Mathematical Programming Qualifying Exam

Fall 1994

Answer any 6 out of the following 8 questions.

1. The following optimal tableau has been obtained by processing a linear programming problem whose objective function was $c^T x$ and whose constraints were of the form $Ax \leq a$, $x \geq 0$. Here s is a vector of non-negative slack variables. The tableau represents a set of linear equations, of which the right-hand side is shown in the rightmost column. The first equation began as $z - c^T x = 0$.

z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	$=$
1	2	5	0	0	0	1	3	48
0	3	-1	0	0	1	2	1	4
0	1	4	1	0	0	-1	4	1
0	2	-2	0	1	0	1	0	2

Answer the following questions, justifying your answers.

(a) What were the dimensions of c, A and a? Was the original problem a maximization or a minimization problem?

(b) What are the optimal values of the decision variables $x_1, \ldots x_4$, and what is the optimal objective value z_*?

(c) Suppose that a_2 were increased to $a_2' := a_2 + 0.5$. What are the new values of $x_1, \ldots, x_4$ and z_*?

(d) Suppose that a_2 were increased to $a_2'' := a_2 + 2.0$. What are the new values of $x_1, \ldots, x_4$ and z_*?

2. Suppose that the problem

$$\min_{x} \ cx \qquad s.t. \ Ax = b, \ 0 \leq x,$$

where x is a vector in R^n and b is a vector in R^m , has an optimal solution.

Prove that the problem

$$\min_{x,y} \ cx \qquad s.t. \ Ax - y = 0, \ 0 \leq x, \ -\mid b \mid \leq y \leq \mid b \mid$$

(where $\mid b \mid$ is the vector whose components are $\mid b_i \mid$) also has an optimal solution.

3. Consider the problem

$$\min_x f(x) \qquad s.t.\ g(x) \leq 0,\ \ -b \leq x \leq b$$

where $f\colon R^n \to R$, $g\colon R^n \to R^m$ are differentiable convex functions on R^n and $b \in R^n$. Suppose that the feasible region is nonempty. Can the minimum value be lower than $f(0) - b[(\nabla f(0))_+ + (-\nabla f(0))_+]$? Justify your answer. (For a vector $c \in R^n$, $(c_+)_i = \max\{c_i, 0\}$, $i = 1, \ldots, n$.)

4. Let $f\colon R^n \to R$ and $g\colon R^n \to R^m$ be convex functions on R^n, let b and c be points in R^m, such that $b < c$ and such that there exists a point in R^n satisfying $g(x) < b$. Suppose that

$$x_b \in \arg\min_x \{f(x)|g(x) \leq b\} \tag{1}$$
$$x_c \in \arg\min_x \{f(x)|g(x) \leq c\} \tag{2}$$

Give an upper bound on $\|u_c\|_1$, the 1-norm of the optimal Lagrange multiplier of (2), in terms of $f(x_b)$, $f(x_c)$, b, and c, and nothing else. **Hint:** Use the KKT saddlepoint optimality criteria.

5. Suppose that $(\bar{x}, \bar{u}) \in R^{n+m}$ is a stationary point of the augmented Lagrangian

$$L(x, u, \alpha) = f(x) + \frac{1}{2\alpha}[\|(\alpha g(x) + u)_+\|^2 - \|u\|^2]$$

where $f\colon R^n \to R$ and $g\colon R^n \to R^m$ are convex, differentiable functions on R^n, $\|\cdot\|$ denotes the 2-norm, $(z_+)_i = \max\{z_i, 0\}, i = 1, \ldots, m$ for $z \in R^m$, and $\alpha > 0$. Relate $\bar{x}$ to

$$\min_x f(x) \qquad s.t.\ g(x) \leq 0,$$

and establish your claim.

6. Consider the piecewise-linear optimization problem

$$\min_x \sum_{i,u,j,v} f(x_{iu}, x_{jv}) \qquad s.t.\ Ax \leq b$$

where x is a vector of flows on a digraph, $Ax \leq b$ is a set of linear constraints which imply $0 \leq x_{kl} \leq M$ for some constant M for all flow variables x_{kl}, and $f(x_{iu}, x_{jv})$ is an interaction penalty defined to be 1 if $x_{iu} \cdot x_{jv} > 0$ for $i \neq j$ and $(u, v) \in R$, where R is a given set of "related" nodes, and defined to be 0 otherwise.

Formulate this problem as a *linear mixed-integer program.*

7. Consider the network flow problem (with n nodes):

$$\min_x \; cx \quad s.t. \; div(i) = b_i \quad (i = 1, ..., n-1), \;\; div(n) \geq b_n, \;\; 0 \leq x$$

where div(i)=$\sum_j x_{ij} - \sum_j x_{ji}$.

Show that the existence of an optimal solution to this problem implies

1) $\sum_{i=1}^{n-1} b_i \leq -b_n$, and

2) that there exists an equivalent problem involving n nodes in which all of the node constraints are divergence equations (state this problem and explain why it is equivalent).

8. Let f be a proper convex function on $\boldsymbol{R}^2$. You are given that f is finite at the origin and at the points $(1, 2)$ and $(2, 1)$. Answer the following questions, justifying your answers.

   (a) Prove that there exist numbers α_1 and α_2 such that the function

   $$g(x) := f(x) - \alpha_1 x_1 - \alpha_2 x_2$$

   has a minimum at the point $(1, 1)$.

   (b) Prove that f satisfies a Lipschitz condition on some neighborhood of $(1, 1)$.

   (c) Give the best lower bound you can for the value of the recession function of the conjugate f^* at the point $(0, 5)$. Exhibit a function f satisfying the above conditions, for which your bound actually equals the value in question.