Theory Qual

Fall 2014

Please answer all four questions below.

1. In the *Most Neighbors* problem, you are given an undirected graph $G = (V, E)$ and asked to find a set $S \subset V$ such that the number of neighbors of S outside of itself, $|\Gamma(S) \setminus S|$, is maximized. Here $\Gamma(v)$ for a node v denotes all of the nodes u such that $(u, v) \in E$, and $\Gamma(S) = \cup_{v \in S} \Gamma(v)$. Prove that the Most Neighbors problem is NP-Hard.

2. You are given a tree on n nodes and your goal in this question is to generate a vertex cover of the tree uniformly at random. The tree has many vertex covers, e.g., the set of all nodes, the set of all nodes on even levels, etc. Give an efficient algorithm for generating a uniformly random vertex cover of the tree, that is, all vertex covers should be equally likely to be generated. For simplicity you may assume that the tree is binary (but not necessarily complete).

3. We call a Turing machine M deciding a language L *optimal* if for every Turing machine N deciding L there exists a polynomial p such that
$$(\forall x) \, t_M(x) \leq p(t_N(x)),$$
where $t_M(x)$ denotes the running time of M on input x.

   We call an infinite sequence of inputs $x_1, x_2, \ldots$ hard for M if x_i can be generated in time polynomial in i but $t_M(x_i)$ is not polynomially bounded in i.

   (a) Show that an optimal Turing machine has no hard sequences $x_1, x_2, \ldots$ in L with $|x_i| \geq i$ for all i.

   (b) Prove the same result without any restriction on $|x_i|$.

4. Let P be a set of n points in $[0, 1]^2$. A point (x, y) in P is called undominated if for every other point (x', y') in P, either $x' < x$ or $y' < y$. For simplicity, you may assume that all of the x- and y-coordinates of the n points in P are distinct.

   (a) Develop an algorithm for finding all of the undominated points in a given set P. The running time of your algorithm should be $O(n \log n)$.

   (b) Can you modify your algorithm (or its analysis) to obtain a running time of $O(n \log m)$, where m is the number of undominated points?