Theory Qual

Fall 2015

Please answer all four questions below.

1. Show that the following problem is NP-complete: Given a graph $G = (V, E)$ and a subset $R \subseteq V$, does there exist a subset $S \subseteq V$ such that every vertex in R has exactly one neighbor in S.

2. You are given a digraph $G = (V, E)$, two vertices $s, t \in V$, and two disjoint subsets $A, B \subseteq V$. You want to know whether on every (not necessarily simple) path from s to t the total number of occurrences of vertices in A is at least as large as the total number of occurrences of vertices in B.

   Show how to answer this question as efficiently as you can:

   (a) in the case where G is acyclic, and

   (b) in the general case.

3. Passengers with tickets on a full flight with n places are seated as follows.

   - The first passenger goes to a random seat.
   - The second passenger takes his assigned seat, unless it is occupied, in which case he takes a random seat.
   - This continues in like fashion for passengers 3, 4, …, n.

   Note that the last passenger has no choice and must take the only seat remaining.

   (a) Compute the probability that the last passenger gets his assigned seat.

   (b) Compute the expected number of passengers who get their assigned seats.

4. See next page.

4. This problem makes use of the following notions:

   - An *oracle circuit* C is a standard Boolean circuit with one additional type of gate, namely an oracle gate. When given a language $K \subseteq \{0,1\}^*$ as the oracle, an oracle gate with input x outputs a bit indicating whether $x \in K$. The oracle circuit C with K as the oracle for all oracle gates is denoted by C^K. The size of an oracle circuit C is the number of connections in C.
   - Given languages $K, L \subseteq \{0,1\}^*$, we define the language $\text{Succinct}^K(L)$ as consisting of all oracle circuits C such that $\text{tt}(C^K) \in L$, where $\text{tt}(C^K)$ denotes the truth-table of C^K, i.e., the string obtained by concatenating the output of C^K on all inputs from $\{0,1\}^m$ in lexicographical order, where m denotes the number of inputs of C.

   Fix K to be a language that is complete for EXP under polynomial-time mapping reductions. Define M to be the language consisting of all concatenations of the form $\text{tt}(C^K)y$, where C is an oracle circuit on m inputs, y is a binary string of length 2^m, and the size of C is at most the value of y (interpreting y as a binary number, possibly with leading zeroes).

   Show that:

   (a) $\text{Succinct}^K(M) \in (\Sigma_2^p)^K \subseteq \text{EXP}$.

   (b) If M is complete for NP under polynomial-time mapping reductions, then NEXP=EXP.

   You can make use of the fact that $\text{Succinct}^{\emptyset}(\text{SAT})$ is complete for NEXP under polynomial-time mapping reductions.

GOOD LUCK!!