Spring 2010 Qualifier Exam:
OPTIMIZATION

February, 2010

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer 5 out of 8 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. Consider the following linear program:

$$\begin{array}{lrcr} \min & 6x_1 - x_2 & & \\ \text{subject to} & x_1 - 2x_2 & \geq & -4, \\ & x_1 - x_2 & \geq & -7, \\ & x_1, x_2 & \geq & 0. \end{array}$$

   (a) Solve this problem.

   (b) Write down the dual of the given problem and the KKT conditions.

   (c) Find a dual solution u^* (by inspection of the KKT conditions).

   (d) Suppose that the objective function is replaced by the following quadratic:

   $$ax_1^2 + bx_2^2 + 6x_1 - x_2,$$

   where a and b are nonnegative parameters. Write down the modified KKT conditions for the resulting problem.

   (e) How large can we make a and b before the solution of the quadratic problem becomes different from the solution of the original linear program?

2. Let $X = \{x \in \mathbb{R}^n \mid Ax \leq b\} \neq \emptyset$ be a non-empty polyhedron, and let $\pi^T x \leq \pi_0$ be a valid inequality for X. (i.e. $\pi^T x \leq \pi_0 \; \forall x \in X$).

   Prove: There exists a $\pi_0' \in \mathbb{R}$ such that $\pi^T x \leq \pi_0'$ is a non-negative linear combination of inequalities in the system $Ax \leq b$.

   Hint: Use LP duality, with an appropriately defined objective function

3. Let $f_1(x)$ be a piecewise linear function of a scalar x. We represent the function in the two dimensional plane using n points $a_1 = (x_1, y_1), a_2 = (x_2, y_2), \ldots, a_n = (x_n, y_n)$. Suppose we have ordered the points so that $x_1 \leq x_2 \leq \ldots \leq x_n$. When n is very large we wish to approximate f_1 by another piecewise linear function f_2 that is defined by a subset of these points. This results in a savings in storage space and in use of the approximation.

   We define the error in the approximation by

   $$\beta \sum_{i=1}^{n} [f_1(x_i) - f_2(x_i)]^2$$

for some constant β, and the cost of storing a given line segment of the function as α. We wish to minimize the total cost as measured by the sum of the cost associated with storing the segments and the costs of the errors incurred.

Show how to formulate such a problem in a modeling language of your choice. Hint: the best formulation may be as a shortest path problem.

4. Consider the following set of network flow constraints:

$$Ax \leq b, \qquad 0 \leq x \leq c \qquad\qquad (N),$$

where the rows of $Ax \leq b$ correspond to m divergence constraints of the form $out_i - in_i \leq b_i$ for the flow vector x. You may assume that b has at least one positive and at least one negative component.

   (a) State a condition on the divergence vector b that is necessary in order for the constraints (N) to be feasible.

   (b) Give a numerical example showing that the condition in (a) is not sufficient for feasibility.

   (c) State a max flow problem (M) that may be solved in order to determine the feasibility of (N). (Describe additional nodes, arcs, and capacity constraints needed to define (M) and provide an explicit statement of (M).)

   (d) Prove that (N) is feasible if and only if the optimal value of (M) is a certain value (and state this value).

5. In a time-indexed formulation of a scheduling problem, the binary variables x_{1t} are used to represent the start-time of job 1, where $x_{1t} = 1$ if and only if job 1 starts at time t, for $t = 1, \ldots, T$, and similarly the binary variables x_{2t} are used to represent the start time of job 2. Since a job can only be started once, the formulation includes the constraints

$$\sum_{t=1}^{T} x_{jt} = 1, \quad j = 1, 2. \tag{1}$$

In our model, we require that job 1 should not start until job 2 starts (but it can start at the same time). This can be modeled with the following constraint:

$$\sum_{t=1}^{T} t x_{1t} \leq \sum_{t=1}^{T} t x_{2t}. \tag{2}$$

(a) Prove that the requirement that job 1 not start until job 2 starts can equivalently be modeled with the following set of constraints:

$$\sum_{s=t}^{T} x_{1s} \leq \sum_{s=t}^{T} x_{2s}, \quad t = 1, \ldots, T. \tag{3}$$

(b) Prove that the formulation given by constraints (1) and (3) is a *better* formulation than that given by (1) and (2). Your answer should include both a verification that (1) and (3) yields a relaxation that is always at least as good as that from (1) and (2), as well as an example that shows it can be strictly better.

6. Let $\{x_k\}$ be a sequence of vectors in $\mathbb{R}^n$ and f be a twice continuously differentiable function.

   (a) If $\{\nabla f(x_k)\}$ has an accumulation point at 0, does it follow that the sequence $\{x_k\}$ must have a stationary accumulation point?

   (b) Suppose that $\lim_{k\to\infty} x_k = x^*$ for some x^*, that $\lim_{k\to\infty} \nabla f(x_k) = 0$, with $\nabla^2 f(x_k)$ positive definite for all k sufficiently large. Are second-order necessary conditions always satisfied at x^*? Are second-order sufficient conditions always satisfied?

   (c) Suppose now that $\{\nabla f(x_k)\}$ has an accumulation point at 0 and that there are exactly two accumulation points $\tilde{x}$ and $\bar{x}$ of the sequence $\{x_k\}$. Must *both* of $\tilde{x}$ and $\bar{x}$ be stationary points of f? Must *at least one* of $\tilde{x}$ and $\bar{x}$ be a stationary point? How do your answers change if we have in addition that $\{x_k\}$ is a bounded sequence?

7. Let f and $g_1, \ldots, g_K$ be proper convex functions on $\mathbb{R}^n$, and let

$$C = (\operatorname{dom} f) \cap_{k=1}^{K} \operatorname{dom} g_k.$$

   Write $g(x)$ for the vector whose components are $g_1(x), \ldots, g_K(x)$ and $\mathbb{R}^K_-$ for the nonpositive orthant of $\mathbb{R}^K$. Let $F = \{x \in C \mid g(x) \in \mathbb{R}^K_-\}$, and suppose that $x_0 \in F$ is a local minimizer of f on F.

   Either prove the following set of statements, or give a counterexample. In a proof you may use the proper separation theorem, but no other advanced results.

   (a) x_0 is a global minimizer of f on F.

(b) There are elements $y^* \in \mathbb{R}^K_+$ and $\phi^* \in \mathbb{R}_+$ such that

$$(y^*, \phi^*) \neq 0, \qquad \langle y^*, g(x_0) \rangle = 0,$$

and such that for each $x \in C$ one has

$$\phi^* f(x) + \langle y^*, g(x) \rangle \geq \phi^* f(x_0) + \langle y^*, g(x_0) \rangle. \tag{4}$$

(c) If there is a point $\hat{x} \in C$ with $g(\hat{x}) < 0$, then $\phi^* > 0$.

8. Consider a problem of finding the best way to place bets totalling A dollars ($A > 0$) in a race involving n horses. Assume that we know the probability p_i that the ith horse wins and the amount $s_i > 0$ that the rest of the public is betting on the ith horse. The track keeps a proportion $1 - c$ of the total amount ($0 < c < 1$) and distributes the rest among the public in proportion to the amounts bet on the winning horse. Thus, if we bet x_i on the ith horse we receive

$$c\left(A + \sum_{i=1}^{n} s_i\right) \frac{x_i}{s_i + x_i}$$

if the ith horse wins. Formulate and solve the problem of maximizing the expected net return.

Hint: Assume that

$$\frac{p_1}{s_1} > \frac{p_2}{s_2} > \cdots > \frac{p_n}{s_n}$$

and show there exists a scalar λ^* such that the optimal solution is

$$x_i^* = \begin{cases} \sqrt{\frac{s_i p_i}{\lambda^*}} - s_i & \text{for } i = 1, \ldots, m^* \\ 0 & \text{for } i = m^* + 1, \ldots, n \end{cases}$$

where m^* is the largest index m for which $\frac{p_m}{s_m} \geq \lambda^*$.