Theory Qual: Date: Sept 15, 2008

YOU SHOULD ATTEMPT ALL 4 PROBLEMS

1. A positive number n is called *squarefree* if no square divides it except for 1. Give an $O(n)$ algorithm that lists all the squarefree numbers in $\{1, \ldots, n\}$. You can treat arithmetic operations such as addition or subtraction or multiplication or integer division as operations with unit cost.

2. Suppose there exists a randomized polynomial-time algorithm with the following behavior: On input an unsatisfiable Boolean formula, it outputs "no" with probability at least 3/4; on input a Boolean formula with exactly one satisfying assignment, it outputs "yes" with probability at least 3/4; on any other input, the behavior of the algorithm is unspecified. Prove that the existence of such an algorithm implies that the polynomial-time hierarchy collapses. To which level can you prove the PH collapses to?

3. A pair of vertices s and t in a graph G is called 2-connected if there are at least two edge-disjoint paths between s and t in the graph. Given a connected graph $G = (V, E)$ with weights w_e on edges and vertices s and t, a 2-route s-t cut is a set of edges $E' \subset E$ such that upon removing E' from the graph, s and t are not 2-connected. The weight of a cut E' is given by $w(E') = \sum_{e \in E'} w_e$.

   a. Prove that s and t are 2-connected if and only if every s-t cut in the graph contains at least 2 edges.

   b. Use part (a) to give a polynomial time algorithm for finding the minimum weight 2-route s-t cut in a given graph.

4. In honor of the upcoming election, consider the problem of apportioning seats in a legislature. Suppose there are m states, with populations $p_1, \ldots, p_m$, and the legislature has n seats. (Currently for the U.S. Congress, $m = 50$ and $n = 435$.) If the numbers

$$q_i := n \frac{p_i}{\sum_i p_i}$$

   were all integers, we would give the i-th state q_i seats. However, they usually are not, so some adjustment is necessary.

Alexander Hamilton suggested the following rule: round each q_i down to the closest integer, and provisionally give the i-th state a number of seats equal to this integer. The remaining seats are to be assigned, one each, to the states with the largest values of q_i.

a. Explain why this problem can be solved in $O(m)$ steps. For this problem, a "step" means an arithmetic operation (addition, subtraction, multiplication, division with remainder), or a comparison, on integers.

b. What problems (such as unfairness from certain perspective, or other undesirable features) might arise by the use of Hamilton's rule?

Please be concise.