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Principal Shift-Invariant Spaces Generated by a Radial Basis Function

Abstract. Approximations from the L_2-closure S of the finite linear combinations of the shifts of a radial basis function are considered, and a thorough analysis of the least-squares approximation orders from such spaces is provided. The results apply to polyharmonic splines, multiquadrics, the Gaussian kernel and other functions, and include the derivation of spectral orders. For stationary refinements it is shown that the saturation class is trivial, i.e., no non-zero function in the underlying Sobolev space can be approximated to a better rate. The approach makes an essential use of recent results of de Boor DeVore and the author.

§1. Introduction

A substantial progress in the understanding the L_∞- and L_2- approximation orders of principal shift-invariant spaces was recently obtained in [4] and [2] respectively. While [4] discusses the application of the methods there to radial functions, no such discussion can be found in [2], and the present paper is meant to fill in that gap. Thus, it is devoted to the analysis of the L_2-approximation orders associated with principal spaces generated by a radial function via the ideas, methods and results of [2].

It seems best to start our discussion with an explanation of the title. First, all functions here are assumed to be either real or complex valued and are defined on the real Euclidean space $\mathbb{R}^d$, for some $d \geq 1$. **A shift** means "an integer translate" or "an integer translation", hence **a shift-invariant space** S is a space which is invariant under the shift operation, i.e., satisfies

$$f \in S \quad \Longleftrightarrow \quad f(\cdot - \alpha) \in S, \quad \alpha \in \mathbb{Z}^d. \tag{1.1}$$

We also assume that S is a closed subspace of $L_2(\mathbb{R}^d)$. For $f \in L_2(\mathbb{R}^d)$, we denote by $S(f)$ **the space generated by** f, i.e., $S(f)$ is the L_2-closure of the *finite* linear combinations of the shifts of f:

$$S(f) := \text{closure}\, S_0(f),$$

with

$$S_0(f) := \{ \sum_{\alpha \in \mathbb{Z}^d} a_\alpha f(\cdot - \alpha) : \text{ almost all } a_\alpha \text{ are zero}\},$$

i.e., $S(f)$ is the smallest closed shift-invariant space that contains f. Certainly, $S(f) \subset S$ for every $f \in S$. We say that S is **principal** if it is generated by a single function, i.e., if there exists $\phi \in S$ for which

$$S = S(\phi).$$

We remark that Theorem 2.16 of [2] shows that in case $\phi \in L_2(\mathbb{R}^d)$ is compactly supported, $S(\phi)$ contains all the *infinite* combinations of the shifts of ϕ (calculated pointwise) which happen to be $L_2(\mathbb{R}^d)$-functions.

In order to read and understand the main results and applications in this paper, no previous knowledge on radial basis functions is required. However, some, or even a good, knowledge of the present state-of-art and the present concepts in the area will help in understanding the *novelty* of the approach here. By "present state-of-art" we mean the survey of Powell [17], with the complement of some more recent results from [7], [6] and [15].

What do we mean here by *a radial basis function*? In [17], Powell lists six functions as being the major examples of radial basis functions. These are $|x|$, $|x|^3$, $|x|^2 \log |x|$, $e^{-|x|^2}$, and $(|x|^2 + c^2)^{\pm 1/2}$, where $|x|$ stands for the Euclidean 2-norm

$$|x| := \sqrt{x_1^2 + x_2^2 + ... + x_d^2}.$$

All these functions are, indeed, radially symmetric, some of them grow at ∞ and some of them decay at ∞. For our purposes, the radial symmetry of the basis function plays less than a role, and we use the terminology "a radial basis" function more for convenience, (as a matter of fact, a few of our examples are not radially symmetric). The two basic properties of the basis function ϕ which we employ here are (1) its **smoothness** or, more precisely, the decay at ∞ of its Fourier transform $\widehat{\phi}$, and (2) its **"ellipticity"**. By the latter we mean that the (generalized) Fourier transform of $\widehat{\phi}$ is a well-defined smooth function in the complement $\mathbb{R}^d \backslash 0$ of the origin, and does not vanish there.

To pursue further the discussion, we want to introduce the notion of **approximation orders**. For $f \in L_2(\mathbb{R}^d)$, the distance $E(f, S)$ of f from the (shift-invariant) space S is defined as usual by

$$E(f, S) := \min\{\|f - g\| : \ g \in S\}, \tag{1.2}$$

where $\|\cdot\|$ is the $L_2(\mathbb{R}^d)$-norm. Assume further that we hold in hand not *one* principal shift-invariant space $S(\phi)$, but a collection of them $\{S_h := S(\phi_h)\}_h$, where h varies over either the interval $(0, 1]$ or some discrete subset of this interval. A priori no connection between the generators $\{\phi_h\}$ of the various spaces is assumed. Each of the shift-invariant spaces S_h is then dilated to the h-level as follows:

$$S_h^h := S^h(\phi_h) := \{f(\cdot/h) : \ f \in S_h\}. \tag{1.3}$$

Note that the dilated space $S^h(\phi_h)$ is generated by the $h\mathbb{Z}^d$-shifts of the dilated function $\phi_h(\cdot/h)$. We say that $\{S_h\}_h$ (or $\{\phi_h\}_h$) **provides approximation order** $k > 0$ (in the 2-norm), if

$$E(f, S_h^h) = O(h^k), \quad \forall f \in W, \tag{1.4}$$

where W is some smooth subspace of $L_2(\mathbb{R}^d)$ which depends on k, usually a Sobolev space.

More than any other case, the literature studies the so-called **stationary case** in which only one function ϕ is employed, i.e., $\phi_h = \phi$, all h. In this case $S_h = S(\phi)$, all h, and the scaled spaces in (1.3) are all dilates of one basic space. The study of non-stationary settings was initiated in spline theory (exponential box splines [18], [9]), but *there are also very good reasons for considering non-stationary refinements in radial basis function theory*. This point is so important, that we pause here momentarily to discuss the following example.

Example 1.5. Let ϕ be the Gaussian kernel, i.e., $\phi(x) = e^{-|x|^2}$. Despite of the superior smoothness and decay properties of this function, it has been conceived as a poor choice as far as approximation orders are concerned. The heuristic reason for this is that the dilated function $\phi(x/h) = e^{-h^{-2}|x|^2}$ converges to the δ functional faster than our linear refinement of the translates, and the task of approximating from this space becomes hopeless. But, a small change in the dilation process alters the picture dramatically: we obtain approximation orders as large as are wished by choosing $\lambda(h)$ to be a function that decays to 0 with h (e.g., $\lambda(h) = O(1/|\log h|)$), and defining

$\phi_h := e^{-\lambda(h)|\cdot|^2}$. (Note that $\phi_h(\cdot/h) = e^{-\lambda(h)h^{-2}|\cdot|^2}$, and so the $\lambda(h)$ parameter slows the convergence of $\phi(\cdot/h)$ to the δ functional.) Further, the choice $\lambda(h) := h^\nu$, $\nu > 0$, results in *spectral* approximation orders, i.e., approximation orders that depend only on the smoothness of the approximand.

Before [4] and [2] were written, the standard approach for the analysis of approximation orders went along the quasi-interpolation argument guidelines, which can roughly be divided into three steps. The first of which is **localization** (referred to sometimes as "preconditioning"): since ϕ usually grows at ∞, one applies to ϕ a (finite/infinite) difference operator to obtain a function ψ with nice decay properties at ∞. Then, one tries to **reproduce polynomials**: if ψ decays fast enough at ∞, then, for some $k \geq 1$, the sum

$$\psi *' p := \sum_{\alpha \in \mathbb{Z}^d} p(\alpha)\psi(\cdot - \alpha)$$

converges uniformly on compact sets for every $p \in \Pi_{k-1}$, with Π_k the space of all polynomials of degree $\leq k$ (in d variables). Under certain conditions on ϕ and by a careful choice of the difference scheme employed, it is possible to prove that $\psi *' p = p$, for all $p \in \Pi_{k-1}$. This gives rise to the approximation scheme

$$f \approx \psi *' f, \quad f \in W. \tag{1.6}$$

The third step is the **error analysis** where the polynomial reproduction is shown to imply that the scheme (1.6) provides approximation order k. In case ψ is compactly supported, the conversion of polynomial reproduction to approximation orders provides no difficulty (cf. e.g., [1]), and the same holds in case it is known that ψ decays at ∞ like $O(|\cdot|^{-k-d-\varepsilon})$ for some $\varepsilon > 0$ (cf. Proposition 1.1 and Corollary 1.2 of [8] and the arguments in [13]). However, things become more involved if the above decay holds only with $\varepsilon = 0$, and subtle information on ϕ and ψ is then required.

The focal point of our discussion here is that *we do not employ any step of the quasi-interpolation argument approach*; specifically, we do not reproduce polynomials (nor do we reproduce exponentials or any other "nice" functions). This results in a tremendous relaxation of the localization step, as the function ψ is no longer required to decay in a manner related to the desired approximation order k, but merely to lie in $L_2(\mathbb{R}^d)$.

Example 1.7. Let ϕ be the univariate inverse multiquadric, i.e., $d = 1$ and

$$\phi := (1 + |\cdot|^2)^{-1/2}.$$

In this case $\phi \in L_2(\mathbb{R})$, hence $S := S(\phi)$ is well-defined. We assume that S is refined by dilation, i.e., that $\phi_h = \phi$ for all h. It was conjectured by many (cf.

e.g., [5]) that ϕ above provides no positive approximation orders; this, indeed, will be proved here. On the positive side, we show that

$$E(f, S^h) = o(1), \quad \forall f \in L_2(\mathbb{R}). \tag{1.8}$$

In the terminology of [2] ϕ **provides density order** 0. We will even determine the rate of decay of $E(f, S^h)$ for smooth f (e.g, for f in the Sobolev space $W_1^2(\mathbb{R})$). An L_∞-analogue of (1.8) has been recently established in [6].

We mentioned one drawback of the quasi-interpolation argument, i.e. that it requires high decay rates from the basis function ψ. There is, however, another significant deficiency in this argument: it provides only *lower bounds* on the approximation order, in the sense that by quasi-interpolation one can only conclude that the approximation order is *at least* some k. General methods for the derivation of *upper bounds* on the approximation order were known only for the stationary case, and even there only for weaker versions of approximation orders (the so-called "controlled", "local" and "controlled-local" approximations), and only for basis functions which decay like $O(|\cdot|^{-k-d-\varepsilon})$ at ∞ (cf. [20], [3], [14], [12], [10]). In contrast, [19], [4], [2] as well as this paper employ methods that determine the *exact* approximation order.

It then becomes very interesting to compare the lower bounds on approximation orders provided by quasi-interpolation with the exact orders that will be described. For this purpose, we state (and prove in the next section) the following theorem.

Theorem 1.9. *Let ϕ be some function which grows no faster than polynomially at ∞, and let $\widehat{\phi}$ be its Fourier transform. Assume that ψ_j, $j = 1, 2$, are two $L_2(\mathbb{R}^d)$-functions which satisfy the equations*

$$\widehat{\psi_j} = u_j \widehat{\phi}, \quad j = 1, 2,$$

where u_j, $j = 1, 2$ are some 2π-periodic functions each vanishes only on a set of measure 0. Then

$$S(\psi_1) = S(\psi_2).$$

How is this theorem connected to our discussion? As mentioned, whenever the basis function grows at ∞, a localization process precedes the construction of an approximation scheme. Whatever approach one chooses from the present literature, the connection between the original ϕ and its localized version ψ is given on the Fourier transform domain by an equation of the form $\widehat{\psi} = u\widehat{\phi}$, where u is some 2π-periodic function (the product $u\widehat{\phi}$ should be interpreted in a distributional sense). The above theorem then says that *the type of localization process used is immaterial to the approximation orders provided by the localized function.* In particular, *the decay rates at ∞ of the localized function ψ, as well as the zero that $\widehat{\psi}$ might or might not have at the origin, are irrelevant to approximation orders.* This is in stark contrast with the lower bounds on approximation order suggested by quasi-interpolation, which are improved together with the decay rates at ∞ of the localized function.

At a first look, the above remarks might seem surprising, since several authors (including myself, cf. [11], [5], [8]) proved that (for some specific basis functions) the approximation orders are improved together with the better decay rates of the function, and even proved that the approximation orders stated in their theorems are the exact (i.e., best possible) ones. This does not contradict the present statements: what is established in the above citations and other references is that for the given localization ψ the approximation scheme (1.6) approximates to a certain (exact) order. The right conclusion is that the approximation scheme (1.6) fails to provide optimal approximation orders whenever ψ decays at ∞ too slowly. As a matter of fact, for functions which decay slowly at ∞, *optimal approximation schemes* (i.e., these that realize the approximation order) *are not local.* For example, in case f happens to be compactly supported, (1.6) employs only finitely many shifts of f, and these shifts are determined by $\operatorname{supp} f$. In contrast, the approximation schemes used here employ infinitely many shifts even in case f is compactly supported, and the coefficients associated with these shifts decay sometimes at ∞ in a slow rate determined by the decay rate of the basis function ψ.

Notations: The symbol "const" stands for a generic positive constant, hence constants appearing in the same display need not to be the same. The notation "$f \sim g$ on Ω" means that $\operatorname{supp} f \cap \Omega$ differs from $\operatorname{supp} g \cap \Omega$ by a null-set, and $f/g, g/f \in L_\infty(\Omega)$.

§2. Approximation Orders in the L_2-Norm

The paper [2] provides a complete analysis of approximation orders from closed shift-invariant spaces of $L_2(\mathbb{R}^d)$. We could have applied those general results to the radial functions considered here, but prefer to derive our results more or less directly, since in this way we obtain finer statements and tighter bounds.

Given $\psi \in L_2(\mathbb{R}^d)$, the space $S(\psi)$ is defined as in the introduction. Throughout the paper, we assume that the generator $\widehat{\psi}$ is non-zero a.e. This assumption is not essential, but is satisfied by all examples in radial basis function theory.

Given the spaces $\{S_h := S(\psi_h)\}_h$, $\psi_h \in L_2(\mathbb{R}^d)$, our goal is to provide a realistic estimate for $\{E(f, S_h^h)\}_h$, $f \in L_2(\mathbb{R}^d)$. Since $E(f, S^h) = h^{d/2}E(f(h\cdot), S)$, as can be easily verified by scaling, we might study the identical quantities

$$h^{d/2}E(f(h\cdot), S_h),$$

as we occasionally do here. To make the analysis more concrete, we briefly discuss some of the possible choices for the sequence $\{\psi_h\}_h$.

Example 2.10. (a) The basis function ϕ is chosen to be a fundamental solution of a homogeneous elliptic differential operator (with constant coefficients) $P(D)$ of order $m > d/2$. In case $P(D)$ is the $m/2$-fold iterated Laplacian, (i.e., if $P(x) = |x|^m$), $\phi(x)$ can be chosen to be $c|x|^{m-d}$ or $c|x|^{m-d}\log|x|$, depending on the parity of d. The Fourier transform $\widehat{\phi}$ coincides with the reciprocal of $P(i\cdot)$ on $\mathbb{R}^d\backslash 0$. Since ϕ grows at ∞, we need to localize it before entering the discussion of approximation orders, and thus we assume that ψ is a localization of ϕ, which means that $\psi \in L_2(\mathbb{R}^d)$ and $\widehat{\psi} = u\widehat{\phi}$ for some 2π-periodic function (even a trigonometric polynomial may do, and recall from Theorem 1.9 that $S(\psi)$ is independent of the periodic u chosen). In the present example we consider only the stationary case, i.e., defining $S := S(\psi)$, we study the decay rates of

$$h^{d/2}E(f(h\cdot), S),$$

for a smooth f. Since the localization ψ plays a dummy role, it is desirable to analyse the problem in terms of the basis function ϕ, or, if possible, in terms of the underlying polynomial P. We mention that in case $P(D)$ is the iterated Laplacian, the space S above is intimately related to the space of polyharmonic splines studied e.g., in [16].

(b) $\phi(x, c) = (|x|^2 + c^2)^{m/2}$, $(m \ge -d,\ m \notin 2\mathbb{Z}_+)$, or $\phi(x, c) = (|x|^2 + c^2)^{m/2}\log(|x|^2 + c^2)$ $(m \in 2\mathbb{Z}_+)$. This contains the multiquadrics and inverse multiquadrics which correspond to the values $m = +1, -1$ respectively. The present example has the following important advantage over the previous one: since ϕ here is infinitely smooth, its Fourier transform decays rapidly (as a matter of fact, exponentially) at ∞, and further this transform is known to vanish nowhere on $\mathbb{R}^d\backslash 0$. Because of these two properties, we will show that an appropriate change of the parameter c from one h-level to another results in an improvement of the approximation properties of the corresponding spaces. Thus, we have two alternatives to choose from:

(b1) The stationary case: as before, we might localize ϕ to get ψ, define $S := S(\psi)$, and do not change ψ with h. This case becomes very similar in its analysis to the one considered in (a). In both of them the approximation orders are determined by the rate of growth of ϕ at ∞, or, more precisely, by the singularity order of $\widehat{\phi}$ at the origin.

(b2) The non-stationary case: here we change the parameter c with h, i.e., define $\phi_h := \phi(\cdot, c_h)$. Each ϕ_h is then localized to obtain a sequence $\{\psi_h\}_h$ of $L_2(\mathbb{R}^d)$-functions (again the type of localization used is insignificant, but it can be shown that the same periodic function u can be used for all ϕ_h). By letting $\{c_h\}_h$ grow to ∞, the fast decay of $\widehat{\phi}$ would provide approximation orders that supersede the orders obtained in the stationary case (b1).

It should be observed that for the ϕ considered in (a), the trade-of between singularity order of $\widehat{\phi}$ at 0 and its decay rate at ∞ provides no benefit. Because of the homogeneity of $\widehat{\phi}$, the order of its pole at 0 is the same as its decay rate at ∞.

(c) Here we consider again a one-parametric family $\phi(\cdot, c)$ of very smooth functions, which, further, are in L_2. For example, $\phi(\cdot, c) = e^{-c|\cdot|^2}$, or $\phi = (|\cdot|^2 + c^2)^{-(d+1)/2}$. In such a case $\widehat{\phi}$ admits no singularity at 0, and the only way to obtain positive approximation orders is as in (b2) above. In this regard, examples of the present type are advantageous over the examples in (b) since we do not need to localize our function.

In all the above examples, the Fourier transform $\widehat{\phi}$ of the basis function ϕ could have been identified on $\mathbb{R}^d \backslash 0$ with some smooth function. Since this is typical of radial basis functions, we adopt such an assumption from now on. In particular, in all subsequent analysis the notation $\widehat{\phi}$ stands also for the function which is defined on $\mathbb{R}^d \backslash 0$ and coincides there with the Fourier transform of ϕ.

§2.1. The PSI space $S(\phi)$ and the function Λ_ϕ

From the definition of $S(\psi)$ it is clear that $S(\psi)$ contains any finite linear combination of the shifts of ψ, and, furthermore, any function $s \in S(\psi)$ can be arbitrarily close approximated by these finite linear combinations. In terms of $\widehat{\psi}$ (which is known to be in $L_2(\mathbb{R}^d)$ since ψ is *assumed* to be so) we know that $\widehat{S(\psi)}$ contains all functions of the form $\tau\widehat{\psi}$, where τ is a trigonometric polynomial. The following characterization of *all* elements of $S(\psi)$, ($\psi \in L_2(\mathbb{R}^d)$) has been obtained in [2]:

$$f \in S(\psi) \quad \Longleftrightarrow \quad (f \in L_2(\mathbb{R}^d),\ \widehat{f} = \tau\widehat{\psi},\ \tau \text{ is } 2\pi-\text{periodic}). \tag{2.11}$$

We want to emphasize that the 2π-periodic τ in the above characterization is not assumed to be integrable or square integrable or even measurable (although it can be proved to be measurable). Further, as any L_2-function, the product $\tau\widehat{\psi}$ is defined almost everywhere, and consequently τ might be defined only a.e.

Proof of Theorem 1.9. To prove that $S(\psi_1) = S(\psi_2)$ it suffices to show that $\psi_1 \in S(\psi_2)$ and vice versa. Defining $\tau := u_1/u_2$, we know by the assumption on u_2 that τ is defined almost everywhere, and because the u_j's are 2π-periodic, so is τ. On the other hand,

$$\widehat{\psi}_1 = \tau\widehat{\psi}_2,$$

and $\psi_1 \in L_2(\mathbb{R}^d)$ by assumption, and therefore, by (2.11), $\psi_1 \in S(\psi_2)$, while the converse holds by symmetry. ■

The approximation properties of the space $S(\psi)$ are determined by behaviour of the function

$$\Lambda_\psi := (1 - \frac{|\widehat{\psi}|^2}{\widetilde{\psi}^2})^{1/2}, \tag{2.12}$$

where $\widetilde{\psi}$ is the following 2π-periodization of $\widehat{\psi}$:

$$\widetilde{\psi} := (\sum_{\beta \in 2\pi\mathbb{Z}^d} |\widehat{\psi}(\cdot + \beta)|^2)^{1/2}. \tag{2.13}$$

The convergence of the sum in the last definition can be taken in the L_1-sense. It is easy to see that, with

$$C := [-\pi, \pi]^d,$$

$\widetilde{\psi} \in L_2(C)$ if and only if $\psi \in L_2(\mathbb{R}^d)$. Note that Λ_ψ is non-negative and bounded by 1.

We already know, by Theorem 1.9, that, at least from a theoretical point of view, the specific choice of the localization process is not important. This choice lacks also any significance in the practical computation of approximation orders: the approximation orders depend on the behaviour of Λ_ψ (see below), but we observe that, because $\widehat{\psi} = \tau\widehat{\phi}$ and τ is 2π-periodic, the function Λ_ϕ is also well-defined and coincides a.e. with Λ_ψ, (subject to the assumption that τ is non-zero a.e.). Thus, to dispense entirely with ψ, we define

$$S(\phi) := S(\psi),$$

with ψ some (any) localization of ϕ. Note that, because of (2.11), the Fourier transform of every function f in $S(\phi)$ can be written in the form $\widehat{f} = \tau\widehat{\phi}$, for some 2π-periodic τ.

We could have defined $S(\phi)$ directly, without any recourse to localization, by an appropriate distributional interpretation of the product $\tau\widehat{\phi}$, τ 2π-periodic. There are two reasons for the indirect definition chosen above: first, the approximation map and its error analysis require the use a localization ψ; second, there is no real loss in the indirect definition, since by our assumption on ϕ, $\widehat{\phi}$ has an isolated singularity at the origin, hence of finite order, and consequently this singularity can always be removed, e.g., by an application to ϕ of a finite difference operator which annihilates polynomials of sufficiently high degree.

§2.2. The stationary case

For the sake of clarity, we first consider the stationary case. Thus, the space $S := S(\phi)$ is fixed and, for the given smooth f, we need to study the quantities

$$E(f, S^h) = h^{d/2} E(f(h\cdot), S). \tag{2.14}$$

The space of smooth functions is chosen as the potential space $W_2^k(\mathbb{R}^d)$:

$$W_2^k(\mathbb{R}^d) := \{f \in L_2(\mathbb{R}^d) : \ \|f\|_{W_2^k(\mathbb{R}^d)} := (2\pi)^{-d/2} \|(1+|\cdot|)^k \widehat{f}\|_{L_2(\mathbb{R}^d)} < \infty\}.$$

In case k is an integer, $W_2^k(\mathbb{R}^d)$ is the usual Sobolev space of the functions whose derivatives up to order k are in $L_2(\mathbb{R}^d)$.

Since the Fourier transform is an isometry on $L_2(\mathbb{R}^d)$, we might alternatively study the quantities $h^{d/2} E(\widehat{f(h\cdot)}, \widehat{S})$, with $\widehat{S}$ the range of $S(\phi)$ under the Fourier transform.

Our first step is **truncation**: instead of approximating $\widehat{f(h\cdot)}$, we approximate only the portion of it that is supported on some 0-neighborhood B, and add the rest of $\widehat{f(h\cdot)}$ to the error bound. Since $\widehat{f(h\cdot)} = h^{-d}\widehat{f}(\cdot/h)$, it is easy to prove (cf. Lemma 3.8 of [2]) that, for any fixed 0-neighborhood B,

$$h^{d/2} \|\widehat{f(h\cdot)}\|_{L_2(\mathbb{R}^d \backslash B)} \le c_B h^k \varepsilon_f(h) \|f\|_{W_2^k(\mathbb{R}^d)}, \tag{2.15}$$

for whatever $k \ge 0$ we choose, with $\varepsilon_f(h) \le 1$ and decays to 0 with h. Hence, for $f \in W_2^k(\mathbb{R}^d)$,

$$|E(f, S^h) - (2\pi)^{-d/2} h^{-d/2} E(\chi_B \widehat{f}(\cdot/h), \widehat{S})| = o(1) h^k \|f\|_{W_2^k(\mathbb{R}^d)},$$

with the $o(1)$ factor bounded independently of f, and where χ_B is the characteristic function of any fixed neighborhood of the origin. Therefore, the truncation process is harmless to the task of determining the approximation orders.

The function Λ_ϕ then enters the discussion because of the following result of [2]:

Result 2.16. *Let $f \in L_2(\mathbb{R}^d)$ and assume that* $\operatorname{supp} \widehat{f} \subset B \subset C$. *Then*

$$E(\widehat{f}, \widehat{S}) = \|\widehat{f}\Lambda_\phi\|_{L_2(\mathbb{R}^d)} = \|\widehat{f}\Lambda_\phi\|_{L_2(B)}.$$

From the last result we conclude that, if $B \subset C$, then

$$E(\chi_B \widehat{f}(\cdot/h), \widehat{S}) = \|\widehat{f}(\cdot/h)\Lambda_\phi\|_{L_2(B)}.$$

Thus, since $h^{-d/2}\|\widehat{f}(\cdot/h)\Lambda_\phi\|_{L_2(B)} = \|\Lambda_\phi(h\cdot)\widehat{f}\|_{L_2(B/h)}$, we arrive at the following:

Corollary 2.17. *Let $B \subset C$ be a neighborhood of the origin, and let $f \in W_2^k(\mathbb{R}^d)$, $k \geq 0$. Then*

$$E(f, S^h) = (2\pi)^{-d/2}\|\Lambda_\phi(h\cdot)\widehat{f}\|_{L_2(B/h)} + o(1)h^k\|f\|_{W_2^k(\mathbb{R}^d)},$$

with the $o(1)$ factor bounded independently of f.

We will make now specific assumptions on the basis function ϕ which will allow us to replace Λ_ϕ in the last corollary by a simpler expression. Throughout the rest of this subsection we assume that ϕ satisfies the following conditions:

(a) Smoothness condition: The function M_ϕ which is defined by

$$\mathrm{M}_\phi{}^2 := \sum_{\beta \in 2\pi\mathbb{Z}^d \backslash 0} |\widehat{\phi}(\cdot + \beta)|^2 \tag{2.18}$$

is (essentially) bounded on some neighborhood B of the origin.

Condition (a) is satisfied by all the functions ϕ we consider, since they all enjoy the stronger property $|\widehat{\phi}(w)| = O(|w|^{-(d/2+\varepsilon)})$ as $w \to \infty$. Without loss, we assume that the set B appearing in the above condition is identical with B in Corollary 2.17.

(b) "Ellipticity" condition: the function M_ϕ is bounded below away of zero around the origin, while the function $|\widehat{\phi}|$ (which, by assumption, is defined on $\mathbb{R}^d \backslash 0$) converges to ∞ at 0.

In all the examples considered here the boundness below of M_ϕ follows from the fact that $\widehat{\phi}$ is continuous on $\mathbb{R}^d \backslash 0$ and does not vanish either there or in some neighborhood of ∞.

Corollary 2.19. *Let $f \in W_2^k(\mathbb{R}^d)$, $k \ge 0$. If ϕ satisfies condition (a) above, then*

$$E(f, S^h) \le \text{const}\|\widehat{f}/(\widehat{\phi}(h\cdot))\|_{L_2(B/h)} + o(1)h^k\|f\|_{W_2^k(\mathbb{R}^d)}, \tag{2.20}$$

with const *independent of f and h, and with $o(1)$ being bounded independently of f. Furthermore, if ϕ also satisfies condition (b) above, then the converse inequality holds as well (with, possibly, a different constant).*

Proof: With M_ϕ as in (2.18), we observe that

$$\Lambda_\phi^2 = \frac{\mathrm{M}_\phi^2}{|\widehat{\phi}|^2 + \mathrm{M}_\phi^2} \le \frac{\mathrm{M}_\phi^2}{|\widehat{\phi}|^2}.$$

Assuming (a), it thus follows that $\Lambda_\phi|\widehat{\phi}|$ is bounded on B, and an application of Corollary 2.17 yields (2.20).

If, further, we assume Condition (b), then, assuming also (without loss, since we can change B if necessary) that $1/|\widehat{\phi}|$ is bounded on B, we conclude that

$$\mathrm{M}_\phi^2 + |\widehat{\phi}|^2 \le c|\widehat{\phi}|^2, \quad \text{on } B.$$

Since we also know that $1/\mathrm{M}_\phi$ is bounded on some 0-neighborhood, it follows that around the origin

$$\Lambda_\phi^2 = \frac{\mathrm{M}_\phi^2}{\mathrm{M}_\phi^2 + |\widehat{\phi}|^2} \ge \text{const}|\widehat{\phi}|^{-2}.$$

Again, an application of Corollary 2.17 proves that the converse inequality holds as well. ∎

Example 2.21. We proceed with case (a) of Example 2.10, i.e., assume that $|\widehat{\phi}| = 1/P$ (on $\mathbb{R}^d\backslash 0$) with $P(D)$ an elliptic operator, $\deg P > d/2$. It is then easy to verify that conditions (a) and (b) that were required in Corollary 2.19 hold, and therefore we obtain the following result.

Theorem 2.22. *Let ϕ be a fundamental solution of a constant coefficient homogeneous elliptic operator $P(D)$ of order $m > d/2$. Then ϕ provides approximation order m in the L_2-norm for every function $f \in W_2^m(\mathbb{R}^d)$. Further, for any non-trivial such f, $E(f, S^h) \ne o(h^m)$.*

Proof: By Corollary 2.19, the first statement of the theorem will be established as soon as we show that $\|\widehat{f}/\widehat{\phi}(h\cdot)\|_{L_2(B/h)} = O(h^m)$ for every

$f \in W_2^m(\mathbb{R}^d)$. Since $P = 1/\widehat{\phi} \sim |\cdot|^m$ on B (and, as a matter of fact everywhere), due to the ellipticity of $P(D)$, we can replace $\|\widehat{f}/\widehat{\phi}(h\cdot)\|_{L_2(B/h)}$ by

$$\||h\cdot|^m \widehat{f}\|_{L_2(B/h)} = h^m \||\cdot|^m \widehat{f}\|_{L_2(B/h)} \leq (2\pi)^{d/2} h^m \|f\|_{W_2^m(\mathbb{R}^d)},$$

and the desired result follows.

If we assume that $E(f, S^h) = o(h^m)$, then, because of Corollary 2.19,

$$\|\widehat{f}/\widehat{\phi}(h\cdot)\|_{L_2(B/h)} = o(h^m),$$

which implies, as above, that

$$h^m \||\cdot|^m \widehat{f}\|_{L_2(B/h)} = o(h^m).$$

Consequently, $\||\cdot|^m \widehat{f}\|_{L_2(B/h)} = o(1)$, which can happen only if $|\cdot|^m \widehat{f} = 0$. Therefore, f is a polynomial, and hence null, since $L_2(\mathbb{R}^d)$ contains no non-trivial polynomials. ■

The reader should observe that the assumption $m > d/2$ is essential: if $m \leq d/2$, ϕ is not locally in L_2.

An examination of the proof of Theorem 2.22 reveals that the actual choice of ϕ there played only a minor role. The properties of ϕ used were the satisfaction of conditions (a) and (b) and the fact that $\widehat{\phi} \sim |\cdot|^{-m}$ near the origin. Therefore, by arguments identical to those used in the last proof we obtain the following:

Theorem 2.23. *Assume that ϕ satisfies the conditions stated before Corollary 2.19, and that $\widehat{\phi} \sim |\cdot|^{-k}$ on (say, the same) 0-neighborhood B, for some $k > 0$. Then ϕ provides approximation order k for all functions in the potential space $W_2^k(\mathbb{R}^d)$. Moreover, for every non-trivial $f \in W_2^k(\mathbb{R}^d)$, $E(f, S^h) \neq o(h^k)$.*

Example 2.24. We now revisit case (b1) of Example 2.10, and since c is fixed here, we denote $\phi := \phi(\cdot, c)$. The common feature to all the basis functions considered in (b) of Example 2.10 is their Fourier transform (on $\mathbb{R}^d \backslash 0$):

$$\widehat{\phi}(w) = \text{const}(c, m, d)|w|^{-(m+d)/2} K_{(m+d)/2}(c|w|),$$

with K_ν being the modified Bessel function of third kind and order ν. The Bessel function is positive on $\mathbb{R}^d \backslash 0$ and decays exponentially at ∞, and from this we conclude that M_ϕ is bounded on C above and below by positive constants. Further, for $\nu > 0$ K_ν is known to a have a pole of order ν at the origin, and therefore, in case $m + d > 0$, we conclude that $\widehat{\phi} \sim |\cdot|^{-(m+d)}$ around the origin. Thus, we can apply Theorem 2.23 to the present ϕ with $k := m + d$ to obtain:

Corollary 2.25. *Let ϕ be as in Example 2.10 (b), and assume that $m+d > 0$. Then the results of Theorem 2.23 hold for this ϕ with $k = m + d$.*

Note that $\phi \in L_2(\mathbb{R}^d)$ whenever $-d < m < -d/2$, and hence for such a choice of m the definition of $S(\phi)$ does not require localization.

We now want to consider in the present example the extreme case when $m+d = 0$. Our analysis still applies to this case in the sense that conditions (a) and (b) required in Corollary 2.19 still hold here, and therefore this corollary reduces the study of $E(f, S^h)$ to the study of $\|\widehat{f}/\widehat{\phi}(h\cdot)\|_{L_2(B/h)}$. The difference between this case and the case $m + d > 0$ is that the singularity of the Bessel function is now of logarithmic type. i.e., $|\widehat{\phi}(w)| \sim |\log |w||$ around the origin, and thus the decay rates of $E(f, S^h)$ require the examination of

$$\|\widehat{f}(w)/\log |h|w||\,\|_{L_2(B/h)}.$$

Our result with respect to basis functions whose Fourier transform has a logarithmic singularity at the origin is as follows.

Theorem 2.26. *Assume that ϕ satisfies the following two conditions:*

(a) M_ϕ *is essentially bounded below and above by positive constants on some 0-neighborhood.*

(b) $\widehat{\phi}(w) \sim \log |w|$ *around the origin.*

Then:

(i) ϕ *provides no positive approximation order k for any $f \in W_2^k(\mathbb{R}^d)$ and any $k > 0$.*

(ii) $E(f, S^h) = o(1)$, *for all $f \in L_2(\mathbb{R}^d)$.*

(iii) For every $k > 0$, and every $f \in W_2^k(\mathbb{R}^d)$,

$$E(f, S^h) \le \mathrm{const}|\log h|^{-1}\|f\|_{W_2^k(\mathbb{R}^d)},$$

for all $h \le h_0$, where const *and h_0 depend on k but not on f.*

Proof: Statement (ii) follows from Theorem 1.7 of [2]. That theorem says that the property $E(f, S^h) = o(1)$, $\forall f \in L_2(\mathbb{R}^d)$ (referred to as "the density property") is equivalent to Λ_ϕ^2 having a Lebesgue value 0 at the origin. The result applies here since, by the assumption made on ϕ, it is clear that Λ_ϕ is continuous at the origin and vanishes there.

Now fix $k > 0$. To prove (i) and (iii), we follow the remarks preceding this theorem and consider the quantities

$$\|\widehat{f}/\log |h\cdot|\,\|_{L_2(B/h)}.$$

We already know that, up to a term of order $o(h^k)$, these numbers determine the decay rates of $E(f,S^h)$ (as $h\to 0$). Without loss we assume that B is the ball of radius $1/e$. For simplicity, we also assume that $h=e^{-l}$ for some integer l (other values of h are treated as below with some obvious modifications). We divide the ball $B/h=e^lB$ into annuli as follows:

$$R_0 := eB, \quad R_j := \{w : e^{j-1} \le |w| \le e^j\},\ j=1,...,l-1. \tag{2.27}$$

On R_j, $j>0$, we have the estimate $(\log(h|w|))^{-2} \le (j-l)^{-2}$, and thus, for $f\in W_2^k(\mathbb{R}^d)$,

$$\begin{aligned}\|\widehat{f}/\log(h|\cdot|)\|^2_{L_2(R_j)} \le& e^{-2k(j-1)}(j-l)^{-2}\||\cdot|^k|\widehat{f}|\|^2_{L_2(R_j)}\\ \le& \text{const}\, e^{-2kj}(j-l)^{-2}\|f\|^2_{W_2^k(\mathbb{R}^d)}.\end{aligned}$$

Summing this last estimate for $j=1,...,l-1$, we arrive at

$$\begin{aligned}\int_{(B/h)\backslash R_0} |\widehat{f}|^2(\log(h|\cdot|))^{-2} \le& \text{const}\|f\|^2_{W_2^k(\mathbb{R}^d)}\sum_{j=1}^{l-1} e^{-2kj}(l-j)^{-2}\\ =& \text{const}\|f\|^2_{W_2^k(\mathbb{R}^d)} e^{-2kl}\sum_{m=1}^{l-1} e^{2km}/m^2.\end{aligned}$$

Elementary integral tests show that the sum in the last expression is $O(e^{2kl}/l^2)$

$= O(h^{2k}/\log^2 h)$, and therefore

$$\int_{(B/h)\backslash R_0} |\widehat{f}|^2(\log(h|\cdot|))^{-2} \le \mathrm{const}\|f\|^2_{W_2^k(\mathbb{R}^d)}|\log h|^{-2}.$$

Also, on R_0 we have

$$\int_{R_0} |\widehat{f}|^2(\log(h|\cdot|))^{-2} \le (\log h)^{-2}\|\widehat{f}\|^2_{L_2(\mathbb{R}^d)} \le \mathrm{const}(\log h)^{-2}\|f\|^2_{W_2^k(\mathbb{R}^d)}.$$

We conclude that for some f-independent h_0 and const, and for every $h \le h_0$,

$$\|\widehat{f}/\widehat{\phi}(h\cdot)\|_{L_2(B/h)} \le \mathrm{const}\|f\|_{W_2^k(\mathbb{R}^d)}/|\log h|.$$

Substituting this into Corollary 2.19, we obtain (iii).

We now prove (i): let $f \in W_2^k(\mathbb{R}^d)$. Upon assuming that ϕ provides approximation order k to f, we conclude from Corollary 2.19 that

$$\|\widehat{f}/\log(h|\cdot|)\|_{L_2(B/h)} = O(h^k). \tag{2.28}$$

Let $j \in \mathbb{Z}$ and let R_j be the annulus in (2.27). For sufficiently small h, $R_j \subset B/h$, hence $\|\widehat{f}/\log(h|\cdot|)\|_{L_2(R_j)} = O(h^k)$ and since $(\log(h|\cdot|))^2 \le (\log h - j + 1)^2$ on R_j, we conclude that

$$\|\widehat{f}/\log(h|\cdot|)\|_{L_2(R_j)} \ge |\log h - j + 1|^{-1}\|\widehat{f}\|_{L_2(R_j)}.$$

Combining (2.28) with the last inequality, we arrive at

$$\limsup_{h\to 0} \frac{h^{-k}}{|\log h - j + 1|}\|\widehat{f}\|_{L_2(R_j)} < \infty,$$

which can happen only if $\widehat{f} = 0$ a.e. on R_j. Since j was arbitrary, $\widehat{f} = 0$ a.e. on $\mathbb{R}^d\backslash 0$, hence $f = 0$. ■

We have discussed cases (a) and (b) in Example 2.10. Let us briefly review case (c) there. In the two examples considered in (c) $\widehat{\phi}$ is a continuous positive function with exponential decay at ∞. Therefore, Λ_ϕ is a continuous positive function. If now $f \in L_2(\mathbb{R}^d) = W_2^0(\mathbb{R}^d)$ then, by Corollary 2.19, (with $S := S(\phi)$),

$$\begin{aligned} E(f, S^h) =& \mathrm{const}\|\Lambda_\phi(h\cdot)\widehat{f}\|_{L_2(C/h)} + o(1) \\ \ge& \mathrm{const}\|\widehat{f}\|_{L_2(C/h)} + o(1) \to \mathrm{const}\|\widehat{f}\|_{L_2(\mathbb{R}^d)}. \end{aligned}$$

Therefore, unless $f = 0$,

$$E(f, S^h) \ne o(1),$$

i.e., ϕ does not provide even density order zero for any non-zero L_2-function. Here, the only information used is the fact that Λ_ϕ is non-zero on C (and actually C could have been replaced by any neighborhood of the origin). Therefore, we have

Corollary 2.29. *Assume that* Λ_ϕ *is bounded below by a positive constant in some neighborhood of the origin. Then, for every non-trivial* $f \in L_2(\mathbb{R}^d)$,

$$E(f, S^h) \neq o(1).$$

In particular, this holds for $\phi = e^{-c|\cdot|^2}$ *and* $\phi = (|\cdot|^2 + c^2)^{-(d+1)/2}$.

§2.3. The non-stationary case

In the context of radial basis functions the notion of "non-stationary case" is connected with "spectral approximation orders". Here, we employ a sequence $\{\phi_h\}_h$ of basis functions each of which is some dilate $\phi(\lambda(h)\cdot)$ of one fixed function ϕ. In the analysis of the stationary case the approximation orders provided by ϕ were determined by the behaviour of $\widehat{\phi}$ on small neighborhoods of the lattice $2\pi\mathbb{Z}^d$. This is no longer the case here, and our conditions on $\widehat{\phi}$ are of global nature.

Define $S_h := S(\phi_h)$. The approximation order provided by $\{\phi_h\}_h$ to the function f is determined by the rate of decay (as $h \to 0$) of the numbers

$$E(f, S_h^h), \quad h > 0. \tag{2.30}$$

For $f \in W_2^k(\mathbb{R}^d)$, Corollary 2.17 shows that

$$E(f, S_h^h) = (2\pi)^{-d/2} \|\Lambda_{\phi_h}(h\cdot)\widehat{f}\|_{L_2(B/h)} + o(1)h^k \|f\|_{W_2^k(\mathbb{R}^d)}.$$

It is important to note that second term in the last equation, **the truncation error**, is independent of $\{\phi_h\}_h$.

The basic idea in the derivation of spectral orders is very simple: we estimate

$$\begin{aligned}\|\Lambda_{\phi_h}(h\cdot)\widehat{f}\|_{L_2(B/h)} \leq & \|\widehat{f}\|_{L_2(\mathbb{R}^d)} \|\Lambda_{\phi_h}(h\cdot)\|_{L_\infty(B/h)} \\ = & (2\pi)^{d/2} \|f\|_{L_2(\mathbb{R}^d)} \|\Lambda_{\phi_h}\|_{L_\infty(B)}.\end{aligned}$$

We do not want to specify in advance the smoothness class from which f is selected, and therefore we are unable to bound the truncation error in the way we did in Corollary 2.17. Instead, we recall, (2.15), that this truncation error has the form

$$h^{d/2} \|\widehat{f(h\cdot)}\|_{L_2(\mathbb{R}^d \setminus B)} = \|\widehat{f}\|_{L_2(\mathbb{R}^d \setminus (B/h))}.$$

In summary, we obtain the following seemingly coarse estimate for $E(f,S_h^h)$:

$$E(f,S_h^h) \le \|f\|_{L_2(\mathbb{R}^d)} \|\Lambda_{\phi_h}\|_{L_\infty(B)} + \text{const}\|\widehat{f}\|_{L_2(\mathbb{R}^d\backslash(B/h))}. \tag{2.31}$$

Our objective is to make the first term above decaying to zero so fast that, unless f is exceptionally smooth, the approximation rate provided by $\{\phi_h\}_h$ to f will be determined by the second term, i.e., by the smoothness of f. Recall that by (2.15) the second term here is $o(h^k)$ for every $f \in W_2^k(\mathbb{R}^d)$, hence in particular we have the following:

Proposition 2.32. *Assume that the sequence $\{\phi_h\}_h$ satisfies, for some 0-neighborhood B,*

$$\|\Lambda_{\phi_h}\|_{L_\infty(B)} = O(h^k), \quad \forall k \in \mathbb{R}_+.$$

Then, with $S_h := S(\phi_h)$,

$$E(f,S_h^h) = o(h^k)$$

for every $f \in W_2^k(\mathbb{R}^d)$.

In order to estimate $\|\Lambda_{\phi_h}\|_{L_\infty(B)}$, we write again

$$\Lambda_{\phi_h}^2 = \frac{\mathrm{M}_h^2}{|\widehat{\phi}_h|^2 + \mathrm{M}_h^2} \le \frac{\mathrm{M}_h^2}{|\widehat{\phi}_h|^2},$$

with $\mathrm{M}_h^2 := \sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} |\widehat{\phi}_h(\cdot+\beta)|^2$. In the stationary case we dispensed with M_h by assuming that in some small neighborhood of the origin M_h is bounded above; the desired properties of Λ_ϕ were then derived from the behaviour of $1/\widehat{\phi}$ at the origin. This is no longer the case: choosing

$$\phi_h := \lambda(h)^d \phi(\lambda(h)\cdot),$$

we have

$$\widehat{\phi}_h = \widehat{\phi}(\cdot/\lambda(h)),$$

and therefore, if $\lambda(h) \to 0$ with h and $\widehat{\phi}$ decays fast at ∞, the values M_h assumes around the origin tend to zero, with their decay rates being controlled by $\{\lambda(h)\}_h$. At the same time, $\widehat{\phi}_h$ also tend to 0, and thus an estimate of the form

$$\|\Lambda_{\phi_h}\|_{L_\infty(B)} \le \text{const}\|\mathrm{M}_h\|_{L_\infty(B)}$$

is not valid.

In order to focus our discussion and in view of the examples that initiate this discussion, we assume that

$$|\widehat{\phi}(w)|^2 \sim \sigma(|w|),$$

for some univariate positive function σ which is non-increasing on $[0,\infty)$. Let $\rho < 2\pi$; then for w in

$$B_\rho := \{w : \ |w| < \rho\}$$

we have:

$$\begin{aligned} \mathrm{M}_h(w)^2 = &\sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} |\widehat{\phi}((w+\beta)/\lambda(h))|^2 \\ \leq &\mathrm{const} \sum_{\beta\in 2\pi\mathbb{Z}^d\backslash 0} \sigma((|\beta|-\rho)/\lambda(h)) \\ \leq &\mathrm{const} \sum_{j=1}^{\infty} \sigma((2\pi j-\rho)/\lambda(h)) j^{d-1} \qquad (2.33) \\ \leq &\mathrm{const} \int_{2\pi-\rho}^{\infty} \sigma(t/\lambda(h)) t^{d-1}\, dt \\ = &\mathrm{const} \lambda(h)^d \int_{(2\pi-\rho)/\lambda(h)}^{\infty} \sigma(t) t^{d-1}\, dt. \end{aligned}$$

On the other hand, we can also estimate

$$|\widehat{\phi}(w/\lambda(h))|^2 \geq \mathrm{const}\, \sigma(\rho/\lambda(h)), \quad w \in B_\rho$$

which, together with (2.33), yields the following bound for $\|\Lambda_{\phi_h}\|_{L_\infty(B_\rho)}$:

$$\|\Lambda_{\phi_h}\|^2_{L_\infty(B_\rho)} \leq \mathrm{const} \lambda(h)^d \frac{\int_{(2\pi-\rho)/\lambda(h)}^{\infty} \sigma(t) t^{d-1}\, dt}{\sigma(\rho/\lambda(h))}.$$

In view of (2.31) and Proposition 2.32, we arrive at the following:

Theorem 2.34. *Assume that* $\widehat{\phi}(w)^2 \sim \sigma(|w|)$ *on* $\mathbb{R}^d$, *where* σ *is a univariate non-increasing positive function defined on* $[0,\infty)$. *Let* $S_h := S(\phi(\lambda(h)\cdot))$, $\lambda(h) > 0$. *Then, for* $0 < \rho < 2\pi$ *and* $f \in L_2(\mathbb{R}^d)$,

$$E(f, S_h^h) \leq c_\rho \|f\|_{L_2(\mathbb{R}^d)} \Big(\frac{\lambda(h)^d \int_{(2\pi-\rho)/\lambda(h)}^{\infty} \sigma(t) t^{d-1}\, dt}{\sigma(\rho/\lambda(h))} \Big)^{1/2} + \|\widehat{f}\|_{L_2(\mathbb{R}^d \backslash B_{\rho/h})}.$$

In particular, for $f \in W_2^k(\mathbb{R}^d)$,

$$E(f, S_h^h) = o(h^k)$$

if

$$\lambda(h)^d \frac{\int_{(2\pi-\rho)/\lambda(h)}^{\infty} \sigma(t) t^{d-1}\, dt}{\sigma(\rho/\lambda(h))} = o(h^{2k}).$$

The point in this theorem is to choose $\rho < \pi$ and to rely on the decay of σ (equivalently, $|\widehat{\phi}|$) at ∞. In order to capture Examples (b2) and (c) in Example 2.10, we assume $\widehat{\phi}$ to decay exponentially with order $r > 0$ and type $n > 0$:

$$\sigma(t) := e^{-2n|t|^r}.$$

Assuming that $\lambda(h) < 1$, we have for this σ that

$$\lambda(h)^d \int_{(2\pi-\rho)/\lambda(h)}^{\infty} \sigma(t) t^{d-1}\, dt \le \mathrm{const}_{n,r,d,\rho}\, \lambda(h)^r \sigma((2\pi-\rho)/\lambda(h)).$$

Consequently

$$\lambda(h)^d \frac{\int_{(2\pi-\rho)/\lambda(h)}^{\infty} \sigma(t) t^{d-1}\, dt}{\sigma(\rho/\lambda(h))} \le \mathrm{const} e^{-2nc\lambda(h)^{-r}},$$

with $c := (2\pi-\rho)^r - \rho^r$. Therefore, Theorem 2.34 reads in this case as follows:

Corollary 2.35. *If* $\widehat{\phi}(w) \sim e^{-n|w|^r}$ *on* $\mathbb{R}^d$, *and* $S_h := S(\phi(\lambda(h)\cdot))$, $0 < \lambda(h) < 1$, *then, for* $0 < \rho < \pi$ *and* $f \in L_2(\mathbb{R}^d)$,

$$E(f, S_h^h) \le \mathrm{const}\big(\|f\|_{L_2(\mathbb{R}^d)} e^{-nc\lambda(h)^{-r}} + \|\widehat{f}\|_{L_2(\mathbb{R}^d \backslash B_{\rho/h})}\big),$$

where $c = (2\pi-\rho)^r - \rho^r$. *In particular, for* $f \in W_2^k(\mathbb{R}^d)$,

$$E(f, S_h^h) = o(h^k)$$

if

$$e^{-nc\lambda(h)^{-r}} = o(h^k).$$

The last result indicates that the approximation properties of $\{S(\phi_h)\}_h$ are improved with the acceleration of the decay of $\lambda(h)$ to 0. However, when choosing $\{\lambda(h)\}_h$ it is good to keep in mind the effects this choice might have on the numerical stability of the approximation process: as $\lambda(h)$ becomes small, the function ϕ_h flattens, and approximation schemes from $S(\phi_h)$ become less and less stable.

Corollary 2.35 covers the examples of the Gaussian kernel $\phi = e^{-|\cdot|^2}$ ($r = 2$ $n = 1/4$), and $\phi = (|\cdot|^2+1)^{-(d+1)}$ ($r = n = 1$). Also, with some simple modifications, it can be used to cover the examples considered in Example 2.10(b). However, for the case $m + d > 0$ there, an improved version of Corollary 2.35 is available. This version takes a simultaneous account of the positive effect of the singularity of $\widehat{\phi}$ at the origin and its decay at ∞. We first state and prove a general result along these lines, and then apply it to Example 2.10(b2). In the theorem below, we use the notation

$$q_l(w) := \begin{cases} 1, & |w| \le 1, \\ |w|^l, & |w| \ge 1. \end{cases}$$

Theorem 2.36. *Assume that $\widehat{\phi}(w) \sim |w|^{-j} e^{-n|w|^r} q_l(w)$ on $\mathbb{R}^d$ for some positive j, n, r and real $l \le j$. Let $\phi_h := \lambda(h)^d \phi(\lambda(h)\cdot)$, $S_h := S(\phi_h)$. Then, for $0 < c < (2\pi)^r$ and for every $f \in W_2^k(\mathbb{R}^d)$, we have*

$$E(f, S_h^h) \le o(h^k) + \text{const}\lambda(h)^{-(j+l-r)} e^{-nc\lambda(h)^{-r}} \begin{cases} \|f\|_{W_2^k(\mathbb{R}^d)} h^k, & k \le j, \\ \|f\|_{W_2^j(\mathbb{R}^d)} h^j, & k \ge j. \end{cases}$$

Proof: For $f \in W_2^k(\mathbb{R}^d)$, $k \le j$, we estimate $\|\Lambda_{\phi_h}(h\cdot)\widehat{f}\|_{L_2(B/h)}$ as follows:

$$\begin{aligned} &\|\Lambda_{\phi_h}(h\cdot)\widehat{f}\|_{L_2(B/h)} \\ \le & h^k \, \| \, |\cdot|^k \widehat{f}\|_{L_2(B/h)} \, \| \, |\cdot|^{-k} \Lambda_{\phi_h}\|_{L_\infty(B)} \\ \le & \text{const}\, h^k \, \|f\|_{W_2^k(\mathbb{R}^d)} \, \| \frac{\mathrm{M}_{\phi_h}}{\widehat{\phi}_h} |\cdot|^{-k}\|_{L_\infty(B)}. \end{aligned}$$

To estimate $\|\frac{\mathrm{M}_{\phi_h}}{\widehat{\phi}_h}|\cdot|^{-k}\|_{L_\infty(B)}$, we use the fact that for $\beta \in 2\pi\mathbb{Z}^d \backslash 0$ and w sufficiently small, since $l \le j$,

$$\frac{q_l((w+\beta)/\lambda(h))}{|w+\beta|^j q_l(w/\lambda(h))} \le \frac{(\frac{|w+\beta|}{\lambda(h)})^l}{|w+\beta|^j} \le \text{const}\,\lambda(h)^{-l}\,|\beta|^{l-j} \le \text{const}\lambda(h)^{-l}.$$

This together with the assumptions made in theorem imply that

$$\frac{\widehat{\phi}_h(w+\beta)}{\widehat{\phi}_h(w)} = \frac{\widehat{\phi}((w+\beta)/\lambda(h))}{\widehat{\phi}(w/\lambda(h))} \le \text{const}|w|^j \lambda(h)^{-(j+l)} \frac{e^{-n|w+\beta|^r \lambda(h)^{-r}}}{e^{-n|w|^r \lambda(h)^{-r}}}.$$

Thus a bound on $\frac{M_{\phi_h}}{\widehat{\phi}_h}$ requires a bound on $\frac{M_{\psi_h}}{\widehat{\psi}_h}$, with $\psi_h := \psi(\lambda(h)\cdot)$, and ψ the inverse transform of $e^{-n|\cdot|^r}$. Such a bound has been computed in the proofs of Theorem 2.34 and Corollary 2.35, where it was shown that

$$\|\frac{M_{\psi_h}}{\widehat{\psi}_h}\|_{L_\infty(B_\rho)} \le \text{const}\, \lambda(h)^r e^{-nc\lambda(h)^{-r}},$$

with $c = (2\pi - \rho)^m - \rho^m$. Thus our combined estimate is the following

$$\frac{M_{\phi_h}(w)}{\widehat{\phi}_h(w)}|w|^{-k} \le \text{const}\lambda(h)^{-(j+l-r)}|w|^{j-k}e^{-nc\lambda(h)^{-r}}, \quad w \in B_\rho.$$

Since $k \le j$, $|w|^{j-k}$ is bounded on B_ρ, and our final estimate becomes

$$\|\Lambda_{\phi_h}(h\cdot)\widehat{f}\|_{L_2(B/h)} \le \text{const} h^k \lambda(h)^{-(j+l-r)}e^{-nc\lambda(h)^{-r}}\|f\|_{W_2^k(\mathbb{R}^d)}.$$

Invoking Corollary 2.17, we obtain the desired result for $k \le j$.

If $k > j$, we alternatively use the bound

$$\begin{aligned}&\|\Lambda_{\phi_h}(h\cdot)\widehat{f}\|_{L_2(B/h)}\\ \le& h^j \,\| \,|\cdot|^j \widehat{f}\|_{L_2(B/h)} \,\| \,|\cdot|^{-j}\Lambda_{\phi_h}\|_{L_\infty(B)}\\ \le& \text{const}\, h^j \,\|f\|_{W_2^j(\mathbb{R}^d)} \,\|\frac{M_{\phi_h}}{\widehat{\phi}_h}|\cdot|^{-j}\|_{L_\infty(B)},\end{aligned}$$

and follow the proof of the first case. The only change is the disappearance of the factor $|w|^{k-j}$. ■

We can now revisit case (b2) in Example 2.10, where we choose the parameter c to be 1. Recall that for ϕ there,

$$\widehat{\phi} = \text{const}|\cdot|^{-(m+d)/2}K_{(m+d)/2}(|\cdot|),$$

with K_ν the modified Bessel function of order ν. Since $K_\nu(w) \sim e^{-w}w^{-1/2}$ when $w \to \infty$, is positive on $\mathbb{R}_+$, and, in case $m + d > 0$, has a pole of order $(m + d)/2$ at the origin, we conclude that, for $m + d > 0$ and on $\mathbb{R}_+$,

$$w^{(m+d)/2}K_\nu(w) \sim e^{-w}q_{(m+d-1)/2}(w).$$

This shows that ϕ here satisfies the conditions required in Theorem 2.36 for the choice $j = m + d$, $n = r = 1$, and $l = (m + d - 1)/2$, thereby proving the following:

Corollary 2.37. *Let ϕ be any of the functions considered in Example 2.10(b), with $m + d > 0$. Define $S_h := S(\phi(\lambda(h)\cdot))$, $\lambda(h) > 0$. If $f \in W_2^k(\mathbb{R}^d)$ then, for $0 < c < 2\pi$ and with $\nu := -3(m + d - 1)/2$,*

$$E(f, S_h^h) \le o(h^k) + \text{const}\lambda(h)^\nu e^{-c/\lambda(h)} \begin{cases} \|f\|_{W_2^k(\mathbb{R}^d)}h^k, & k \le m + d,\\ \|f\|_{W_2^{m+d}(\mathbb{R}^d)}h^{m+d}, & k > m + d.\end{cases}$$

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