Fall 1997 Qualifier Exam:
MATHEMATICAL PROGRAMMING

Instructions: Answer 5 of the following 7 questions.

1. If the following LP is solvable, find a lower bound to its objective function on the feasible region:

$$\begin{array}{lrrrrrrr} \min & - & 47x_1 & + & 13x_2 & + & 22x_3 & \\ \text{s.t.} & - & 4x_1 & + & x_2 & - & 17x_3 & \geq 2 \\ & - & x_1 & + & x_2 & + & 39x_3 & \geq 1 \\ & & x_1 & , & x_2 & , & x_3 & \geq 0 \end{array}$$

   If the LP is unsolvable give a reason for that.

2. What is the radius of the largest ball that can be inscribed in the n-dimensional simplex:

$$\left\{x \middle| x \in \mathbb{R}^n,\ x \geq 0,\ -\frac{e'x}{\sqrt{n}} + \frac{1}{\sqrt{n}} \geq 0\right\},$$

   where e is a vector of ones?

3. Let $f: \mathbb{R}^n \to \mathbb{R}$, $g: \mathbb{R}^n \to \mathbb{R}^m$ be convex differentiable functions on $\mathbb{R}^n$ and let $(\bar{x}, \bar{u}) \in \mathbb{R}^{n+m}$ be a Karush-Kuhn-Tucker point of

$$\min_x f(x) \qquad \text{s.t. } g(x) \leq 0.$$

   (a) Give a lower bound to

$$\inf_x f(x) \qquad \text{s.t. } g(x) \leq b,$$

   for $b \in \mathbb{R}^m$ in terms of b, $\bar{x}$ and $\bar{u}$.

   (b) Give simple conditions on $\bar{x}$, $\bar{u}$ and b that ensure the lower bound of part (a) is equal to $\inf_x\{f(x)|g(x) \leq b\}$.

4. Consider the function f defined on $\mathbb{R}$ by

$$f(x) = (1/2)x^2 + \cos \alpha x + \sin \beta x,$$

where α and β are real constants with $|\alpha| + |\beta| < 1$.

(a) Show that f has a unique global minimizer x_*.

(b) Given a point $\tilde{x} \in \mathbb{R}$, explain how you could determine whether or not $\tilde{x} = x_*$.

(c) Determine whether or not f is convex on $\mathbb{R}$.

(d) Given $x_0 \in \mathbb{R}$, state an algorithm that will produce a sequence $\{x_0, x_1, \ldots\}$ *guaranteed* to converge to x_*.

(e) Determine the rate of convergence of your algorithm.

Justify all of your statements.

5. Suppose f is a closed proper convex function on $\mathbb{R}^n$ and λ is a fixed positive number. Define f_λ by

$$f_\lambda(x) = \inf_y g(x, y),$$

where

$$g(x, y) = f(y) + (2\lambda)^{-1}\|y - x\|^2.$$

(a) Show that f_λ is a convex function.

(b) Show that the infimum in y of $g(x, y)$ is attained at a unique point of $\mathbb{R}^n$.

Suggestion. Consider investigating the following questions:

- Is $g(x, \cdot)$ closed proper convex?
- Are its level sets bounded?
- Is it strictly convex?

6. A university wishes to order a total of at least t tablets of paper. It may divide this order among n suppliers $S_1, \ldots, S_n$. The prices charged by S_i are: S_i^1 per single tablet and S_i^{100} per box of 100 tablets where

$$S_i^1 < S_i^{100} < 100 S_i^1.$$

Each supplier also charges a fixed fee of K_i for handling an order and can supply at most $u_i < \infty$ tablets.

(a) Represent the problem of determining the amount to be ordered from each supplier in order to minimize the total cost of obtaining t tablets as an integer program (IP).

(b) Show that the optimal value $v_I(t)$ of (IP) is *not* a convex function of t.

(c) Show that the optimal value $v_C(t)$ of the continuous relaxation of (IP) is a convex function of t.

7. Consider a project consisting of three tasks, A, B and C, which must be accomplished subject to the constraints shown in the following table, where NET means "Not earlier than" and NLT means "Not later than":

Task	Duration	Constraints
A	5 weeks	
B	8 weeks	Start NET start of A and NLT 3 weeks after start of A. Finish NLT 5 weeks after finish of A.
C	2 weeks	Start NET 2 weeks after start of B and NET finish of A. Finish NLT finish of B.

Construct a network formulation of this project and its constraints and carry out the steps of an algorithm of your choice (with starting point of your choice) to show that the project can be completed in 8 weeks. What is the critical path for this project? In which task would any slippage affect project completion time?