Spring 2006 Qualifier Exam:
OPTIMIZATION

January, 2006

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book the area of the exam, your code number, and the question answered in that book. On *one* of your books list the numbers of *all* the questions answered. *Do not write your name on any answer book.*
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer 5 out of 8 questions.

POLICY ON MISPRINTS AND AMBIGUITIES:

The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems during the *first hour* of the exam. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. Solve:

$$\begin{array}{rl} \max_{x,y} & \min\{2x+y+1, y-x-4\} \\ \text{subject to} & x-y \geq 3 \\ & x+y \leq 1 \\ & x \geq 2 \end{array}$$

   Is the solution unique? Describe all optimal solutions. What is the optimal solution set if the constraint $x \geq 2$ is replaced by $x \geq 1$? Be sure to justify any reformulations that you carry out.

2. Let p be a given row vector in R^n and let $\mathcal{B}$ be a set of $m > 1$ given points in R^n. Use duality to demonstrate that either (1) p is in the convex hull of $\mathcal{B}$ or (2) there exists a vector v such that $b^i v < pv$ for all $b^i \in \mathcal{B}$. (Hint: consider the problem max z s.t. $b^i v + z \leq pv$ for $i = 1, \ldots m$.)

3. A load of 20 tonnes needs to be transported on a route passing through five cities, with a choice of three different modes of tranportation: road, rail and air. In any of the three intermediate cities it is possible to change the mode of tranport but the load uses a single mode of transport between two consecutive cites. The cost of transport in $ per tonne between the pairs of cities is:

	1-2	2-3	3-4	4-5
rail	30	25	40	60
road	25	40	45	50
air	40	20	50	45

   The costs for changing the mode of transport in $ per tonne is:

	rail	road	air
rail	0	5	12
road	8	0	10
air	15	10	0

   Formulate a model in AMPL or GAMS that determines how to organize the transport of the load at least cost. More credit will be given for solutions that lead to a tighter linear programming relaxation.

4. Existence of a conformal decomposition of a given flow x on a digraph G may be demonstrated by augmenting the original data by a node and arcs with appropriate flows, and then applying a decomposition process to the resulting flow $\bar{x}$ on the augmented digraph $\bar{G}$ (which is a circulation in which all divergences are 0). For simplicity in the following parts, you may assume $x > 0$ and that x was not a circulation.

   (a) If c is cycle flow in the decomposition of $\bar{x}$, show that the corresponding path flow p in G conforms to the original flow x. (Be sure to consider the two cases in which p is and is not a cycle flow and to verify all requirements of conformance.)

   (b) Show (by establishing an appropriate relationship to the decomposition in part (a)) that a conformal decomposition of x can also be obtained directly from G (without transforming it into a circulation) by using a decomposition process that takes into account the minimum of the absolute values of source and sink divergences as well as flow values.

   (c) Give a two-node, two-arc numerical example that illustrates failure of the modified version of the procedure in (b) in which the minimum of the absolute values of source and sink divergences is not taken into account when setting source-sink path flow values in the decomposition (i.e., source-sink path flow values are set in a manner analogous to part (a), based on path residual flow values only).

5. Consider the following two sets:

$$
\begin{aligned}
A \;=\; & \Big\{(x,y) \in \mathbf{R}^4 \times \mathbb{Z}^4 : \\
& x_1 + x_2 + x_3 + x_4 \le 20; \\
& x_1 \le 13y_1, x_2 \le 9y_2, x_3 \le 8y_3, x_4 \le 6y_4; \\
& x_i \ge 0, i = 1,...,4; y_i \in \{0,1\}, i = 1,...,4\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
B \;=\; & \Big\{(x,y) \in \mathbf{R}^4 \times \mathbb{Z}^4 : \\
& x_1 + x_2 + x_3 + x_4 \le 20; \\
& x_1 = 13y_1, x_2 = 9y_2, x_3 = 8y_3, x_4 = 6y_4; \\
& x_i \ge 0, i = 1,...,4; y_i \in \{0,1\}, i = 1,...,4\Big\}.
\end{aligned}
$$

Note that the only difference in A and B is the change from inequalities to equations in the third line of the definitions of the two sets.

(a) Is

$$y_1 + y_2 \le 1$$

a valid inequality for A? For B? For both sets, either give a short justification that the inequality is valid, or give a feasible point that shows that the inequality is not valid.

(b) Consider the point

$$x_1 = y_1 = 0; x_2 = 9, y_2 = 1; x_3 = 8, y_3 = 1; x_4 = 3, y_4 = \frac{1}{2}$$

Note that this point satisfies all the constraints of B *except for* the integrality restrictions on y. Give a valid inequality for B that cuts off this point (i.e., that this point does *not* satisfy).

(c) Consider again the point

$$x_1 = y_1 = 0; x_2 = 9, y_2 = 1; x_3 = 8, y_3 = 1; x_4 = 3, y_4 = \frac{1}{2}$$

Give a valid inequality for A that cuts off this point (i.e., that this point does *not* satisfy).

6. (a) Consider the following quadratic program in two variables:

$$\min\,(x_1-1)^2+(x_2-2)^2 \text{ subject to } -x_1-x_2\geq 0.$$

Find the function $q(\lambda)$ such that the (standard QP) dual of this problem is

$$\max_{\lambda\geq 0} q(\lambda).$$

(λ is a scalar variable.)

(b) Solve this dual formulation from part (a) and use it to deduce a solution to the original problem.

(c) By verifying the first-order necessary and second-order sufficient conditions, solve the following nonlinear programming problem:

$$\min\, -x_1x_2 \text{ subject to } 1-x_1^2-x_2^2\geq 0.$$

7. Let C be a closed convex set in $\mathbb{R}^n$ containing the origin in its relative interior, let Q be a positive semidefinite $n\times n$ matrix, and let q be some element of $\mathbb{R}^n$. Consider the optimization problem

$$\inf_x\{q^Tx \mid x^TQx\leq 1,\ x\in C\}. \tag{1}$$

(a) Develop a duality structure for (1) based on the embedding

$$F(x,\alpha)=\begin{cases} q^Tx & \text{if } x^TQx\leq 1+\alpha,\ x\in C\\ +\infty & \text{otherwise.}\end{cases}$$

You must exhibit the Lagrangian and the dual objective function, showing enough of your calculations to satisfy the grader that your analysis is correct.

(b) Now assume that $x_0\in\mathbb{R}^n$ attains the infimum in (1). Show that then there is an $\alpha_0^*\in\mathbb{R}_+$ such that for each $x\in C$ one has

$$q^Tx_0=q^Tx_0+\alpha_0^*(x_0^TQx_0-1)\leq q^Tx+\alpha_0^*(x^TQx-1).$$

You may use standard theorems of convex analysis, but if you do so then you must state the theorem you are using.

8. Let $\bar{x}$ be a solution of the minimization problem:

$$\min_{x} \{f(x) \mid g(x) \leq 0\}, \qquad \textbf{(MP)}$$

where $f : R^n \rightarrow R^1$ and $g : R^n \rightarrow R^m$. Consider the associated unconstrained penalty function minimization problems:

$$\min_{x \in R^n} P(x, \alpha) = \min_{x \in R^n} f(x) + \alpha \| \max(g(x), 0) \|_1, \qquad \textbf{(P1)}$$

$$\min_{x \in R^n} Q(x, \beta) = \min_{x \in R^n} f(x) + \beta \| \max(g(x), 0) \|_2^2, \qquad \textbf{(P2)}$$

where $\| \cdot \|_1$, $\| \cdot \|_2$ denote the 1-norm and 2-norm respectively.

(i) Do there exist finite values of α, β for which $\bar{x}$ solves (P1) and/or (P2)?

(ii) State precisely the conditions under which your answer to (i) holds and the threshold values $\bar{\alpha}$ and/or $\bar{\beta}$ if such exist, so that $\bar{x}$ solves (P1) and/or (P2). Prove your claim.