MATHEMATICAL PROGRAMMING

Fall Qualifying Exam
September 24, 2001
Instructions: Answer any 5 of the following 8 questions.

1. A professor needs to order books for three classes he is teaching: cs195, ie595, and ma695. All three books he needs can be obtained from any of three online booksellers: desert.com, jungle.com, and ocean.com. Each bookseller calculates shipping charges as a fixed charge, plus an marginal cost for each book shipped, according to the following table:

	fixed charge ($)	marginal charge ($)
desert.com	20	.50
jungle.com	10	1.0
ocean.com	5	1.0

(For example, if no books are ordered from desert.com, then of course there is no charge from them, while if a total of 6 books are ordered, the shipping charge is $20+6(.50) = 23$ dollars.) The prices charged by the booksellers for the three books (in dollars) are given in the following table:

	cs195	ie595	ma695
desert.com	2	4	2
jungle.com	2.5	3	2
ocean.com	2	3	2.5

The professor needs to order a total of 10 books for cs195, 10 books for ie595, and 15 books for ma695. He wants to minimize the total cost (shipping charges plus purchase prices).

(a) Formulate the professor's problem as an integer programming problem. You may use real variables to represent the quantity of each book ordered from each bookseller (though in practice we would need a solution with integer values for these variables).

(b) Formulate the problem in GAMS.

(c) Suppose that jungle.com decide to raise its price for the ma695 book from 2 dollars to 2.5 dollars. Can you tell by inspection whether or not it is possible for the solution to include any purchases from jungle.com? Explain your reasons.

2. Let (x,u) be a fixed point of the augmented Lagrangian iteration:

$$\begin{aligned} \nabla_x L(x^{i+1}, u^i, \alpha) &= 0 \\ u^{i+1} &= u^i + \alpha \nabla_u L(x^{i+1}, u^i, \alpha), \end{aligned}$$

where

$$L(x,u,\alpha) = f(x) + \frac{1}{2}[\|(\alpha g(x) + u)_+\|_2^2 - \|u\|_2^2],$$

and $f : R^n \to R$ and $g : R^n \to R^m$ are differentiable functions on R^n, $(z)_+ :=\max\{0,z\}$ and α is a fixed positive real number. How is (x,u) related to:

$$\min_x f(x) \text{ s.t. } g(x) \leq 0?$$

3. Consider an instance (IP) [min cx s.t. Ax=b, x binary] of a linear integer program in which x is a vector of 10 binary variables. When branch-and-bound is applied to this problem instance, node 0 with a non-binary solution and an objective of 104.8 is obtained. Branching is then done on the first variable, yielding node 1 with a binary solution and objective 110 and node 2 with a non-binary solution and objective 106.5 . Answer the following questions about the continuation of the branch-and-bound process, which you should assume proceeds one node at a time:

(a) If the solution at node 1 is optimal for (IP), what is the smallest number of additional nodes that need to be considered in order to prove its optimality? Explain.

(b) If the solution at node 1 is optimal for (IP), what is the largest number of additional nodes that may need to be considered in order to prove its optimality? Explain.

(c) What is the smallest number of additional nodes that could be considered in order to terminate with a feasible solution guaranteed (via branch-and-bound information) to have an objective within 0.1 (absolute gap) of optimal ? (Hint: the binary solution at node 1 does not correspond to the terminating feasible solution in this instance.) Illustrate with a possible branch-and-bound tree for this problem instance. Give an explicit statement of the relaxed problem whose optimal value yields the lower bound in this instance.

4. Suppose that f is an extended-real-valued, closed proper convex function on $\mathbb{R}^n$. Assume also that there is some positive number τ such that for each $y \in \mathbb{R}^n$ with $||y|| = \tau$ there is some $x \in \mathbb{R}^n$ and some $\xi > 0$ (both of which may depend on y) such that $f(x+\xi y) > f(x)$. Show that for any positive real number ρ there is a positive σ such that at each $u \in \mathbb{R}^n$ with $||u|| = \sigma$ one has $f(u) > \rho$.

5. We are manufacturing three products, which we will call Products 1, 2, and 3. These require two materials, A and B, which are in short supply. Making 1 unit of Product 1 requires 1 unit of A and 2 units of B; 1 unit of Product 2 requires 4 units of A and 3 units of B; and 1 unit of Product 3 requires 1 unit of A and 1 unit of B. We have a total of 10 units of A and 4 units of B available. Profit contributions from Products 1, 2, and 3 are \$3, \$6, and \$4 per unit respectively. We would like to find a product mix with the greatest possible profit contribution. We cannot use more units of A and B than are available, but we do not need to use all units available. You have determined that a linear programming model will adequately represent this problem.

   Someone has suggested that the best strategy to follow is to make only Product 3, because of the high profit contribution per unit of each resource used. Answer the following questions, justifying your answers:

   - Determine what is in fact the best product mix to use. You must specify the mix and its profit contribution, and also demonstrate that it is the best.
   - Is it possible that there is another feasible product mix that yields as much profit contribution as your optimal mix?
   - Suppose that, for reasons of business policy, we would like to manufacture more of Product 1 than is specified in your optimal product mix. Is it possible to do so without using more resources than we have? How much profit contribution would we lose for each additional unit of Product 1 made?

6. Vehicles are to be shipped (at minimum total cost) from distribution centers (DCs) 1 and 2 to dealers 3 and 4. There are 30 available vehicles at DC1

and 40 at DC2. Dealer 3 requires exactly 45 vehicles, and dealer 4 requires exactly 20 vehicles. DC1 has a delivery charge of $120 per vehicle, but an additional $55 per vehicle surcharge is imposed on deliveries to dealer 3 for each vehicle shipped in excess of 15 vehicles. The corresponding figures for DC2 shipments are $100 per vehicle and a $50 per vehicle surcharge to dealer 3 for each vehicle shipped in excess of 25.

(a) Model this problem as a linear network flow problem with an equality constraint at each node.

(b) For this equivalent linear network, state the optimal value, the optimal flows, and the optimal dual values (node prices).

7. Suppose that $a_0, a_1, \ldots, a_m$ are fixed elements of $\mathbb{R}^n$, such that for any $y \in \mathbb{R}^n$ the inner product $\langle a_0, y\rangle$ is not greater than the maximum element of the set $\{\langle a_1, y\rangle, \ldots, \langle a_m, y\rangle\}$. Is it necessarily the case that a_0 belongs to the convex hull of $a_1, \ldots, a_m$? If so, prove it; if not, exhibit a counterexample.

8. Consider the linear program:

$$\min_{x,y} c'x + d'y \text{ s.t. } Ax + By \geq b,\ x \geq 0,\ y \geq 0,$$

where A is square matrix such that $A \geq I$, $c \leq 0$ and I is the identity matrix. Suppose that this linear program has a solution. What is the minimum value of the objective function? Justify.