Theory Qual

Fall 2005

Please answer all 4 questions below.

1. Show that the following problem is NP-complete: Given a 3-CNF formula, does there exist an assignment that satisfies exactly one literal of every clause?

   Hint: Use 3-colorability.

2. Recall that a path in a directed graph G is a sequence of $\ell \geq 0$ vertices $v_1, v_2, \ldots, v_\ell$ such that (v_i, v_{i+1}) is an edge of G for $1 \leq i < \ell$.

   You are given a directed acyclic graph and want to partition the vertices into as few paths as possible. More precisely, you would like to construct paths $P_1, P_2, \ldots, P_k$ with k as small as possible such that every vertex appears in exactly one P_j, $1 \leq j \leq k$.

   Give a polynomial-time algorithm for this problem.

3. Let χ denote the characteristic sequence of a language L over $\{0, 1\}$, i.e., the i-th bit of χ indicates whether the ith string over $\{0, 1\}$ in the lexicographic order belongs to L.

   Show that if the number $.\chi$ (a period followed by the binary sequence χ) is rational then L is regular.

   Hint: The implication also holds for the alphabet $\{0\}$ instead of $\{0, 1\}$. Think about that case first. A proof for that case will give you partial credit.

4. Let L denote a language over the alphabet Σ. Recall that $L \in \mathrm{MA}$ iff there exists an integer c and a probabilistic polynomial-time machine M such that for any nonnegative integer n and any $x \in \Sigma^n$,

   $$\begin{aligned} x \in L &\Rightarrow (\exists\, y \in \Sigma^{n^c})\, \Pr[M(x, y) \text{ accepts }] > 2/3 \\ x \notin L &\Rightarrow (\forall\, y \in \Sigma^{n^c})\, \Pr[M(x, y) \text{ accepts }] < 1/3, \end{aligned}$$

   and that $L \in \mathrm{PP}$ iff there exists a probabilistic polynomial-time machine N such that for any nonnegative integer n and any $x \in \Sigma^n$,

   $$\begin{aligned} x \in L &\Rightarrow \Pr[N(x) \text{ accepts }] > 1/2 \\ x \notin L &\Rightarrow \Pr[N(x) \text{ accepts }] < 1/2. \end{aligned}$$

   Show that MA is contained in PP.

G O O D   L U C K ! !