MATHEMATICAL PROGRAMMING
Spring 1996 Qualifying Exam

Answer 5 questions from the following 7 questions.

1. Solve the following quadratic program using Lemke's method:

$$\begin{array}{ll} \min & \frac{1}{2}x_1^2 + x_1x_2 + \frac{1}{2}x_2^2 + x_1 + 2x_2 + 7 \\ \text{such that} & x_1 + x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

2. Use linear programming duality theory to show that exactly one of the following two systems has a solution

   (I) $Ax = b$, $x \geq 0$.

   (II) $A'u \leq 0$, $b'u > 0$.

   Quote any theorems you use accurately. What is an alternative system for

   (III) $Cx \geq d$, $x \geq 0$.

3. Consider the problem $\min_{x\in X}\ f(x)$, where $f : R^n \to R$ is a differentiable convex function on R^n and X is a convex subset of R^n. Let $\bar{x} \in \bar{X}$ where $\bar{X}$ is the solution set $\text{argmin}_{x\in X}\ f(x)$, and let

$$\bar{S} := \{x \mid x \in X,\ \nabla f(x)(x - \bar{x}) = 0\}.$$

   Relate $\bar{S}$ to $\bar{X}$ in the strongest way possible.

4. Consider the interior penalty solution

$$x(\alpha) \in \arg\min_x\{f(x) - \alpha \sum_{j=1}^{m} \log\left(-g_j(x)\right) \Big| g(x) < 0\},\quad \alpha > 0,$$

   for the convex program:

$$\min_x\{f(x) \Big| g(x) \leq 0\},$$

   where $f : R^n \longrightarrow R$, $g : R^n \longrightarrow R^m$, are differentiable convex functions on R^n. Give the sharpest estimate you can for $\inf_x\{f(x) \Big| g(x) \leq 0,\}$ in terms of $x(\alpha), \alpha$ and m. What is the maximum error in your estimate?

5. Consider the set of constraints $\{ax \geq b,\quad x \quad binary\}$ where a and x are n-vectors.

   a) Construct an integer program (IP) that determines the maximum number of variables with value 1 in any binary vector satisfying the constraints.

   b) Consider the LP relaxation of the (IP) of part a). If this LP has finite optimal value z^*, show that the optimal value of (IP) is the largest integer $\leq z^*$. (Hint: consider rounding down an appropriate extreme point of the LP relaxation.)

   c) Does the result in b) hold if $ax \geq b$ is a system of constraints? Explain.

6. Consider the linear program:

$$(DNF) \qquad \max_{x,y} \sum_{i=1}^{n} w_i(x_i + y_i) \qquad s.t. \quad x_i + y_j \leq K_{ij} \quad \text{for} \quad (i,j) \in P, \quad x, y \geq 0$$

where P is a non-empty subset of $\{1, ..., n\} \times \{1, ..., n\}$.

a)Construct a network flow problem (with inequality constraints allowed at the nodes) which has the above problem as its dual. (State the constraints of the network flow problem in terms of node and arc constraints only, and its objective in terms of the flow variables. You may make a change of variables in (DNF) if you wish and construct a network flow problem equivalent to the transformed problem.)

b)Suppose that the set P has the property that for each i in $\{1, ..., n\}$ there is a j and and a j' such that $(i,j) \in P$ and $(j', i) \in P$. Give a closed form expression for a feasible solution of the network flow problem in terms of the data of (DNF), and explain why it satisfies the network constraints.

7. Suppose f is a closed proper convex function on $\mathbf{R}^n$. Show that in order for f to be Lipschitzian on $\mathbf{R}^n$ it is necessary and sufficient that the effective domain $\text{dom} f^*$ of the conjugate function f^* be bounded. Show further that any bound for $\text{dom} f^*$ furnishes a Lipschitz constant for f.