OPTIMIZATION

Fall 2005 Qualifying Exam
September, 2005
Instructions: Answer 5 out of 8 questions

1. (a) Consider the approximation problem

$$\min \left\| \begin{bmatrix} x_1 + 2x_2 - 1 \\ 3x_1 + x_2 + 3 \end{bmatrix} \right\| \text{ subject to } x_1 \geq 0,\, x_2 \geq 0.$$

Express this problem as a linear program for the cases in which the norm in the objective is an ℓ_1-norm or an ℓ_∞-norm. (Do not try to solve these problems.)

(b) Consider the problem

$$\begin{array}{rcl} \min 3x_1 - 2x_2 & & \text{subject to} \\ x_1 - x_2 & \geq & 7 + t, \\ -x_1 & \geq & -5 + t \\ x_1, x_2 & \geq & 0. \end{array}$$

Find a value of t for which this problem is feasible. Starting from this value, find the solutions of this problem for *all* values of t.

2. Consider the following linear program (LP) for $x \in \mathbb{R}^n$:

$$\min_x \quad cx \text{ s.t. } Ax = b,\ x \geq 0.$$

Assume that (LP) has an optimal solution. Given any row vector $d \in \mathbb{R}^n$, prove that there exists an $\varepsilon > 0$ such that if the perturbed problem (LPP) defined by

$$\min_x \quad (c + \rho d)x \text{ s.t. } Ax = b,\ x \geq 0,$$

with $0 < \rho < \varepsilon$ has a unique optimal solution $\bar{x}$, then $\bar{x}$ is also optimal for (LP). For a given d, discuss how ε could be computed by considering the set of all extreme points of the feasible set of (LP).

3. In the game of Sudoku, a nine by nine square is given with some of the 81 squares filled in with digits $(1, 2, \ldots, 9)$, for example:

	c1	c2	c3	c4	c5	c6	c7	c8	c9
r1	2			6	7				
r2			6				2		
r3	4						8		1
r4	5					9	3		
r5		3						5	
r6			2	8					7
r7			1						
r8	7		8				6		
r9					5	3			8

The game involves simply placing digits in all the remaining squares so that:

(a) In each row, each of the digits $1, 2, \ldots, 9$ appears exactly once.

(b) In each column, each of the digits $1, 2, \ldots, 9$ appears exactly once.

(c) In each 3 by 3 subblock (as outlined above), each of the digits $1, 2, \ldots, 9$ appears exactly once.

So for example, a solution of the above game is:

	c1	c2	c3	c4	c5	c6	c7	c8	c9
r1	2	8	3	6	7	1	5	9	4
r2	1	9	6	5	4	8	2	7	3
r3	4	7	5	3	9	2	8	6	1
r4	5	1	7	4	6	9	3	8	2
r5	8	3	9	2	1	7	4	5	6
r6	6	4	2	8	3	5	9	1	7
r7	3	2	1	9	8	6	7	4	5
r8	7	5	8	1	2	4	6	3	9
r9	9	6	4	7	5	3	1	2	8

Set up a model in AMPL or GAMS whose solution will solve this game.

4. Consider the following n-node network flow problem in which all data are assumed to be integer and u is assumed to be positive:

$$\min_{x} \quad cx \text{ s.t. } Ax = s, \; 0 \leq x \leq u.$$

Denote by $q(p)$ the dual function given by the optimal value of the relaxed problem:

$$\min_{x} \quad (cx + p(s - Ax)) \text{ s.t. } 0 \leq x \leq u.$$

For a given integer price vector p, derive an expression in terms of the problem data for the directional derivative of q at p in the direction of the i *th* unit vector $d_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ If none of these directional derivatives is positive, is p guaranteed to be optimal for the dual problem max $q(p)$? Explain.

5. Consider the two-variable set

$$S = \{(x, y) : x + y \geq 1.6; x, y \geq 0; y \in \mathbb{Z}\}.$$

Let $\text{conv}(S)$ be the convex hull of S, and let $S' = \{(x, y) : x + y \geq 1.6; x, y \geq 0\}$; that is, S' is the set S with the integrality restriction on y relaxed.

   (a) List the extreme points of S' and $\text{conv}(S)$.

   (b) There is one nontrivial facet-defining inequality for $\text{conv}(S)$. State this inequality, and prove 1) that it is valid for S, and 2) that it is facet-defining for $\text{conv}(S)$.

   (c) Consider the generally defined set

   $$T = \{(x, y) : x + y \geq b; x, y \geq 0; y \in \mathbb{Z}\},$$

   where $0 < \lfloor b \rfloor < b < \lceil b \rceil$. State a nontrivial, facet-defining valid inequality for this set in terms x, y, and b. (*Hint:* It may help you to define and use $f = b - \lfloor b \rfloor$ in the definition.)

6. Consider the minimization problem:

$$\min_x \{f(x) \mid Ax \leq b\}, \qquad (*)$$

where $f : R^n \to R^1$ is convex and differentiable on R^n, $A \in R^{m\times n}$ and $b \in R^m$. Assume that the feasible region is nonempty and that:

$$Ax \leq 0 \text{ has } \textbf{no} \text{ solution } x \neq 0. \qquad (**)$$

   (a) Does $(*)$ have a solution? Prove your claim.

   (b) Set up a linear program that will give a lower bound to:

$$\inf_x \{f(x) \mid Ax \leq b\}. \qquad (***)$$

   (c) Prove that your linear program provides a lower bound to $(***)$.

7. Suppose f is a function from $\mathbb{R}^n$ to $(-\infty,+\infty]$ that is finite at some point. For a real number $\lambda > 0$ define the *Moreau envelope function* e^f_λ for each $x \in \mathbb{R}^n$ by

$$e^f_\lambda(x) = \inf_{y\in\mathbb{R}^n} \{f(y) + (2\lambda)^{-1}\|y-x\|^2\}. \qquad (1)$$

In answering the following questions you may use known theorems if you wish, but you must state what theorem you are using.

   (a) Show that e^f_λ minorizes f; that is, for each $x \in \mathbb{R}^n$ $e^f_\lambda(x) \leq f(x)$.

   (b) Is e^f_λ everywhere finite? Prove, or give a counterexample.

For the rest of this question, assume in addition that f is a closed convex function.

   (c) Show that e^f_λ is convex.

   (d) Show that some affine function a minorizes e^f_λ.

   (e) Show that for each $x \in \mathbb{R}^n$ the infimum in the definition of $e^f_\lambda(x)$ is attained.

8. (a) Write down the augmented Lagrangian function (with parameter α) for the nonlinear programming problem

$$\min f(x) \text{ such that } g(x) \leq 0$$

where f and g are differentiable functions on $\mathbf{R}^n$.

(b) For what values of the parameter α does a stationary point of the augmented Lagrangian satisfy the Karush-Kuhn-Tucker conditions of the original nonlinear programming problem? Prove your claim.

(c) For what reasons would you take a large value of α in your computations?