Theory Qual

Fall 2004

Please answer all 4 questions below.

1. This question deals with a simple form of gerrymandering, a trick used to bias the outcome of elections. (Don't worry if you have never heard the term before.)

   You are given the list of registered voters for a number of precincts. Each voter belongs to one precinct and is a member of one of two parties. The list contains the precinct and party membership for each voter. You want to figure out whether it is possible to group the precincts into two districts such that (i) each district contains the same number of precincts and (ii) one party has a strict majority in both districts.

   Devise a polynomial-time algorithm for this problem.

2. Consider the following randomized process to grow a tree on n vertices. We begin in step 1 with a single isolated vertex v_1. In each subsequent step $k \geq 2$, we introduce a new vertex v_k and draw an edge from v_k to a vertex chosen uniformly at random from $v_1, v_2, \ldots, v_{k-1}$. We stop after step n.

   What is the expected number of leaves in the resulting tree? Your answer should be a simple exact expression.

3. Show that the following problem is #P-complete: Given a directed graph G, compute the number of directed simple cycles in G, where "simple" means that the cycle does not repeat any vertex. You can use the fact that counting the number of Hamiltonian cycles in a directed graph is #P-complete.

4. Recall that for a complexity class $\mathcal{C}$, the class $\mathcal{C}/\text{poly}$ consists of all languages L for which there exists a language $L' \in \mathcal{C}$, an integer c, and a sequence $a_0, a_1, a_2, \ldots$ with $|a_n| = O(n^c)$ such that $L = \{\, x \, : \, \langle x, a_{|x|} \rangle \in L' \,\}$.

   (a) Show that $\text{NEXP}/\text{poly} = \text{coNEXP}/\text{poly}$.

   (b) Show that if $\text{NP}/\text{poly} = \text{coNP}/\text{poly}$ then the polynomial-time hierarchy collapses. Try to deduce a collapse as far down as you can.

G O O D   L U C K ! !