CS810: Homework 2 Due date: Thursday, March 13th, 2003

1. The class $S_p^2$, “Symmetric second level” of the Polynomial Time Hierarchy”, was defined by Russell and Sundaram in 1995 as follows: $L \in S_p^2$ iff there is a P-time computable 0-1 function $P$ on three arguments, such that

\[ x \in L \implies (\exists y)(\forall z)[P(x, y, z) = 1] \tag{1} \]
\[ x \notin L \implies (\exists z)(\forall y)[P(x, y, z) = 0] \tag{2} \]

where as usual “$\exists y$” stands for “$\exists y \in \{0, 1\}^{p(j(x))}$” for some polynomial $p(\cdot)$. Similarly “$\forall z$” stands for “$\forall z \in \{0, 1\}^{q(j(x))}$” for some polynomial $q(\cdot)$.

Prove that $S_p^2 \subseteq \Sigma_2^p \cap \Pi_2^p$.

What is the difference of $S_p^2$ and $\Sigma_2^p \cap \Pi_2^p$ in their definition? In other words, why can’t we immediately claim $S_p^2 = \Sigma_2^p \cap \Pi_2^p$? (This “equality” is in fact open.)

2. Strengthen the Karp-Lipton Theorem as follows: If NP has polynomial size circuits, then PH collapses to $S_p^2$.

3. A set $T \subseteq 1^*$ is called a tally set. Show that SAT $\in P^T$ for some sparse set $S$ iff SAT $\in P^T$ for some tally set $T$.

4. In our proof of Mahaney’s theorem, we used the Left-Cut, and “focused” on the left-most satisfying assignment if one exists.

Define a Right-Cut set for SAT, and give an analogous proof for Mahaney’s theorem.

If we do not define the Right-Cut set for SAT, but still use Left-Cut, can we still argue in terms of right-most satisfying assignment? In particular, when we considered at a certain level $\ell$ in the tree of binary assignments, if we found two nodes have the same label (by the reduction), can we drop the left node? Prove your answer.

5. Suppose we have a p-time reduction from SAT to a co-sparse set $T$ (a set $T$ is co-sparse if its complement $T^c$ is sparse). Prove that NP= P.

6. For any p-time 1-1 function $f$, prove that $f(SAT)$ is NP-complete.

Can you cook up such a function based on the exponentiation function $f$ (whose inverse is some version of the discrete log function), such that the proof of our theorem by Berman-Hartmanis on isomorphism of NP-complete sets does not apply?
7. One can define log-space reduction for class P (as well as NP etc.) Define this, and show in particular that this reduction is also transitive. (Note in log-space, if you have reduction from $A$ to $B$ and from $B$ to $C$, in order to compute the composition, you don’t have space to write down the intermediate results, which is beyond log-space.)

Define P-completeness for problems in P under log-space reductions.

Prove that the similar results of Berman-Hartmanis also hold for P-complete sets.

Note:

Please be concise.