**CS810: Homework 2** Due date: Thursday , March 13th, 2003

1. The class $S_2^p$, "Symmetric second level" of the Polynomial Time Hierarchy", was defined by Russell and Sundaram in 1995 as follows: $L \in S_2^p$ iff there is a P-time computable 0-1 function $P$ on three arguments, such that

$$x \in L \implies (\exists^p y)(\forall^p z)[P(x,y,z)=1] \quad (1)$$
$$x \notin L \implies (\exists^p z)(\forall^p y)[P(x,y,z)=0] \quad (2)$$

   where as usual "$\exists^p y$" stands for "$\exists y \in \{0,1\}^{p(|x|)}$" for some polynomial $p(\cdot)$. Similarly "$\forall^p z$" stands for "$\forall z \in \{0,1\}^{q(|x|)}$" for some polynomial $q(\cdot)$.

   Prove that $S_2^p \subseteq \Sigma_2^p \cap \Pi_2^p$.

   What is the difference of $S_2^p$ and $\Sigma_2^p \cap \Pi_2^p$ in their definition? In other words, why can't we immediately claim $S_2^p = \Sigma_2^p \cap \Pi_2^p$? (This "equality" is in fact open.)

2. Strengthen the Karp-Lipton Theorem as follows: If NP has polynomial size circuits, then PH collapses to $S_2^p$.

3. A set $T \subseteq 1^*$ is called a tally set. Show that SAT $\in P^S$ for some sparse set $S$ iff SAT $\in P^T$ for some tally set $T$.

4. In our proof of Mahaney's theorem, we used the Left-Cut, and "focused" on the left-most satisfying assignment if one exists.

   Define a Right-Cut set for SAT, and give an analogous proof for Mahaney's theorem.

   If we do not define the Right-Cut set for SAT, but still use Left-Cut, can we still argue in terms of right-most satisfying assignment? In particular, when we considered at a certain level $\ell$ in the tree of binary assignments, if we found two nodes have the same label (by the reduction), can we drop the left node? Prove your answer.

5. Suppose we have a p-time reduction from SAT to a co-sparse set $T$ (a set $T$ is co-sparse if its complement $T^c$ is sparse). Prove that NP= P.

6. For any p-time 1-1 function $f$, prove that $f$(SAT) is NP-complete.

   Can you cook up such a function based on the exponentiation function $f$ (whose inverse is some version of the discrete log function), such that the proof of our theorem by Berman-Hartmanis on isomorphism of NP-complete sets does not apply?

7. One can define log-space reduction for class P (as well as NP etc.) Define this, and show in particular that this reduction is also transitive. (Note in log-space, if you have reduction from $A$ to $B$ and from $B$ to $C$, in order to compute the composition, you don't have space to write down the intermediate results, which is beyond log-space.)

   Define P-completeness for problems in P under log-space reductions.

   Prove that the similar results of Berman-Hartmanis also hold for P-complete sets.

**Note:**

**Please be concise.**