MATHEMATICAL PROGRAMMING

Fall 2000 Qualifying Exam
September 18, 2000
Instructions: Answer 5 of the following 8 questions.

1.
- A bookcase requires three hours of work, one unit of metal, and four units of wood, and it brings in a profit of \$19.
- A desk requires two hours of work, one unit of metal, and three units of wood and it brings in a profit of \$13.
- A chair requires one hour of work, one unit of metal, and three units of wood, and it brings in a profit of \$12.
- A bedframe requires two hours of work, one unit of metal, and four units of wood, and it brings in a profit of \$17.
- Only 225 hours of labor, 117 units of metal, and 420 units of wood are available per day.

The optimal solution calls for 39 bookcases, 48 chairs, and 30 bedframes to be produced every day. Solve the following variations:

(a) The profit brought in by each desk increases from \$13 to \$15.

(b) The availability of metal increases from 117 to 125 units per day.

(c) The company may also produce coffee tables, each of which requires three hours of work, one unit of metal, two units of wood, and brings in a profit of \$14.

(You should assume that the changes are cumulative. Thus, for example, the profit for desks is \$15 in the second and third questions).

2. Suppose that:

$$\sup_{u \in R^n} \{-q'u \mid M'u \leq 0,\ u \geq 0\} = +\infty.$$

What can you say about the linear complementarity problem:

$$0 \leq x \ \perp \ Mx + p \geq 0,$$

where $M \in R^{n \times n}$, $p,\ q \in R^n$ and $p \leq q$?

3. Suppose that the problem:

$$\min_{x \in R^n} x' F(x) \text{ s.t.} F(x) \geq 0, \; x \geq 0,$$

where $F : R^n \to R^n$ is differentiable on R^n, has a Karush-Kuhn-Tucker point $(\bar{x}, \bar{u}, \bar{v}) \in R^{n+n+n}$ such that $\bar{u} = \bar{x}$, where $\bar{u}$ is optimal multiplier associated with $F(x) \geq 0$. What can you say about the nonlinear complementarity problem:

$$0 \leq x \;\perp\; F(x) \geq 0?$$

4. Consider the problem:

$$\min_{x \in S} f(x),$$

where $f : R^n \to R$ and $S \in R^n$, which may possibly be empty. Suppose that some **fixed** $\bar{x} \in R^n$ solves the penalty problem:

$$\min_{x \in R^n} P(x, \alpha) = f(x) + \alpha Q(x), \text{ for all } \alpha \in \{\alpha^i \uparrow \infty\},$$

where $Q(x)$ is some penalty function such that:

$$Q(x) = 0 \text{ for } x \in S, \text{ else } Q(x) > 0.$$

What can you say about:

(a) $Q(\bar{x})$?

(b) $f(\bar{x})$?

Make sure that you consider the case when S is empty.

5. Suppose that f is an extended-real-valued convex function on $\mathbb{R}^n$, and let x be a point of $\operatorname{ri}\operatorname{dom} f$ with $f(x) > -\infty$. Write S for the subspace parallel to $\operatorname{dom} f$.

   (a) Show that $\partial f(x) \neq \emptyset$.

   (b) Show from first principles that the set $D := \partial f(x) \cap S$ is nonempty, compact, and convex. You may use without proof elementary results about dimensionality, relative interiors, and orthogonal decompositions, as well as the fact that f is continuous on $\operatorname{ri}\operatorname{dom} f$.

   (c) Show that $\partial f(x) = D + S^{\perp}$.

6. Consider the following (not necessarily feasible) complementarity problem (CP): determine an n vector x such that x has the minimum number of non-zero components subject to the constraints: $0 \leq x$ and $0 \leq Ax + b$ and $x'(Ax + b) = 0$, where A is a given $n \times n$ matrix and b is a given n vector. Assume that a value $M > 0$ is known such that any possible solution of (CP) must satisfy $x \leq Me$ where e is a vector of ones.

   (a) Introducing an n vector u of binary variables, formulate (CP) as a linear MIP (P) with objective $min\;\; e'u$ and appropriate constraints in x and u. (Discuss how any data needed to set up (P) may be easily determined.) Show that (P) and (CP) are equivalent.

   (b) If the integrality constraints on u are deleted, and the resulting LP is solved, discuss two outcomes of this LP solution process that would allow conclusions to be drawn about (CP) without any additional computation.

7. Let N and A be non-empty sets of nodes and arcs, respectively, that define a digraph. Let $s \neq t$ be elements of N and consider the maximum flow problem from s to t corresponding to the digraph and flow vector bounds $0 \leq x \leq c$, where c is non-negative.

   (a) State the maximum flow problem, using divergence constraints only at the intermediate nodes i ($s \neq i \neq t$), and show that this problem always has an optimal solution.

   (b) Show that this problem always has an optimal solution in which $x_{ti} = 0$ for all pairs (t,i) corresponding to arcs in the original digraph. (Be sure to establish feasibility for any solution that you construct.)

8. A company must complete three jobs. The amounts of processing time (in minutes) required to complete the jobs are shown in the table below:

```
set machine /1*4/;
set job /j1*j3/;

table proctime(job,machine)
        1       2       3       4
j1      20              25      30
j2      15      20              18
j3              35      28
;
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   A job cannot be processed on machine m unless for all $i < m$ the job has completed its processing on machine i. Once a job begins its processing on machine m, the job cannot be preempted on machine m. The flow time for a job is the difference between the job's completion time and the time at which the job begins its first stage of processing. Formulate a GAMS or AMPL model whose solution can be used to minimize the average flow time of the three jobs.

   Ensure that you carefully describe the purpose of each variable and equation in your model. (Hint: two types of contraints will be needed: one to ensure a job cannot begin to be processed until all earlier portions of the job are completed. The other ensures that only one job will occupy a machine at a given time.)

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525Theory The LCP is infeasible, and hence unsolvable, because by the Strong Duality Theory of Linear Programming the following linear program is infeasible:

$$\min_{x} \{0'x \mid Mx \geq -q,\ x \geq 0\},$$

and so is:

$$\min_{x} \{0'x \mid Mx \geq -p,\ x \geq 0\}.$$

726 solution The point $\bar{x}$ solves the NCP, because the KKT conditions are:

$$F(\bar{x}) + \nabla F(\bar{x})'(\bar{x} - \bar{u}) - \bar{v} = 0,\ 0 \leq \bar{u} \perp F(\bar{x}) = 0,\ 0 \leq \bar{v} \perp \bar{x} \geq 0,$$

from which follows:

$$\bar{x}'F(\bar{x}) = \bar{x}'\bar{v} = 0,$$

if we set $\bar{u} = \bar{x}$ in the KKT conditions.

730 solution (a)

$$Q(\bar{x}) = \min_{x \in R^n} Q(x)$$

(b)

$$f(\bar{x}) = \min_{x \in R^n} \{f(x) \mid Q(x) = \min_{x \in R^n} Q(x)\}$$

Proof:

(a) Fix $x \in R^n$ and let $\alpha^i \to \infty$. Then:

$$f(\bar{x}) + \alpha^i Q(\bar{x}) \leq f(x) + \alpha^i Q(X)$$

or:

$$Q(\bar{x}) \leq Q(x) + \frac{f(x) - f(\bar{x})}{\alpha^i}.$$

Letting $\alpha^i \to \infty$ gives (a).

(b) Let

$$x \in \operatorname*{arg\,min}_{x \in R^n} Q(x).$$

Then:

$$\begin{array}{llll} f(x) & = f(x) + \alpha^0(Q(x) - Q(x)) & = P(x, \alpha^0) - \alpha^0 Q(x) & \\ & = P(x, \alpha^0) - \alpha^0 Q(\bar{x}) & \geq P(\bar{x}, \alpha^0) - \alpha^0 Q(\bar{x}) & = f(\bar{x}) \end{array}$$

This gives (b).

727 solution If f were improper it would take $-\infty$ everywhere on $\operatorname{ri}\operatorname{dom} f$, but we are told that its value at x is not $-\infty$. Therefore f is proper convex, and such a function is subdifferentiable everywhere on the relative interior of its effective domain. Therefore $\partial f(x) \neq \emptyset$. This also shows that $f(x)$ must be finite.

The set $\partial f(x)$ is the intersection over all $z \in \mathbb{R}^n$ of the sets

$$\{d \mid f(z) \geq f(x) + \langle d, z - x \rangle\},$$

each of which is closed and convex. This shows that $\partial f(x)$ is closed and convex, and as S is also closed and convex so is the intersection D. For compactness, we need only show that D is bounded. Let V be the set of points in S having unit length. As $x \in \operatorname{ri}\operatorname{dom} f$, for small positive ϵ we have $x + \epsilon V \subset \operatorname{ri}\operatorname{dom} f$. Accordingly, f takes a maximum ϕ on $x + \epsilon V$. If d is any nonzero point of D, then the point $y := (\epsilon / \|d\|)d$ belongs to ϵV, so

$$\phi \geq f(x + y) \geq f(x) + \langle d, y - x \rangle = f(x) + \epsilon \|d\|.$$

This yields the bound $\|d\| \leq \epsilon^{-1}[\phi - f(x)]$, and therefore D is bounded.

Next, let w be any point of $\partial f(x)$ and let q be any point of $S^\perp$. For each $z \in \mathbb{R}^n$ we have $f(z) \geq f(x) + \langle w + q, z - x \rangle$ (consider the two cases $z \in \operatorname{dom} f$ and $z \notin \operatorname{dom} f$). So $S^\perp$ is contained in the lineality space of $\partial f(x)$, and then

$$\partial f(x) = S^\perp + [\partial f(x) \cap S] = S^\perp + D.$$

This shows simultaneously that $D \neq \emptyset$ (because $\partial f(x)$ is nonempty) and that $\partial f(x) = D + S^\perp$.

720 solution First compute a $K \geq 0$ such that $Ax + b \leq Ke$ for $0 \leq x \leq Me$. This may be done by choosing K to the be the largest k_i such that $k_i = max\{A_i x + b_i | 0 \leq x \leq Me\}$. If the value for any k_i is negative, (CP) is infeasible and no further computation is needed.

The required MIP may then be written as:

min $\quad e'u \;\; s.t. \;\; 0 \leq x \leq Mu, 0 \leq Ax + b \leq K(e - u)$, u binary.

It is easy to verify this is equivalent to (CP) since the objective of the MIP simply counts the non-zeroes in x and the constraints of MIP only allow complementary solutions.

When the integrality constraints on u are relaxed and the LP is solved, a conclusion about (CP) is available if the LP is infeasible (implying the infeasibility of (CP)) or if the LP has an integer solution, in which case the x values will be optimal for (CP).

719 solution The maximum flow problem is always feasible (0 is always feasible) and is bounded above by the sum of the capacities, so it always has an optimal solution. To show that there is an optimal solution with all x_{ti}=0, consider an optimal solution with an $x_{ti} \neq 0$. Use the conformal decomposition theorem (or logically trace flows beginning with this arc) to obtain a path flow containing this arc. Since t is the only node at which divergence may be negative in an optimal solution, this path must be a cycle, and flow on the cycle can be reduced (and x_{ti} eventually driven to 0 by repeated applications of this process) without affecting the objective.