ON THE ERROR IN SURFACE SPLINE INTERPOLATION OF A COMPACTLY SUPPORTED FUNCTION

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Abstract. We show that the $L_p(\Omega)$-norm of the error in surface spline interpolation of a compactly supported function in the Sobolev space W_2^{2m} decays like $O(\delta^{\gamma_p+m})$ where $\gamma_p := \min\{m, m+d/p-d/2\}$ and m is a parameter related to the smoothness of the surface spline. In case $1 \le p \le 2$, the achieved rate of $O(\delta^{2m})$ matches that of the error when the domain is all of $\mathbb{R}^d$ and the interpolation points form an infinite grid.

1. Introduction

Let Ξ be a finite set of scattered points in $\mathbb{R}^d$ and let $f : \mathbb{R}^d \to \mathbb{C}$ be a function which is known only on Ξ. A problem of practical importance is that of constructing a smooth function which interpolates the known data $f_{|\Xi}$ and provides a good approximation to f on any domain which is near Ξ. There are a number of methods which are currently being investigated in the literature for which the reader is referred to the surveys [10], [6], and [18]. In this paper we restrict ourselves to the method known as surface spline interpolation which we now describe.

Let m be an integer greater than $d/2$, and let H be the set of continuous functions $s : \mathbb{R}^d \to \mathbb{C}$ all of whose derivatives of total order m are square integrable. Let $|||\cdot|||$ be the semi-norm defined on H by

$$|||s||| := \||\cdot|^m \widehat{s}\|_{L_2(\mathbb{R}^d)},$$

where $\widehat{s}$ denotes the Fourier transform of s given formally by $\widehat{s}(w) := \int_{\mathbb{R}^d} s(x)e^{-iw\cdot x}\,dx$, $w \in \mathbb{R}^d$. Duchon [7] has shown that if $f \in H$ and Ξ is a bounded subset of $\mathbb{R}^d$ satisfying

$$\forall q \in \Pi_{m-1}(q_{|\Xi} = 0 \Rightarrow q = 0), \tag{1.1}$$

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where $\Pi_k := \{\text{polynomials of total degree } \le k\}$, then there exists a unique $s \in H$ which minimizes $|||s|||$ subject to the interpolation conditions $s_{|_\Xi} = f_{|_\Xi}$. The function s is called the *surface spline interpolant to f at Ξ* and will be denoted by $T_\Xi f$. When Ξ contains only finitely many points, Duchon further shows that $T_\Xi f$ is the unique function in $S(\phi;\Xi)$ which interpolates f at Ξ. Here $\phi : \mathbb{R}^d \to \mathbb{R}$ is the radially symmetric function given by

$$\phi := \begin{cases} |\cdot|^{2m-d} & \text{if } d \text{ is odd} \\ |\cdot|^{2m-d} \log|\cdot| & \text{if } d \text{ is even,} \end{cases}$$

and $S(\phi;\Xi)$ denotes the space of all functions of the form

$$q + \sum_{\xi\in\Xi} \lambda_\xi \phi(\cdot - \xi),$$

where $q \in \Pi_{m-1}$ and the λ_ξ's satisfy[1]

$$\sum_{\xi\in\Xi} \lambda_\xi r(\xi) = 0, \quad \forall r \in \Pi_{m-1}. \tag{1.2}$$

In order to discuss the extent to which $T_\Xi f$ approximates f, let us assume that $\Omega \subset \mathbb{R}^d$ is an open bounded domain over which the error between f and $T_\Xi f$ is measured. We assume that $\Xi \subset \overline{\Omega}$ and define the 'density' of Ξ in Ω to be the number

$$\delta := \delta(\Xi;\Omega) := \sup_{x\in\Omega} \inf_{\xi\in\Xi} |x - \xi| .$$

A common means of describing the asymptotic approximation attributes of an interpolation method is via the notion of L_p-approximation orders. Surface spline interpolation in Ω is said to provide *L_p-approximation of order γ* if

$$\|f - T_\Xi f\|_{L_p(\Omega)} = O(\delta^\gamma) \quad \text{as } \delta \to 0$$

for all sufficiently smooth functions f. Duchon [8] has shown that if Ω is connected, has the cone property, and has a Lipschitz boundary, then surface spline interpolation in Ω provides L_p-approximation of order at least

$$\gamma_p := \min\{m, m + d/p - d/2\}$$

for $p \in [1\,..\,\infty]$. More precisely, it was shown that for all $f \in H$ and $p \in [1\,..\,\infty]$,

(1.3)
$$\|f - T_\Xi f\|_{L_p(\Omega)} \le \text{const}(m,\Omega)\, \delta^{\gamma_p}\, |||T_\Omega f - T_\Xi f|||, \quad \text{for sufficiently small } \delta, \text{ and}$$

(1.4)
$$|||T_\Omega f - T_\Xi f||| \to 0 \text{ as } \delta \to 0.$$

[1] In case Ξ is infinite, we require additionally that only finitely many of the λ_ξ's are nonzero.

The lengthy assumptions on Ω were employed because Duchon only wanted to assume that $f \in W_2^m(\Omega)$. These assumptions assured the existence of a function in H whose restriction to Ω agreed with f. If one assumes straight off that $f \in H$, then (1.3) and (1.4) hold provided that Ω is a bounded open subset of $\mathbb{R}^d$ having the cone property. In the limiting case when the points Ξ are taken as the infinite grid $h\mathbb{Z}^d$ and Ω is taken as all of $\mathbb{R}^d$, it is known (cf. [5], [13]) that $\|f - T_\Xi f\|_{L_p(\mathbb{R}^d)} = O(h^{2m})$ for all sufficiently smooth f.

The gap between γ_p and $2m$ is rather substantial, and it has been my aim of late to narrow this gap. An upper bound on the possible L_p-approximation order of surface spline interpolation is obtained in [15] for the special case when $\Omega = B := \{x \in \mathbb{R}^d : |x| < 1\}$. It is shown that there exists a C^∞ function f such that

$$\|f - T_\Xi f\|_{L_p(\Omega)} \neq o(\delta^{m+1/p}) \quad \text{as } \delta \to 0.$$

Interestingly, what is actually proved is that $\|f - T_\Xi f\|_{L_p(B\backslash(1-h)B)} \neq o(\delta^{m+1/p})$ where $B\backslash(1-h)B$ can be interpreted as the boundary layer within Ω of depth h. Thus it appears that our inability to achieve L_p-approximation of order $2m$ is due primarily to boundary effects. This corroborates experimental evidence reported by Powell and Beatson [19]. It becomes interesting now to see if it is possible to approach L_p-approximation of order $2m$ is one changes the rules of the game so as to disabe the boundary effects. One approach is to measure the error not on all of Ω, but rather on a compact subset of Ω. Bejancu [1] has considered the case when Ω is the open unit cube $(0\,.\,.\,1)^d$ and the interpolation points are those points of the grid $h\mathbb{Z}^d$ which lie in the closed cube $[0\,.\,.\,1]^d$. He shows that if K is a compact subset of $(0\,.\,.\,1)^d$ and f is sufficiently smooth, then

$$\|f - T_\Xi f\|_{L_\infty(K)} = O(h^{2m}) \quad \text{as } h \to 0.$$

In the present work, we use an alternate means of disabling the boundary effects. We assume that f, the function being interpolated, is compactly supported within Ω. Before stating our main result (see Corollary 5.1 for a more general statement), we define the Sobolev spaces W_2^γ.

Definition 1.5. The Sobolev space W_2^γ, $\gamma \geq 0$, is the set of all $f \in L_2 := L_2(\mathbb{R}^d)$ for which

$$\|f\|_{W_2^\gamma} := \left\|(1 + |\cdot|^2)^{\gamma/2} \widehat{f}\right\|_{L_2} < \infty.$$

Theorem 1.6. *Let Ω be an open bounded subset of $\mathbb{R}^d$ having the cone property. If $\Xi \subset \overline{\Omega}$ satisfies (1.1) and $f \in W_2^{2m}$ is supported in $\overline{\Omega}$, then*

$$\|f - T_\Xi f\|_{L_p(\Omega)} \leq \text{const}(\Omega, m)\delta^{\gamma_p + m} \|f\|_{W_2^{2m}},$$

for sufficiently small $\delta := \delta(\Xi; \Omega)$.

Note that, for $p \in [1\,.\,.\,2]$, the exponent of δ is $2m$. Although $\gamma_p + m < 2m$ when $2 < p \leq \infty$, we at least have $\gamma_p + m > m + 1/p$. Our proof of Theorem 1.6 is accomplished by showing that the factor $|||T_\Omega f - T_\Xi f|||$, on the right side of (1.3), decays like $O(\delta^m)$.

For this, it suffices to show that there exists $s \in S(\phi;\Xi)$ such that $|||T_\Omega f - s||| = O(\delta^m)$. We do this by first showing, in Section 3, that there exists an $s_h \in S(\phi; h\mathbb{Z}^d)$ such that $|||T_\Omega f - s_h||| = O(\delta^m)$, where h is a multiple of δ. Then, in Section 4, we show that there exists $s \in S(\phi;\Xi)$ such that $|||s_h - s||| = O(\delta^m)$. The final result, Corollary 5.1, is then proved in Section 5.

Throughout this paper we use standard multi-index notation: $D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$. The natural numbers are denoted $\mathbb{N} := \{1,2,3,\dots\}$, and the non-negative integers are denoted $\mathbb{N}_0$. For multi-indices $\alpha \in \mathbb{N}_0^d$, we define $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_d$, while for $x \in \mathbb{R}^d$, we define $|x| := \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$. For multi-indices α, we employ the notation $()^\alpha$ to represent the monomial $x \mapsto x^\alpha$, $x \in \mathbb{R}^d$. The space of polynomials of total degree $\le k$ can then be expressed as $\Pi_k := \operatorname{span}\{()^\alpha : |\alpha| \le k\}$. For $x \in \mathbb{R}^d$, we define the complex exponential e_x by $e_x(t) := e^{ix\cdot t}$, $t \in \mathbb{R}^d$. The Fourier transform of a function f can then be expressed as $\widehat{f}(w) := \int_{\mathbb{R}^d} e_{-w}(x) f(x)\, dx$. The space of compactly supported C^∞ functions is denoted $C_c^\infty(\mathbb{R}^d)$. If μ is a distribution and g is a test function, then the application of μ to g is denoted $\langle g, \mu\rangle$. We employ the notation const to denote a generic constant in the range $(0\,..\,\infty)$ whose value may change with each occurence. An important aspect of this notation is that const depends only on its arguments if any, and otherwise depends on nothing.

2. Preliminaries

The Besov spaces, which we now define, play an essential role in our theory.

Definition 2.1. Let $A_0 := \overline{B}$, and for $k \in \mathbb{N}$, let $A_k := 2^k\overline{B}\backslash 2^{k-1}B$. The Besov space $B_{2,q}^\gamma$, $\gamma \in \mathbb{R}$, $1 \le q \le \infty$, is defined to be the set of all tempered distributions f for which

$$\|f\|_{B_{2,q}^\gamma} := \left\| k \mapsto 2^{k\gamma} \left\|\widehat{f}\right\|_{L_2(A_k)} \right\|_{\ell_q(\mathbb{N}_0)} < \infty.$$

The spaces $B_{2,q}^\gamma$ are Banach spaces; the reader is refered to [17] for a general reference.

Definition. For $\gamma \in (0\,..\,m]$, let $\mathcal{M}_\gamma$ be the set of all compactly supported distributions μ which satisfy

$$\langle q, \mu\rangle = 0 \quad \forall q \in \Pi_{m-1} \tag{2.2}$$

and $\|\mu\|_{\mathcal{M}_\gamma} < \infty$, where $\|\mu\|_{\mathcal{M}_\gamma} := \begin{cases} \|\mu\|_{B_{2,\infty}^{\gamma-m}} & \text{if } 0 < \gamma < m \\ \|\mu\|_{L_2} & \text{if } \gamma = m. \end{cases}$

The set of all $\mu \in \mathcal{M}_\gamma$ for which $\operatorname{supp}\mu \subset A$ is denoted $\mathcal{M}_\gamma(A)$.

For $\mu \in \mathcal{M}_\gamma$, we define the convolution $\phi * \mu$ by

$$(\phi * \mu)\widehat{\ } := \widehat{\phi}\widehat{\mu}.$$

Proposition 2.3. *Let $\gamma \in (0\,..\,m]$. If $\mu \in \mathcal{M}_\gamma$, $q \in \Pi_{m-1}$, and $0 < h \le 1$, then*

$$\begin{aligned} &(i) && \phi * \mu + q \in H, \\ &(ii) && \left\| |\cdot|^{-m}\,\widehat{\mu} \right\|_{L_2(\mathbb{R}^d \setminus h^{-1}B)} \le \mathrm{const}(m,\gamma) h^\gamma \|\mu\|_{\mathcal{M}_\gamma}, \qquad \textit{and} \\ &(iii) && \|\widehat{\mu}\|_{L_2(h^{-1}B)} \le \mathrm{const}(m,\gamma) h^{\gamma-m} \|\mu\|_{\mathcal{M}_\gamma}. \end{aligned}$$

Proof. The proofs of [16; Lem. 2.3, Prop. 2.4] can be adapted in a straightforward fashion to obtain (i). For (ii),(iii) we have

$$\begin{aligned} &\left\| |\cdot|^{-m}\,\widehat{\mu} \right\|_{L_2(\mathbb{R}^d \setminus h^{-1}B)} \le h^m \|\widehat{\mu}\|_{L_2} = h^m \|\mu\|_{\mathcal{M}_m}, \qquad \text{and} \\ &\|\widehat{\mu}\|_{L_2(h^{-1}B)} \le \|\widehat{\mu}\|_{L_2} = \|\mu\|_{\mathcal{M}_m} \end{aligned}$$

which proves (ii) and (iii) for the case $\gamma = m$. So assume $0 < \gamma < m$, and let l be the least integer for which $2^l > h^{-1}$. Then

$$\left\| |\cdot|^{-m}\,\widehat{\mu} \right\|_{L_2(\mathbb{R}^d \setminus h^{-1}B)} \le \sum_{k=l}^{\infty} \left\| |\cdot|^{-m}\,\widehat{\mu} \right\|_{L_2(A_k)} \le 2^m \sum_{k=l}^{\infty} 2^{-km} \|\widehat{\mu}\|_{L_2(A_k)}$$

$$\le 2^m \sum_{k=l}^{\infty} 2^{-km} 2^{k(m-\gamma)} \|\mu\|_{\mathcal{M}_\gamma} \le \mathrm{const}(m,\gamma) 2^{-l\gamma} \|\mu\|_{\mathcal{M}_\gamma} \le \mathrm{const}(m,\gamma) h^\gamma \|\mu\|_{\mathcal{M}_\gamma}, \qquad \text{and}$$

$$\|\widehat{\mu}\|_{L_2(h^{-1}B)} \le \sum_{k=0}^{l} \|\widehat{\mu}\|_{L_2(A_k)} \le \sum_{k=0}^{l} 2^{k(m-\gamma)} \|\mu\|_{\mathcal{M}_\gamma}$$

$$\le \mathrm{const}(m,\gamma) 2^{l(m-\gamma)} \|\mu\|_{\mathcal{M}_\gamma} \le \mathrm{const}(m,\gamma) h^{\gamma-m} \|\mu\|_{\mathcal{M}_\gamma}$$

which completes the proof of (ii) and (iii). □

3. The gridded surface spline $s_h(\mu)$

Let $\eta \in C_c(\mathbb{R}^d)$ and $\sigma \in C_c^\infty(\mathbb{R}^d)$ satisfy

$$\sup_{j \in \mathbb{Z}^d} |\delta_{0,j} - \widehat{\eta}(w - 2\pi j)| \le \mathrm{const}(d,m)|w|^m, \quad w \in \mathbb{R}^d \tag{3.1}$$

$$|1 - \widehat{\sigma}(w)| \le \mathrm{const}(d,m)\frac{|w|^m}{1+|w|^{3m}}, \quad w \in \mathbb{R}^d, \tag{3.2}$$

and put

$$\psi := \eta * \sigma.$$

The existence of such functions η and σ is known. For example, η can be realized as a finite linear combination of the translates of a box spline (see [3]) and σ can be realized as a finite linear combination of the translates of any function in $C_c^\infty(\mathbb{R}^d)$ having nonzero mean.

For $\mu \in \mathcal{M}_\gamma$ and $h > 0$, we define

$$s_h(\mu) := \sum_{j \in \mathbb{Z}^d} [\psi(\cdot/h) * \mu](hj)\,\phi(\cdot - hj).$$

The proof of the following result is motivated by the techniques developed in [2].

Proposition 3.3. *Let* $\gamma \in (0\,..\,m]$, $h \in (0\,..\,1]$. *If* $\mu \in \mathcal{M}_\gamma$, $q \in \Pi_{m-1}$, *and* $f := \phi * \mu + q$, *then*

$$\begin{aligned} &(i) \qquad s_h(\mu) \in S(\phi; h\mathbb{Z}^d \cap (h \operatorname{supp} \psi + \operatorname{supp} \mu)) \qquad \textit{and} \\ &(ii) \qquad |||f - s_h(\mu)||| \le \operatorname{const}(m,\gamma) h^\gamma \|\mu\|_{\mathcal{M}_\gamma}. \end{aligned}$$

Proof. Put $\mu_h := \psi(\cdot/h) * \mu$. Since $\operatorname{supp} \mu_h \subset h \operatorname{supp} \psi + \operatorname{supp} \mu$, it is clear that $s_h(\mu) \in \operatorname{span}\{\phi(\cdot - \xi) : \xi \in h\mathbb{Z}^d \cap (h \operatorname{supp} \psi + \operatorname{supp} \mu)\}$. Hence, in order to prove (i), it remains only to show that $\sum_{j \in \mathbb{Z}^d} \mu_h(hj) r(j) = 0$ for all $r \in \Pi_{m-1}$. If we put $g := \mu_h(h\cdot) r$, then we obtain from Poisson's summation formula (cf. [20], Chapter 7) that $\sum_{j\in\mathbb{Z}^d} g(j) = \sum_{j\in\mathbb{Z}^d} \widehat{g}(2\pi j)$. Now $\widehat{\mu_h} = h^d \widehat{\psi}(h\cdot)\widehat{\mu}$; hence, if $r = \sum_{|\alpha|<m} i^{-|\alpha|} a_\alpha ()^\alpha$, then

$$\widehat{g} = \sum_{|\alpha|<m} a_\alpha D^\alpha (h^{-d} \widehat{\mu_h}(\cdot/h)) = \sum_{|\alpha|<m} a_\alpha D^\alpha [\widehat{\eta}\widehat{\sigma}\widehat{\mu}(\cdot/h)].$$

Condition (3.1) ensures that $D^\alpha[\widehat{\eta}\widehat{\sigma}\widehat{\mu}(\cdot/h)] = 0$ at $2\pi j$ whenever $j \in \mathbb{Z}^d \backslash 0$ and $|\alpha| < m$. On the other hand, (2.2) ensures that $D^\alpha[\widehat{\eta}\widehat{\sigma}\widehat{\mu}(\cdot/h)] = 0$ at 0 for all $|\alpha| < m$. Hence, $\sum_{j\in\mathbb{Z}^d} \mu_h(hj) r(j) = \sum_{j\in\mathbb{Z}^d} \widehat{g}(2\pi j) = 0$ which proves (i). We turn now to (ii). For brevity, let us write s_h in place of $s_h(\mu)$. According to [11], $\widehat{\phi}$ can be identified on $\mathbb{R}^d \backslash 0$ with $c_\phi |\cdot|^{-2m}$ where c_ϕ is a nonzero constant depending only on m and d. For $w \in \mathbb{R}^d \backslash 0$, we have $\widehat{s_h}(w) = \sum_{j\in\mathbb{Z}^d} \widehat{\phi}(w) \mu_h(hj) e^{-ihj\cdot w}$. If we define $g := \mu_h(h\cdot) e_{-hw}$, then we obtain from Poisson's summation formula that $\sum_{j\in\mathbb{Z}^d} g(j) = \sum_{j\in\mathbb{Z}^d} \widehat{g}(2\pi j)$. Hence,

$$\begin{aligned} \widehat{s_h}(w) &= \widehat{\phi}(w) \sum_{j\in\mathbb{Z}^d} g(j) = \widehat{\phi}(w) \sum_{j\in\mathbb{Z}^d} \widehat{g}(2\pi j) \\ &= \widehat{\phi}(w) \sum_{j\in\mathbb{Z}^d} h^{-d} \widehat{\mu_h}(w + 2\pi j/h) = \widehat{\phi}(w) \sum_{j\in\mathbb{Z}^d} \widehat{\psi}(hw + 2\pi j) \widehat{\mu}(w + 2\pi j/h). \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{1}{|c_\phi|} |||f - s_h||| = \frac{1}{|c_\phi|} \left\| |\cdot|^m (\widehat{f} - \widehat{s_h}) \right\|_{L_2(\mathbb{R}^d \backslash 0)} \\ &= \left\| |\cdot|^{-m} [\widehat{\mu} - \sum_{j\in\mathbb{Z}^d} \widehat{\psi}(h\cdot + 2\pi j)\widehat{\mu}(\cdot + 2\pi j/h)] \right\|_{L_2} \\ &\le \left\| |\cdot|^{-m} (1 - \widehat{\psi}(h\cdot))\widehat{\mu} \right\|_{L_2} + \left\| |\cdot|^{-m} \sum_{j\in\mathbb{Z}^d\backslash 0} \widehat{\psi}(h\cdot + 2\pi j)\widehat{\mu}(\cdot + 2\pi j/h)) \right\|_{L_2} =: I + II. \end{aligned}$$

We consider first I. It follows from (3.1) and (3.2) that $\left|1 - \widehat{\psi}(w)\right| \le \operatorname{const}(d,m) \dfrac{|w|^m}{1 + |w|^m}$,

$w \in \mathbb{R}^d$. Consequently,

$$
\begin{aligned}
I^2 &= \left\| |\cdot|^{-m} (1-\widehat{\psi}(h\cdot))\widehat{\mu} \right\|^2_{L_2(h^{-1}B)} + \left\| |\cdot|^{-m} (1-\widehat{\psi}(h\cdot))\widehat{\mu} \right\|^2_{L_2(\mathbb{R}^d\setminus h^{-1}B)} \\
&\le \operatorname{const}(d,m) \left\| |\cdot|^{-m} |h\cdot|^m \widehat{\mu} \right\|^2_{L_2(h^{-1}B)} + \operatorname{const}(d,m) \left\| |\cdot|^{-m} \widehat{\mu} \right\|^2_{L_2(\mathbb{R}^d\setminus h^{-1}B)} \\
&= \operatorname{const}(d,m) h^{2m} \|\widehat{\mu}\|^2_{L_2(h^{-1}B)} + \operatorname{const}(d,m) \left\| |\cdot|^{-m} \widehat{\mu} \right\|^2_{L_2(\mathbb{R}^d\setminus h^{-1}B)} \le \operatorname{const}(d,m,\gamma) h^{2\gamma} \|\mu\|^2_{\mathcal{M}_\gamma}
\end{aligned}
$$

by Proposition 2.3 (ii), (iii). Let $C := [-\frac{1}{2} \mathinner{\ldotp\ldotp} \frac{1}{2})^d$. In order to estimate II, we employ the partition $\mathbb{R}^d = \cup_{k\in\mathbb{Z}^d} 2\pi h^{-1}(k+C)$ to write

$$
II^2 = \sum_{k\in\mathbb{Z}^d} \left\| |\cdot|^{-m} \sum_{j\in\mathbb{Z}^d\setminus 0} \widehat{\psi}(h\cdot+2\pi j)\widehat{\mu}(\cdot+2\pi j/h) \right\|^2_{L_2(2\pi h^{-1}(k+C))}.
$$

For $j \in \mathbb{Z}^d\setminus 0$ and $k \in \mathbb{Z}^d\setminus\{-j\}$, we have

$$
\begin{aligned}
&\left\| |\cdot|^{-m} \widehat{\psi}(h\cdot+2\pi j)\widehat{\mu}(\cdot+2\pi j/h) \right\|_{L_2(2\pi h^{-1}(k+C))} \\
&= \left\| |\cdot-2\pi j/h|^{-m} \widehat{\sigma}(h\cdot)\widehat{\eta}(h\cdot)\widehat{\mu} \right\|_{L_2(2\pi h^{-1}(k+j+C))} \\
&\le \operatorname{const}(d,m) \left\| \frac{|h\cdot-2\pi(k+j)|^m}{|\cdot-2\pi j/h|^m} \widehat{\sigma}(h\cdot)\widehat{\mu} \right\|_{L_2(2\pi h^{-1}(k+j+C))} \\
&\le \operatorname{const}(d,m) \left\| \frac{|h\cdot-2\pi(k+j)|^m}{|\cdot-2\pi j/h|^m} \widehat{\sigma}(h\cdot) |\cdot|^m \right\|_{L_\infty(2\pi h^{-1}(k+j+C))} \left\| |\cdot|^{-m} \widehat{\mu} \right\|_{L_2(2\pi h^{-1}(k+j+C))} \\
&\le \operatorname{const}(d,m) \|\widehat{\sigma}\|_{L_\infty(2\pi(k+j+C))} \left\| |\cdot|^m \frac{|\cdot+2\pi(k+j)|^m}{|\cdot+2\pi k|^m} \right\|_{L_\infty(2\pi C)} \left\| |\cdot|^{-m} \widehat{\mu} \right\|_{L_2(2\pi h^{-1}(k+j+C))} \\
&\le \operatorname{const}(d,m) \|\widehat{\sigma}\|_{L_\infty(2\pi(k+j+C))} \frac{|k+j|^m}{1+|k|^m} \left\| |\cdot|^{-m} \widehat{\mu} \right\|_{L_2(2\pi h^{-1}(k+j+C))}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left\| \sum_{j\in\mathbb{Z}^d\setminus\{0,-k\}} |\cdot|^{-m} \widehat{\psi}(h\cdot+2\pi j)\widehat{\mu}(\cdot+2\pi j/h) \right\|_{L_2(2\pi h^{-1}(k+C))} \\
&\le \operatorname{const}(d,m) \sum_{j\in\mathbb{Z}^d\setminus\{0,-k\}} \|\widehat{\sigma}\|_{L_\infty(2\pi(k+j+C))} \frac{|k+j|^m}{1+|k|^m} \left\| |\cdot|^{-m} \widehat{\mu} \right\|_{L_2(2\pi h^{-1}(k+j+C))} \\
&\le \frac{\operatorname{const}(d,m)}{1+|k|^m} \sqrt{\sum_{j\in\mathbb{Z}^d\setminus\{0,-k\}} \|\widehat{\sigma}\|^2_{L_\infty(2\pi(k+j+C))} |k+j|^{2m}} \sqrt{\sum_{j\in\mathbb{Z}^d\setminus\{0,-k\}} \left\| |\cdot|^{-m} \widehat{\mu} \right\|^2_{L_2(2\pi h^{-1}(k+j+C))}} \\
&\le \operatorname{const}(d,m) \frac{1}{1+|k|^m} \left\| |\cdot|^{-m} \widehat{\mu} \right\|_{L_2(\mathbb{R}^d\setminus 2\pi h^{-1}C)} \le \operatorname{const}(d,m,\gamma) \frac{h^\gamma}{1+|k|^m} \|\widehat{\mu}\|_{\mathcal{M}_\gamma}
\end{aligned}
$$

by Proposition 2.3 (ii). Now if $k \neq 0$ and $j = -k$, then

$$\begin{aligned}
&\left\| |\cdot|^{-m} \widehat{\psi}(h\cdot + 2\pi j)\widehat{\mu}(\cdot + 2\pi j/h)\right\|_{L_2(2\pi h^{-1}(k+C))} = \left\| |\cdot + 2\pi k/h|^{-m} \widehat{\psi}(h\cdot)\widehat{\mu}\right\|_{L_2(2\pi h^{-1}C)} \\
&\leq \mathrm{const}(d,m) \left\| |\cdot + 2\pi k/h|^{-m}\right\|_{L_\infty(2\pi h^{-1}C)} \|\widehat{\mu}\|_{L_2(2\pi h^{-1}C)} \\
&\leq \mathrm{const}(d,m)\frac{h^m}{1+|k|^m} \|\widehat{\mu}\|_{L_2(2\pi h^{-1}C)} \leq \mathrm{const}(d,m,\gamma)\frac{h^\gamma}{1+|k|^m} \|\widehat{\mu}\|_{\mathcal{M}_\gamma}
\end{aligned}$$

by Proposition 2.3 (iii). Therefore,

$$II^2 \leq \mathrm{const}(d,m,\gamma)(h^\gamma \|\widehat{\mu}\|_{\mathcal{M}_\gamma})^2 \sum_{k\in\mathbb{Z}^d} \frac{1}{(1+|k|^m)^2} \leq \mathrm{const}(d,m,\gamma)(h^\gamma \|\mu\|_{\mathcal{M}_\gamma})^2$$

since $m > d/2$; hence, $I + II \leq \mathrm{const}(d,m,\gamma)h^\gamma \|\mu\|_{\mathcal{M}_\gamma}$. $\square$

4. An approximation to $s_h(\mu)$ from $S(\phi;\Xi)$

Let $\mathcal{N}$ be the set $\mathcal{N} := \{\frac{1}{2m}j : j \in \mathbb{Z}^d, j_i \geq 0, \text{ and } j_1 + \cdots + j_d \leq m\}$. It is known [4] that $\mathcal{N}$ is 'correct' for interpolation in Π_m; consequently, we have the following:

Lemma 4.1. *There exists $\epsilon_1 \in (0\,..\,1/4)$ (depending only on d, m) such that if $x \in r\overline{B}$, $\#\widetilde{\mathcal{N}} = \#\mathcal{N}$ and $\delta(\widetilde{\mathcal{N}};\mathcal{N}) \leq \epsilon_1$, then there exists $\{a_\xi\}_{\xi\in\widetilde{\mathcal{N}}}$ such that*

$$\max_{\xi\in\widetilde{\mathcal{N}}} |a_\xi| \leq \mathrm{const}(d,m,r) \quad \text{and} \quad q(x) = \sum_{\xi\in\widetilde{\mathcal{N}}} a_\xi q(\xi) \quad \forall q \in \Pi_m.$$

The following is equivalent to the standard definition of the cone property.

Definition 4.2. A set $\Omega \subset \mathbb{R}^d$ is said to have the *cone property* if there exists $\epsilon_\Omega, r_\Omega \in (0\,..\,\infty)$ such that for all $x \in \Omega$ there exists $y \in \Omega$ such that $|x - y| = \epsilon_\Omega$ and

$$(1-t)x + ty + r_\Omega tB \subset \Omega \quad \forall t \in [0\,..\,1].$$

The purpose of this section is to prove the following

Proposition 4.3. *Let Ω be a bounded, open subset of $\mathbb{R}^d$ having the cone property. If Ξ is a finite subset of $\overline{\Omega}$ satisfying $\delta := \delta(\Xi;\Omega) \leq \epsilon_1 r_\Omega$, then for all $\gamma \in (0\,..\,m]$, $\mu \in \mathcal{M}_\gamma(\overline{\Omega})$, there exists $s \in S(\phi;\Xi)$ such that*

$$|||s_h(\mu) - s||| \leq \mathrm{const}(\Omega,m,\psi,\gamma)\delta^\gamma \|\mu\|_{\mathcal{M}_\gamma},$$

where $h := \delta/\epsilon_1$.

Let r_0 be the smallest positive real number for which

$$\mathrm{supp}\,\psi \subset r_0\overline{B}.$$

Let Ω, μ, and Ξ satisfy the hypothesis of Proposition 4.3. Let $\mu_h \in C_c^\infty(\mathbb{R}^d)$ be given by $\mu_h := \psi(\cdot/h) * \mu$, and note that $\operatorname{supp}\mu_h \subset \operatorname{supp}\mu + h\operatorname{supp}\psi \subset \overline{\Omega} + hr_0\overline{B}$. For $j \in \mathbb{Z}^d$ satisfying $\mu_h(hj) \neq 0$, there exists $x_j \in \Omega$ such that $|x_j - hj| \leq hr_0$. By Definition 4.2, there exists $y_j \in \Omega$ such that $|x_j - y_j| = \epsilon_\Omega$ and

$$(1-t)x_j + ty_j + r_\Omega tB \subset \Omega, \quad \forall t \in [0\,..\,1].$$

Substituting $t = h/r_\Omega$ (necessarily ≤ 1) we obtain $z_j + hB \subset \Omega$, where $z_j := (1-h/r_\Omega)x_j + (h/r_\Omega)y_j$. Note that

$$\left|j - h^{-1}z_j\right| \leq |hj - x_j|/h + |x_j - z_j|/h \leq r_0 + \epsilon_\Omega/r_\Omega =: r_1, \tag{4.4}$$

and $\delta(h^{-1}\Xi; h^{-1}z_j + B) \leq h^{-1}\delta(\Xi;\Omega) = \epsilon_1$. Since $\mathcal{N} \subset B$, there exists $\mathcal{N}_j \subset \Xi$ such that $\#\mathcal{N}_j = \#\mathcal{N}$ and $\delta(h^{-1}\mathcal{N}_j - h^{-1}z_j; \mathcal{N}) \leq \epsilon_1$. By Lemma 4.1, there exists $\{a_{j,\xi}\}_{\xi\in\mathcal{N}_j}$ such that

$$\max_{\xi\in\mathcal{N}_j} |a_{j,\xi}| \leq \operatorname{const}(d,m,r_1) \quad \text{and} \tag{4.5}$$

$$q(j - h^{-1}z_j) = \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}\, q(h^{-1}(\xi - z_j)), \quad \forall q \in \Pi_m. \tag{4.6}$$

Two easily proved consequences of (4.6) are that for all $q \in \Pi_m$,

$$q(0) = \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}\, q(\xi/h - j) \quad \text{and} \quad q = \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}\, q(\cdot - (\xi/h - j)). \tag{4.7}$$

Noting that $s_h(\mu)$ can be written as $s_h(\mu) = \sum_{j\in\mathbb{Z}^d} \mu_h(hj)\phi(\cdot - hj)$, Dyn and Ron [9] have suggested that in order to approximate $s_h(\mu)$ from $S(\phi;\Xi)$, one should first find 'pseudo-shifts' $\phi_j \in \operatorname{span}\{\phi(\cdot - \xi) : \xi \in \Xi\}$ which approximate $\phi(\cdot - hj)$ and then put $s := \sum_{j\in\mathbb{Z}^d} \mu_h(hj)\phi_j$.

Definition. For $j \in \mathbb{Z}^d$ satisfying $\mu_h(hj) \neq 0$, define

$$\phi_j := \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}\,\phi(\cdot - \xi),$$

$$\zeta_j := \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}\,\zeta(\cdot - \xi), \qquad \text{where } \zeta := \begin{cases} |\cdot|^{m-d} & \text{if } m-d \notin 2\mathbb{N}_0 \\ |\cdot|^{m-d}\log|\cdot| & \text{if } m-d \in 2\mathbb{N}_0 \end{cases}.$$

Lemma 4.8. *If* $s := \sum_{j\in\mathbb{Z}^d} \mu_h(hj)\phi_j$, *then* $s \in S(\phi;\Xi)$ *and*

$$|||s_h(\mu) - s||| \leq \operatorname{const}(d,m)\left\| \sum_{j\in\mathbb{Z}^d} \mu_h(hj)(\zeta(\cdot - hj) - \zeta_j)\right\|_{L_2}. \tag{4.9}$$

Proof. It is clear that $s \in \operatorname{span}\{\phi(\cdot-\xi) : \xi \in \Xi\}$, so in order to show that $s \in S(\phi;\Xi)$, it suffices to show that $\sum_{j\in\mathbb{Z}^d} \mu_h(hj) \sum_{\xi\in\mathcal{N}_j} a_{j,\xi} q(\xi) = 0$, for all $q \in \Pi_{m-1}$. It was shown in the proof of Proposition 3.3 that $s_h(\mu) \in S(\phi; h\mathbb{Z}^d \cap \operatorname{supp}\mu_h)$; hence $\sum_{j\in\mathbb{Z}^d} q(hj)\mu_h(hj) = 0$ for all $q \in \Pi_{m-1}$. Therefore, if $q \in \Pi_{m-1}$, then $\sum_{j\in\mathbb{Z}^d} \mu_h(hj) \sum_{\xi\in\mathcal{N}_j} a_{j,\xi} q(\xi) = \sum_{j\in\mathbb{Z}^d} \mu_h(hj)q(hj) = 0$ which proves that $s \in S(\phi;\Xi)$. Now, if $\left\|\sum_{j\in\mathbb{Z}^d} \mu_h(hj)(\zeta(\cdot - hj) - \zeta_j)\right\|_{L_2} = \infty$, then the inequality is clear; so assume $\sum_{j\in\mathbb{Z}^d} \mu_h(hj)(\zeta(\cdot - hj) - \zeta_j) \in L_2$. Then

$$
\begin{aligned}
|||s_h(\mu) - s||| &= ||| \sum_{j\in\mathbb{Z}^d} \mu_h(hj)[\phi(\cdot - hj) - \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}\phi(\cdot - \xi)]||| \\
&= |c_\phi| \left\| |\cdot|^m \sum_{j\in\mathbb{Z}^d} \mu_h(hj)[|\cdot|^{-2m} e_{-hj} - \sum_{\xi\in\mathcal{N}_j} a_{j,\xi} |\cdot|^{-2m} e_{-\xi}] \right\|_{L_2} \\
&= |c_\phi| \left\| \sum_{j\in\mathbb{Z}^d} \mu_h(hj)[|\cdot|^{-m} e_{-hj} - \sum_{\xi\in\mathcal{N}_j} a_{j,\xi} |\cdot|^{-m} e_{-\xi}] \right\|_{L_2} \\
&= \operatorname{const}(d,m) \left\| \sum_{j\in\mathbb{Z}^d} \mu_h(hj)(\zeta(\cdot - hj) - \zeta_j) \right\|_{L_2},
\end{aligned}
$$

since $\left|\widehat{\zeta}\right| = \operatorname{const}(d,m)\, |\cdot|^{-m}$ on $\mathbb{R}^d\backslash 0$ (cf. [11]). $\square$

The problem of estimating the right side of (4.9) would be much simpler if the function $\zeta - \zeta_j(\cdot + hj)$ was independent of j. The following lemma, proposition, and lemma will allow us to carry forth our desired estimate despite the dependence of $\zeta - \zeta_j(\cdot + hj)$ on j.

Let $\rho : \mathbb{R}^d \to [0\,..\,\infty)$ be given by $\rho(x) := 0$ if $x \in (1 + r_1)B$ and

$$
\rho(x) := \max\{\left|\zeta(x) - \sum_{\xi\in\widetilde{\mathcal{N}}} a_\xi \zeta(x-\xi)\right|\}, \qquad \text{if } x \notin (1+r_1)B,
$$

where the maximum is taken over all $z, \widetilde{\mathcal{N}}$ satisfying $z \in r_1\overline{B}$, $\#\widetilde{\mathcal{N}} = \#\mathcal{N}$, $\delta(\widetilde{\mathcal{N}} - z, \mathcal{N}) \le \epsilon_1$, and the coefficients $\{a_\xi\}_{\xi\in\widetilde{\mathcal{N}}}$ are determined by the requirement $q(0) = \sum_{\xi\in\widetilde{\mathcal{N}}} a_\xi q(\xi)$, $\forall q \in \Pi_m$. We will show that ρ belongs to the space $\mathcal{L}_2$ which was first introduced by Jia and Micchelli [14] as the set of all $g \in L_2$ for which

$$
\|g\|_{\mathcal{L}_2} := \left\| \sum_{j\in\mathbb{Z}^d} |g(\cdot - j)| \right\|_{L_2(C)} < \infty,
$$

where $C := [-1/2\,..\,1/2)^d$.

Lemma 4.10. $\|\rho\|_{\mathcal{L}_2} \le \mathrm{const}(d,m,r_1)$.

Proof. Let $x \in \mathbb{R}^d \backslash (1+r_1)B$, and let $z, \widetilde{\mathcal{N}}$ be such that $\rho(x) = \left|\zeta(x) - \sum_{\xi\in\widetilde{\mathcal{N}}} a_\xi \zeta(x-\xi)\right|$, where the coefficients $\{a_\xi\}$ are as described in the definition of ρ. Since $\delta(\widetilde{\mathcal{N}} - z, \mathcal{N}) \le \epsilon_1$, $z \in r_1\overline{B}$, and $q(-z) = \sum_{\xi\in\widetilde{\mathcal{N}}} a_\xi q(\xi - z)$ for all $q \in \Pi_m$, it follows by Lemma 4.1 that $\max_{\xi\in\widetilde{\mathcal{N}}} |a_\xi| \le \mathrm{const}(d,m,r_1)$. Note that since $\mathcal{N} \subset \frac{1}{2}\overline{B}$ and $\epsilon_1 \in (0\,..\,1/4)$, it follows that $\widetilde{\mathcal{N}} \subset (r_1 + \frac{3}{4})B$. Define the difference operator T by $Tg := g - \sum_{\xi\in\widetilde{\mathcal{N}}} a_\xi g(\cdot - \xi)$. It follows from the requirement $q(0) = \sum_{\xi\in\widetilde{\mathcal{N}}} a_\xi q(\xi)\ \forall q \in \Pi_m$ that $Tq = 0\ \forall q\in\Pi_m$. Let $q \in \Pi_m$ be the mth-degree Taylor polynomial of ζ at x. Then

$$\begin{aligned}
\rho(x) = |T\zeta(x)| = |T(\zeta - q)(x)| &= \left|\zeta(x) - q(x) + \sum_{\xi\in\widetilde{\mathcal{N}}} a_\xi(\zeta(x-\xi) - q(x-\xi))\right| \\
&\le \left(\max_{\xi\in\widetilde{\mathcal{N}}} |a_\xi|\right) \sum_{\xi\in\widetilde{\mathcal{N}}} |\zeta(x-\xi) - q(x-\xi)| \\
&\le \mathrm{const}(d,m,r_1) \max\{|D^\alpha\zeta(w)| : |\alpha| = m+1 \text{ and } w \in x + (r_1 + 3/4)\overline{B}\} \\
&\le \mathrm{const}(d,m,r_1)\, |x|^{-d-1}\, (1 + \log|x|).
\end{aligned}$$

It follows from this that $\|\rho\|_{\mathcal{L}_2} \le \mathrm{const}(d,m,r_1)$. □

The following proposition, which demonstrates the utility of the space $\mathcal{L}_2$, was proved in [14].

Proposition 4.11. *If $c \in \ell_2(\mathbb{Z}^d)$ and $g \in \mathcal{L}_2$, then*

$$\left\|\sum_{j\in\mathbb{Z}^d} c_j\, g(\cdot - j)\right\|_{L_2} \le \|c\|_{\ell_2(\mathbb{Z}^d)}\, \|g\|_{\mathcal{L}_2}.$$

Lemma 4.12. *If $j \in \mathbb{Z}^d$ is such that $\mu_h(hj) \ne 0$ and $\rho_j := \zeta - \zeta_j(\cdot + hj)$, then*

$$\begin{aligned}
&(i) \qquad |\rho_j(x)| \le h^{m-d}\rho(x/h) \quad \forall x \in \mathbb{R}^d\backslash h(1+r_1)B \quad \textit{and} \\
&(ii) \qquad \|\rho_j\|_{L_2(h(1+r_1)B)} \le \mathrm{const}(d,m,r_1)h^{m-d/2}.
\end{aligned}$$

Proof. We first establish the identity

$$\rho_j(x) = h^{m-d}[\zeta(x/h) - \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}\zeta(x/h - (\xi/h - j))], \quad x \notin \{0\} \cup (\mathcal{N}_j - hj). \tag{4.13}$$

If $m - d \notin 2\mathbb{N}_0$, then (4.13) is simply a consequence of the fact that $\zeta(y) = h^{m-d}\zeta(y/h)$. If $m - d \in 2\mathbb{N}_0$, then $\zeta(x) = \zeta(hx/h) = h^{m-d}\zeta(x/h) + h^{m-d}\,|x/h|^{m-d}\log h$, and hence

$$\begin{aligned}
\rho_j(x) &= h^{m-d}[\zeta(x/h) - \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}\zeta(x/h - (\xi/h - j))] \\
&\quad + h^{m-d}\log h[|x/h|^{m-d} - \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}\,|x/h - (\xi/h - j)|^{m-d}].
\end{aligned} \tag{4.14}$$

Let T be the difference operator defined by $Tg := g - \sum_{\xi\in\mathcal{N}_j} a_{j,\xi} g(\cdot - (\xi/h - j))$, and note that $Tq = 0\ \forall q \in \Pi_m$ by (4.7). In particular, since $|\cdot|^{m-d} \in \Pi_m$, it follows that $|x/h|^{m-d} - \sum_{\xi\in\mathcal{N}_j} a_{j,\xi} |x/h - (\xi/h - j)|^{m-d} = [T(|\cdot|^{m-d})](x/h) = 0$ which, in view of (4.14), completes the proof of (4.13). In order to establish (i), let $z = h^{-1}z_j - j$. Then $(h^{-1}\mathcal{N}_j - j) - z = h^{-1}\mathcal{N}_j - h^{-1}z_j$ and hence $\delta((h^{-1}\mathcal{N}_j - j) - z, \mathcal{N}) \le \epsilon_1$. (i) now follows from (4.13) since by the definition of ρ and with (4.4),(4.7) in view, $\left|\zeta(x/h) - \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}\zeta(x/h - (\xi/h - j))\right| \le \rho(x/h)$. For (ii), we note that

$$\|\rho_j\|_{L_2(h(1+r_1)B)} = h^{m-d} \left\|\zeta(\cdot/h) - \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}\zeta(\cdot/h - (\xi/h - j))\right\|_{L_2(h(1+r_1)B)}, \quad \text{by (4.13)},$$

$$= h^{m-d/2} \left\|\zeta - \sum_{\xi\in\mathcal{N}_j} a_{j,\xi}\zeta(\cdot - (\xi/h - j))\right\|_{L_2((1+r_1)B)}$$

$$\le \mathrm{const}(d,m,r_1) h^{m-d/2} \|\zeta\|_{L_2(2(1+r_1)B)} = \mathrm{const}(d,m,r_1) h^{m-d/2}, \quad \text{by (4.5)}.$$

□

Proof of Proposition 4.3. Let $s \in S(\phi;\Xi)$ be as in Lemma 4.8, and for brevity, put $\widetilde{B} := (1+r_1)B$. Then

$$\mathrm{const}(d,m)|||s_h(\mu) - s||| \le \left\|\sum_{j\in\mathbb{Z}^d} \mu_h(hj)(\zeta(\cdot - hj) - \zeta_j)\right\|_{L_2} = \left\|\sum_{j\in\mathbb{Z}^d} \mu_h(hj)\rho_j(\cdot - hj)\right\|_{L_2}$$

$$\le \left\|\sum_{j\in\mathbb{Z}^d} \mu_h(hj)\chi_{h(j+\widetilde{B})}\rho_j(\cdot - hj)\right\|_{L_2} + \left\|\sum_{j\in\mathbb{Z}^d} \mu_h(hj)\chi_{\mathbb{R}^d\setminus h(j+\widetilde{B})}\rho_j(\cdot - hj)\right\|_{L_2}$$

$$\le \mathrm{const}(d,r_1)\sqrt{\sum_{j\in\mathbb{Z}^d} |\mu_h(hj)|^2 \|\rho_j(\cdot - hj)\|^2_{L_2(h(j+\widetilde{B}))}} + h^{m-d}\left\|\sum_{j\in\mathbb{Z}^d} \mu_h(hj)\rho(\cdot/h - j)\right\|_{L_2}$$

$$= \mathrm{const}(d,r_1)\sqrt{\sum_{j\in\mathbb{Z}^d} |\mu_h(hj)|^2 \|\rho_j\|^2_{L_2(h\widetilde{B})}} + h^{m-d/2}\left\|\sum_{j\in\mathbb{Z}^d} \mu_h(hj)\rho(\cdot - j)\right\|_{L_2}$$

$$\le \mathrm{const}(d,m,r_1) h^{m-d/2} \|\mu_h\|_{\ell_2(h\mathbb{Z}^d)}, \quad \text{by Lemma 4.12 (ii), Lemma 4.10, and Proposition 4.11.}$$

Therefore,

$$|||s_h(\mu) - s||| \le \mathrm{const}(d,m,r_1) h^{m-d/2} \|\mu_h\|_{\ell_2(h\mathbb{Z}^d)}. \tag{4.15}$$

Claim 4.16. $\|\mu_h\|_{\ell_2(h\mathbb{Z}^d)} = (h/2\pi)^{d/2} \left\|\sum_{j\in\mathbb{Z}^d} \widehat{\psi}(h\cdot + 2\pi j)\widehat{\mu}(\cdot + 2\pi j/h)\right\|_{L_2(2\pi h^{-1}C)}.$

proof. Define $G : 2\pi h^{-1}C \to \mathbb{C}$ by $G := \sum_{j\in\mathbb{Z}^d} \mu_h(hj)e_{-hj}$ and note that $\|G\|_{L_2(2\pi h^{-1}C)} = (2\pi/h)^{d/2}\|\mu_h\|_{\ell_2(h\mathbb{Z}^d)}$. Hence

$$\|\mu_h\|_{\ell_2(h\mathbb{Z}^d)} = (h/2\pi)^{d/2}\|G\|_{L_2(2\pi h^{-1}C)}. \tag{4.17}$$

Fix $x \in 2\pi h^{-1}C$ and put $g := \mu_h(h\cdot)e_{-hx}$ and note that $G(x) = \sum_{j\in\mathbb{Z}^d} g(j) = \sum_{j\in\mathbb{Z}^d}\widehat{g}(2\pi j)$ by Poisson's summation formula (cf. [20], Chapter 7). Now, $\widehat{g} = (\mu_h(h\cdot))\widehat{\ }(\cdot + hx) = h^{-d}\widehat{\mu_h}(\cdot/h + x) = \widehat{\psi}(\cdot + hx)\widehat{\mu}(\cdot/h + x)$. Therefore,
$G(x) = \sum_{j\in\mathbb{Z}^d}\widehat{g}(2\pi j) = \sum_{j\in\mathbb{Z}^d}\widehat{\psi}(2\pi j + hx)\widehat{\mu}(2\pi j/h + x)$ which, in view of (4.17), proves the claim.

Now,

$$\begin{aligned}
\left\|\sum_{j\in\mathbb{Z}^d}\widehat{\psi}(h\cdot+2\pi j)\widehat{\mu}(\cdot+2\pi j/h)\right\|_{L_2(2\pi h^{-1}C)} &\le \sum_{j\in\mathbb{Z}^d}\left\|\widehat{\psi}(h\cdot+2\pi j)\widehat{\mu}(\cdot+2\pi j/h)\right\|_{L_2(2\pi h^{-1}C)}\\
\le \left\|\widehat{\psi}\right\|_{L_\infty(2\pi C)}\|\widehat{\mu}\|_{L_2(2\pi h^{-1}C)} &+ \sum_{j\in\mathbb{Z}^d\setminus 0}\left\|\widehat{\psi}(h\cdot)\,|\cdot|^m\right\|_{L_\infty(2\pi h^{-1}(j+C))}\left\||\cdot|^{-m}\,\widehat{\mu}\right\|_{L_2(2\pi h^{-1}(j+C))}\\
\le \mathrm{const}(d,m,\psi)\|\widehat{\mu}\|_{L_2(2\pi h^{-1}C)} &+ h^{-m}\sqrt{\sum_{j\in\mathbb{Z}^d\setminus 0}\left\|\widehat{\psi}\,|\cdot|^m\right\|^2_{L_\infty(2\pi(j+C))}}\left\||\cdot|^{-m}\,\widehat{\mu}\right\|_{L_2(\mathbb{R}^d\setminus 2\pi h^{-1}C)}\\
\le \mathrm{const}(d,m,\psi,\gamma)h^{\gamma-m}\|\mu\|_{\mathcal{M}_\gamma}, &\quad \text{by Proposition 2.3, (3.1), (3.2),}
\end{aligned}$$

which, in view of (4.15) and Claim 4.16 completes the proof. □

5. The Main Results

Combining Proposition 3.3 and Proposition 4.3 yields the following:

Theorem 5.1. *Let Ω be a bounded, open subset of $\mathbb{R}^d$ having the cone property, and let $\Xi \subset \overline{\Omega}$ satisfy $\delta := \delta(\Xi;\Omega) \le \min\{\epsilon_1 r_\Omega, \epsilon_1\}$. If $f \in C(\mathbb{R}^d)$ is such that there exists $\gamma \in (0\,..\,m]$, $\mu \in \mathcal{M}_\gamma(\overline{\Omega})$, $q \in \Pi_{m-1}$ such that $f = \phi * \mu + q$ on Ω, then*

$$\begin{aligned}
&(i) && \phi * \mu + q = T_\Omega f,\\
&(ii) && |||\phi*\mu + q - T_\Xi f||| \le \mathrm{const}(\Omega,m,\gamma)\delta^\gamma\|\mu\|_{\mathcal{M}_\gamma}, \quad \textit{and}\\
&(iii) && \|f - T_\Xi f\|_{L_p(\Omega)} \le \mathrm{const}(\Omega,m,\gamma)\delta^{\gamma_p+\gamma}\|\mu\|_{\mathcal{M}_\gamma}
\end{aligned}$$

for all $1 \le p \le \infty$.

Proof. Since (i) follows from (ii) via (1.4) and (iii) follows from (i) and (ii) via (1.3), it suffices to prove (ii). It is known [7] that

$$|||\phi*\mu + q - T_\Xi f||| \le |||\phi*\mu - s||| \quad \forall s \in S(\phi;\Xi). \tag{5.2}$$

Put $h = \delta/\epsilon_1$ and recall from Proposition 3.3 (ii) that $|||\phi * \mu - s_h(\mu)||| \le \text{const}(m, \gamma, \psi) h^\gamma \|\mu\|_{\mathcal{M}_\gamma}$. By Proposition 4.3, there exists $s \in S(\phi; \Xi)$ such that $|||s_h(\mu) - s||| \le \text{const}(\Omega, m, \gamma, \psi)\delta^\gamma \|\mu\|_{\mathcal{M}_\gamma}$. Hence, by (5.2),

$$|||\phi * \mu + q - T_\Xi f||| \le |||\phi * \mu - s_h(\mu)||| + |||s_h(\mu) - s||| \le \text{const}(\Omega, m, \gamma, \psi)\delta^\gamma \|\mu\|_{\mathcal{M}_\gamma}$$

which, after a suitable choice of ψ, proves (ii). $\square$

Given a smooth f, the problem of finding $\mu \in \mathcal{M}_\gamma(\overline{\Omega})$, $q \in \Pi_{m-1}$ such that $\phi * \mu + q = f$ on Ω is quite difficult. In the special case $m = d = 2$, $\Omega = B$, it is known [16] that if $f \in C^\infty(\mathbb{R}^2)$, then there exists $\mu \in \mathcal{M}_{1/2}(\overline{\Omega})$, $q \in \Pi_1$ such that $\phi * \mu + q = f$ on Ω. There is one special case in which μ is easily found. That is the case when f is a smooth function supported in $\overline{\Omega}$. The following corollary deals with this special case.

For $\gamma \in (0\,..\,m]$, let $\mathcal{F}_\gamma$ be the space given by

$$\mathcal{F}_\gamma := \begin{cases} B^{\gamma+m}_{2,\infty} & \text{if } \gamma \in (0\,..\,m) \\ W^{2m}_2 & \text{if } \gamma = m. \end{cases}$$

Corollary 5.3. *Let Ω and Ξ be as in Theorem 5.1, and let $\gamma \in (0\,..\,m]$. If $f \in \mathcal{F}_\gamma$ is supported in $\overline{\Omega}$, then*

$$\begin{aligned} &(i) \qquad f = T_\Omega f \quad \textit{and} \\ &(ii) \qquad |||f - T_\Xi f||| \le \text{const}(\Omega, m, \gamma)\delta^\gamma \|f\|_{\mathcal{F}_\gamma}, \end{aligned}$$

where $\delta := \delta(\Xi; \Omega)$. Additionally, if δ is sufficiently small, then

$$(iii) \qquad \|f - T_\Xi f\|_{L_p(\Omega)} \le \text{const}(\Omega, m, \gamma)\delta^{\gamma_p + \gamma} \|f\|_{\mathcal{F}_\gamma},$$

for all $1 \le p \le \infty$.

Proof. As mentioned in the proof of Theorem 5.1, it suffices to prove (ii). Assume $f \in \mathcal{F}_\gamma$ is supported in $\overline{\Omega}$. Put $\mu := \dfrac{(-1)^m}{c_\phi}\Delta^m f$, where $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ denotes the Laplacian operator, and note that $\mu \in \mathcal{M}_\gamma(\overline{\Omega})$ and $\|\mu\|_{\mathcal{M}_\gamma} \le \text{const}(d, m, \gamma)\|f\|_{\mathcal{F}_\gamma}$. We show that $f = \phi * \mu$. Since $\widehat{f} = (\phi * \mu)\widehat{\ }$ on $\mathbb{R}^d \backslash 0$, it follows that the difference $f - \phi * \mu$ is a polynomial. For $x \notin \operatorname{supp} f$, it follows from Green's second identity [12; page 5] that

$$\phi * \mu(x) = \frac{(-1)^m}{c_\phi}\int_{\operatorname{supp} f} \phi(x - t)\Delta^m f(t)\, dt = \frac{(-1)^m}{c_\phi}\int_{\operatorname{supp} f} \Delta^m \phi(x - t) f(t)\, dt = 0$$

since $\Delta^m \phi = 0$ on $\mathbb{R}^d \backslash 0$. Thus the polynomial $f - \phi * \mu = 0$ on $\mathbb{R}^d \backslash \operatorname{supp} f$; hence $f = \phi * \mu$. If $\delta > \min\{\epsilon_1 r_\Omega, \epsilon_1\}$, then choosing $s = 0$ in (5.2) yields

$$|||f - T_\Xi f||| \le |||f||| \le \text{const}(m, \gamma)\|f\|_{\mathcal{F}_\gamma} \le \text{const}(\Omega, m, \gamma)\delta^\gamma \|f\|_{\mathcal{F}_\gamma}.$$

On the other hand, if $\delta \le \min\{\epsilon_1 r_\Omega, \epsilon_1\}$, then by Theorem 5.1 (ii),

$$|||f - T_\Xi f||| \le \text{const}(\Omega, m, \gamma)\delta^\gamma \|\mu\|_{\mathcal{M}_\gamma} \le \text{const}(\Omega, m, \gamma)\delta^\gamma \|f\|_{\mathcal{F}_\gamma}.$$

$\square$

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