponentially with a factor of $1/\alpha \geqslant -\min_i(\lambda_i + \lambda_{k-1-i})/2$ either for increasing j or else for decreasing j. But, I have been unable to prove this so far.

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