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# Strong Branching Inequalities for Convex Mixed Integer Nonlinear Programs

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## Abstract

Strong branching is an effective branching technique that can significantly reduce the size of the branch-and-bound tree for solving Mixed Integer Nonlinear Programming (MINLP) problems. The focus of this paper is to demonstrate how to effectively use “discarded” information from strong branching to strengthen relaxations of MINLP problems. Valid inequalities such as branching-based linearizations, various forms of disjunctive inequalities, and mixing-type inequalities are all discussed. The inequalities span a spectrum from those that require almost no extra effort to compute to those that require the solution of an additional linear program. In the end, we perform an extensive computational study to measure the impact of each of our proposed techniques. Computational results reveal that existing algorithms can be significantly improved by leveraging the information generated as a byproduct of strong branching in the form of valid inequalities.

# 1 Introduction

In this work, we study valid inequalities derived from strong branching for solving the convex mixed integer nonlinear programming (MINLP) problem

$$\begin{aligned} z_{\text{MINLP}} = \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_j(x) \leq 0, \quad \forall j \in J, \\ & x \in X, \quad x_I \in \mathbb{Z}^{|I|}. \end{aligned} \tag{1}$$

The functions  $f : X \rightarrow \mathbb{R}$  and  $g_j : X \rightarrow \mathbb{R} \forall j \in J$  are smooth, convex functions, and the set  $X \stackrel{\text{def}}{=} \{x \in \mathbb{R}_+^n \mid Ax \leq b\}$  is a polyhedron. The set  $I \subseteq \{1, \dots, n\}$  contains the indices of discrete variables, and  $B \subseteq I$  is the index set of binary variables.

In order to have a linear objective function an auxiliary variable  $\eta$  is introduced, and the nonlinear objective function is moved to the constraints, creating the equivalent problem

$$z_{\text{MINLP}} = \text{minimize} \{ \eta : (\eta, x) \in \mathcal{S}, x_I \in \mathbb{Z}^{|I|} \} \tag{2}$$

where

$$\mathcal{S} = \{(\eta, x) \in \mathbb{R} \times X \mid f(x) \leq \eta, g_j(x) \leq 0 \quad \forall j \in J\}.$$

We define the set  $\mathcal{P} = \{(\eta, x) \in \mathcal{S} \mid x_I \in \mathbb{Z}^{|I|}\}$  to be the set of feasible solutions to (2).

Branch and bound forms a significant component of most algorithms for solving MINLP problems. In NLP-based branch and bound, the lower bound on the value  $z_{\text{MINLP}}$  comes from the solution value of the nonlinear program (NLP) that is the *continuous relaxation* of (2):

$$z_{\text{NLPR}} = \min \{ \eta : (\eta, x) \in \mathcal{S} \}. \tag{3}$$

In linearization-based approaches, such as outer-approximation [16] or the LP/NLP branch-and-bound algorithm [30], the lower bound comes from solving a linear program (LP), often called *the master problem* that is based on a polyhedral outer-approximation of  $\mathcal{P}$ :

$$\begin{aligned} z_{\text{MP}(\mathcal{K})} = \min \quad & \eta \\ \text{s.t.} \quad & \eta \geq f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x}) \quad \forall \bar{x} \in \mathcal{K}, \\ & g_j(\bar{x}) + \nabla g_j(\bar{x})^T (x - \bar{x}) \leq 0 \quad \forall j \in J, \forall \bar{x} \in \mathcal{K}, \\ & x \in X, \end{aligned} \tag{4}$$

where  $\mathcal{K}$  is a set of points about which linearizations of the convex functions  $f(\cdot)$  and  $g_j(\cdot)$  are taken. For more details on algorithms for solving convex MINLP problems, the reader is referred to the surveys [11, 20].

Regardless of the bound employed by the branch-and-bound algorithm, algorithms are required to branch. By far the most common branching approach is branching on individual integer variables. In this approach, branching involves selecting a single branching variable  $x_i, i \in I$  such that in the solution  $\hat{x}$  to the relaxation (3) or (4),  $\hat{x}_i \notin \mathbb{Z}$ . Based on the branching variable, the problem is recursively divided, imposing the constraint  $x_i \leq \lfloor \hat{x}_i \rfloor$  for one child subproblem and

$x_i \geq \lceil \hat{x}_i \rceil$  for the other. The relaxation solution  $\hat{x}$  may have many candidates for the branching variable, and the choice of branching variable can have a very significant impact on the size of the search tree [27, 3]. Ideally, the selection of the branching variable would lead to the smallest resulting enumeration tree. However, without explicitly enumerating the trees coming from all possible branching choices, choosing the best variable is difficult to do exactly. A common heuristic is to select the branching variable that is likely to lead to the largest improvement in the children nodes' lower bounds. The reasoning behind this heuristic is that nodes of the branch-and-bound tree are fathomed when the lower bound for the node is larger than the current upper bound, so one should select branching variables to increase the children nodes' lower bounds by as much as possible.

In the context of solving the Traveling Salesman Problem, Applegate *et al.* [4] propose to explicitly calculate the lower bound changes for many candidate branching variables and choose the branching variable that results in the largest change for the resulting child nodes. This method has come to be known as *strong branching*. Strong branching or variants of strong branching, such as reliability branching [3], have been implemented in state-of-the-art solvers for solving mixed integer linear programs, the special case of MINLP where all functions are linear.

For MINLP, one could equally well impose the extra bound constraint on the candidate branching variable in the nonlinear continuous relaxation (3). We call this type of strong branching, *NLP-based strong branching*. In particular, for a fractional solution  $\hat{x}$ , NLP-based strong branching is performed by solving the two continuous nonlinear programming problems

$$\hat{\eta}_i^0 = \text{minimize}\{\eta : (\eta, x) \in \mathcal{S}_i^0\} \tag{NLP_i^0}$$

and

$$\hat{\eta}_i^1 = \text{minimize}\{\eta : (\eta, x) \in \mathcal{S}_i^1\} \tag{NLP_i^1}$$

for each fractional variable index  $i \in F \stackrel{\text{def}}{=} \{i \in I \mid \hat{x}_i \notin \mathbb{Z}\}$ , where  $\mathcal{S}_i^0 = \{(\eta, x) \in \mathcal{S} \mid x_i \leq \lfloor \hat{x}_i \rfloor\}$  and  $\mathcal{S}_i^1 = \{(\eta, x) \in \mathcal{S} \mid x_i \geq \lceil \hat{x}_i \rceil\}$ . The optimal values of the subproblems  $(\eta_i^0, \eta_i^1 \forall i \in F)$  are used to choose a branching variable [2, 27].

In the LP/NLP branch-and-bound algorithm, the NLP continuous relaxation (3) is not solved at every node in the branch-and-bound tree, although it is typically solved at the root node. Instead, the polyhedral outer-approximation (4) is used throughout the branch-and-bound tree. The outer-approximation is refined when an integer feasible solution to the current linear relaxation is obtained. Since the branch-and-bound tree is based on a *linear* master problem, it is not obvious whether strong branching should be based on solving the *nonlinear* subproblems  $(NLP_i^0)$  and  $(NLP_i^1)$  or based on solving the LP analogues to these where the nonlinear constraints are replaced by the current linear outer approximation. However, our computational experience in Section 4.3 is that even when using a linearization-based method, a strong-branching approach based on solving NLP subproblems can yield significant reduction in the number of nodes in a branch-and-bound tree. Leyffer [26] also has given empirical evidence of the effectiveness of NLP-based strong branching for solving convex MINLP problems.

On the other hand, using NLP subproblems for strong branching is computationally more intensive than using LP subproblems, so it makes sense to attempt to use information obtained from NLP-based strong branching in ways besides simply choosing a branching variable. In this

work, we describe a variety of ways to transfer strong branching information into the child node relaxations. The focus of our work will be on improving the implementation of the LP/NLP branch-and-bound algorithm in the software package FilMINT [1]. The information may be transferred to the child relaxations by adding additional linearizations to the master problem (4) or through the addition of simple disjunctive inequalities. We demonstrate the relation of the simple disjunctive inequalities we derive to standard disjunctive inequalities. We derive and discuss many different techniques by which these simple disjunctive strong branching inequalities may be strengthened. The strengthening methods range from methods that require almost no extra computation to methods that require the solution of a linear program. In the end, we perform an extensive computational study to measure the impact of each of our methods. Incorporating these changes in the solver FilMINT results in a significant reduction in CPU time on the instances in our test suite.

The remainder of the paper is divided into 4 sections. Section 2 describes some simple methods for using inequalities generated as an immediate byproduct of the strong branching process. Section 3 concerns methods for strengthening inequalities obtained from strong branching. Section 4 reports on our computational experience with all of our described methods, and Section 5 offers some conclusions of our work.

## 2 Simple Strong Branching Inequalities

In this section, we describe elementary ways that information obtained from the strong branching procedure can be recorded and used in the form of valid inequalities for solving MINLP problems. The simplest scheme is to use linearizations from the NLP subproblems. Alternatively, valid inequalities may be produced from the disjunction, and these inequalities may be combined by mixing.

### 2.1 Linearizations

When using a linearization-based approach for solving MINLP problems, a simple idea for obtaining more information from the NLP strong branching subproblems ( $NLP_i^0$ ) and ( $NLP_i^1$ ) is to add the solutions to these subproblems to the linearization point set  $\mathcal{K}$  of the master problem (4).

There are a number of reasons why using linearizations about solutions to ( $NLP_i^0$ ) and ( $NLP_i^1$ ) may yield significant computational benefit. First, the inequalities are trivial to obtain once the NLP subproblems have been solved; one simply has to evaluate the gradient of the nonlinear functions at the optimal solutions of ( $NLP_i^0$ ) and ( $NLP_i^1$ ). Second, the inequalities are likely to improve the lower bound in the master problem (4) after branching. In fact, if these linearizations are added to (4), then after branching on the variable  $x_i$ , the lower bound  $z_{\text{MP}(\mathcal{K})}$  will be at least as large the bound obtained by an NLP-based branch-and-bound algorithm. Third, optimal solutions to ( $NLP_i^0$ ) and ( $NLP_i^1$ ) satisfy the nonlinear constraints of the MINLP problem. Computational experience with different linearization approaches for solving MINLP problems in [1] suggests that the most important linearizations to add to the master problem (4) are those obtained at points that are feasible to the NLP relaxation. Finally, depending on the

branching strategy employed, using these linearizations may lead to improved branching decisions. For example, our branching strategy, described in detail in Section 4.1, is based on pseudocosts that are initialized using NLP strong branching information, but are updated based on the current polyhedral outer approximation after a variable is branched on. Thus, the improved polyhedral outer approximation derived from these linearizations may lead to improved pseudocosts, and hence better branching decisions.

## 2.2 Simple disjunctive inequalities

Another approach to collecting information from the strong branching subproblems ( $NLP_i^0$ ) and ( $NLP_i^1$ ) is to combine information from the two subproblems using *disjunctive* arguments. We call the first very simple inequality a *strong branching cut* (SBC). We omit the simple proof of its validity.

**Proposition 1** *The strong branching cut (SBC)*

$$\eta \geq \hat{\eta}_i^0 + (\hat{\eta}_i^1 - \hat{\eta}_i^0)x_i \quad (5)$$

is valid for the MINLP (2), where  $i \in B$ , and  $\hat{\eta}_i^0, \hat{\eta}_i^1$  are the optimal solution values to ( $NLP_i^0$ ) and ( $NLP_i^1$ ), respectively.

Similar inequalities can be written for other common branching disjunctions, such as the GUB constraint

$$\sum_{i \in S} x_i = 1, \quad (6)$$

where  $S \subseteq B$  is subset of binary variables.

**Proposition 2** *Let (6) be a constraint for the MINLP problem (2). The GUBSBC inequality*

$$\eta \geq \sum_{i \in S} \hat{\eta}_i^1 x_i \quad (7)$$

is valid for (2), where  $\hat{\eta}_i^1$  is the optimal solution value to ( $NLP_i^1$ ) for  $i \in S$ .

If the instance contains a constraint of the form  $\sum_{i \in S} x_i \leq 1$ , then a slack binary variable can be added to convert it to the form of (6), so that (7) may be used in this case as well.

The simple SBC (5) can be generalized to disjunctions based on general integer variables. The following result follows by using a convexity argument and a disjunctive argument based on the disjunction  $x_i \leq \lfloor \hat{x}_i \rfloor$  or  $x_i \geq \lceil \hat{x}_i \rceil$ , for some integer variable  $x_i$  whose relaxation value  $\hat{x}_i$  is fractional. A complete proof of Lemma 1 can be found in the Ph.D. thesis of Kılınç [24].

**Lemma 1** *Let  $i \in I$  and  $k \in \mathbb{Z}$ . The strong branching cut*

$$\eta \geq \hat{\eta}^0 + (\hat{\eta}^1 - \hat{\eta}^0)(x_i - \lfloor \hat{x}_i \rfloor)$$

is valid for (2), where  $\hat{\eta}^0$  and  $\hat{\eta}^1$  are the optimal solution values to ( $NLP_i^0$ ) and ( $NLP_i^1$ ), respectively.

### 2.3 Mixing Strong Branching Cuts

Mixing sets arose in the study of a lot-sizing problem by Pochet and Wolsey [29] and were systematically studied by Günlük and Pochet [22]. A similar set was introduced as a byproduct of studying the mixed vertex packing problem by Atamtürk, Nemhauser, and Savelsbergh [5].

A collection of strong branching inequalities (5) can be transformed into a mixing set in a straightforward manner. Specifically, let  $\hat{B} \subseteq B$  be the index set of binary variables on which strong branching has been performed, and let  $\delta_i = \hat{\eta}_i^1 - \hat{\eta}_i^0$  be the difference in objective values between the two NLP subproblems ( $NLP_i^0$ ) and ( $NLP_i^1$ ). Proposition 1 states that the SBC inequalities

$$\eta \geq \hat{\eta}_i^0 + \delta_i x_i \quad \forall i \in \hat{B} \quad (8)$$

are valid for the MINLP problem (2). Without loss of generality, assume that  $\delta_i \geq 0$ , for otherwise, one can define  $\tilde{x}_i = 1 - x_i$ ,  $\tilde{\delta}_i = -\delta_i$ , and write (8) as

$$\eta \geq \hat{\eta}_i^1 + \tilde{\delta}_i \tilde{x}_i,$$

which has  $\tilde{\delta}_i \geq 0$ . Since  $\delta_i \geq 0$ , the value

$$\underline{\eta} \stackrel{\text{def}}{=} \max_{\forall i \in \hat{B}} \hat{\eta}_i^0 \leq \eta$$

is a valid lower bound for the objective function variable  $\eta$ . Furthermore, by definition, the inequalities

$$\eta \geq \underline{\eta} + \sigma_i x_i \quad \forall i \in \bar{B} \quad (9)$$

are valid for (MINLP), where  $\sigma_i = \hat{\eta}_i^1 - \underline{\eta}$  and  $\bar{B} = \{i \mid \sigma_i > 0, i \in \hat{B}\}$ . The inequalities (9) define a mixing set

$$\mathcal{M} = \{(\eta, x) \in \mathbb{R} \times \{0, 1\}^{|\bar{B}|} \mid \eta \geq \underline{\eta} + \sigma_i x_i \quad \forall i \in \bar{B}\}. \quad (10)$$

Proposition 3 is a straightforward application of the *mixing inequalities* of [22] or the *star inequalities* of [5] and demonstrates that the inequalities, which we call MIXSBC, are valid for  $\mathcal{M}$ , thus valid for the feasible region  $\mathcal{P}$  of the MINLP problem.

**Proposition 3 ([5, 22])** *Let  $T = \{i_1, \dots, i_t\}$  be a subset of  $\bar{B}$  such that  $\sigma_{i_{(j-1)}} < \sigma_{i_j}$  for  $j = 2, \dots, t$ . Then the MIXSBC inequality*

$$\eta \geq \underline{\eta} + \sum_{i_j \in T} \theta_{i_j} x_{i_j} \quad (11)$$

*is valid for  $\mathcal{M}$ , where  $\theta_{i_1} = \sigma_{i_1}$  and  $\theta_{i_j} = \sigma_{i_j} - \sigma_{i_{j-1}}$  for  $j = 2, \dots, t$ .*

If a MIXSBC inequality (11) is violated by a fractional solution  $\hat{x}$ , it may be identified in polynomial time using a separation algorithm given in [5] or [22].

### 3 Strengthened Strong Branching Inequalities

The valid inequalities introduced in Section 2 can be obtained almost “for free” using strong branching information. In this section, we explore methods for strengthening and combining simple disjunctive inequalities. By doing marginally more work, we hope to obtain more effective valid inequalities. The section begins by examining the relationship between the simple strong branching cut (5) and general disjunctive inequalities. A byproduct of the analysis is a simple mechanism for strengthening the inequalities (5) by using the optimal Lagrange multipliers from the NLP strong branching subproblems. The analysis also suggests the construction of a cut-generating linear program (CGLP) to further improve the (weak) disjunctive inequality generated by strong branching.

#### 3.1 SBC and Disjunctive Inequalities

The SBC (5) is a disjunctive inequality. For ease of presentation, we describe the relationship only for disjunctions of binary variables. The extension to disjunctions on integer variables is straightforward and can be found in [24]. Let  $(\hat{\eta}^0, \hat{x}^0)$  and  $(\hat{\eta}^1, \hat{x}^1)$  be optimal solutions to the NLP subproblems  $(NLP_i^0)$  and  $(NLP_i^1)$ , respectively. Since  $f(\cdot)$  and  $g_j(\cdot)$  are convex, linearizing the nonlinear inequalities about the points  $(\hat{\eta}^0, \hat{x}^0)$  and  $(\hat{\eta}^1, \hat{x}^1)$  gives two polyhedra

$$\mathcal{X}_i^0 = \left\{ (\eta, x) \left| \begin{array}{l} c^0 x - \eta \leq b^0 \\ D^0 x \leq d^0 \\ Ax \leq b \\ x_i \leq 0 \\ x \in \mathbb{R}_+^n \end{array} \right. \right\}, \quad \mathcal{X}_i^1 = \left\{ (\eta, x) \left| \begin{array}{l} c^1 x - \eta \leq b^1 \\ D^1 x \leq d^1 \\ Ax \leq b \\ -x_i \leq -1 \\ x \in \mathbb{R}_+^n \end{array} \right. \right\} \quad (12)$$

that outerapproximate the feasible region of the two strong branching subproblems. In the description of the polyhedra (12), we use the following notation for the gradient  $\nabla f(x) \in \mathbb{R}^{n \times 1}$ , and Jacobian  $\nabla g(x) \in \mathbb{R}^{n \times |J|}$  of the objective and constraint functions at various points:

$$\begin{aligned} c^0 &= \nabla f(\hat{x}^0)^T, & c^1 &= \nabla f(\hat{x}^1)^T, \\ b^0 &= \nabla f(\hat{x}^0)^T \hat{x}^0 - \hat{\eta}^0, & b^1 &= \nabla f(\hat{x}^1)^T \hat{x}^1 - \hat{\eta}^1, \\ D^0 &= \nabla g(\hat{x}^0)^T, & D^1 &= \nabla g(\hat{x}^1)^T, \\ d^0 &= \nabla g(\hat{x}^0)^T \hat{x}^0 - g(\hat{x}^0) & d^1 &= \nabla g(\hat{x}^1)^T \hat{x}^1 - g(\hat{x}^1). \end{aligned}$$

We assume that the sets  $\mathcal{X}_i^0$  and  $\mathcal{X}_i^1$  are non-empty, for if one of the sets is empty, the bound on the variable  $x_i$  may be fixed to its alternative value. Since  $\mathcal{X}_i^0$  and  $\mathcal{X}_i^1$  are polyhedra, we may apply known disjunctive theory to obtain the following theorem.

**Theorem 1** [7] *The disjunctive inequality*

$$\alpha x - \sigma \eta \leq \beta \quad (13)$$

is valid for  $\text{conv}(\mathcal{X}_i^0 \cup \mathcal{X}_i^1)$  and hence for the MINLP problem (2) if there exists  $\lambda^0, \lambda^1 \in \mathbb{R}_+^{|J|}, \mu^0, \mu^1 \in \mathbb{R}_+^m, \theta^0, \theta^1 \in \mathbb{R}_+$  and  $\sigma \in \mathbb{R}_+$  such that

$$\alpha \leq \sigma c^0 + \lambda^0 D^0 + \mu^0 A + \theta^0 e_i, \quad (14a)$$

$$\alpha \leq \sigma c^1 + \lambda^1 D^1 + \mu^1 A - \theta^1 e_i, \quad (14b)$$

$$\beta \geq \sigma b^0 + \lambda^0 d^0 + \mu^0 b, \quad (14c)$$

$$\beta \geq \sigma b^1 + \lambda^1 d^1 + \mu^1 b - \theta^1, \quad (14d)$$

$$\lambda^0, \lambda^1, \mu^0, \mu^1, \theta^0, \theta^1, \sigma \geq 0. \quad (14e)$$

One specific choice of multipliers  $\lambda^0, \lambda^1, \mu^0, \mu^1, \theta^0, \theta^1, \sigma$  in (14) leads to the strong branching inequality (5).

**Proposition 4** *Let  $(\hat{\eta}^0, \hat{x}^0)$  and  $(\hat{\eta}^1, \hat{x}^1)$  be optimal solutions to the NLP subproblems  $(NLP_i^0)$  and  $(NLP_i^1)$ , respectively, satisfying a constraint qualification. Then,*

$$\eta \geq \hat{\eta}^0 + (\hat{\eta}^1 - \hat{\eta}^0)x_i$$

is a disjunctive inequality (13).

**Proof.**

Since both  $(\hat{\eta}^0, \hat{x}^0)$  and  $(\hat{\eta}^1, \hat{x}^1)$  satisfy a constraint qualification, there exists Lagrange multiplier vectors  $\hat{\lambda}^h, \hat{\mu}^h, \hat{\phi}^h \geq 0$  and a Lagrange multiplier  $\hat{\theta}^h \geq 0$ , for each  $h \in \{0, 1\}$  satisfying the Karush-Kuhn-Tucker (KKT) conditions

$$\nabla f(\hat{x}^h) + \nabla g(\hat{x}^h)^T \hat{\lambda}^h + A^T \hat{\mu}^h - \hat{\phi}^h + \hat{\theta}^h e_i = 0, \quad (15a)$$

$$g(\hat{x}^h)^T \hat{\lambda}^h = 0, \quad (15b)$$

$$(A\hat{x}^h - b)^T \hat{\mu}^h = 0, \quad (15c)$$

$$\hat{\phi}^h(\hat{x}^h - h) = 0, \quad (15d)$$

$$\hat{x}_i^h \hat{\theta}^h = 0 \quad (15e)$$

We assign multipliers  $\sigma^0 = 1, \lambda^0 = \hat{\lambda}^0, \mu^0 = \hat{\mu}^0, \theta^0 = \hat{\theta}^0 - \hat{\eta}^0 + \hat{\eta}^1$  into (14a) and (14c) and  $\sigma^1 = 1, \lambda^1 = \hat{\lambda}^1, \mu^1 = \hat{\mu}^1, \theta^1 = \hat{\theta}^1 + \hat{\eta}^0 - \hat{\eta}^1$  into (14b) and (14d) in Theorem 1. Substituting these multipliers into (14) and simplifying the resulting inequalities using the KKT conditions (15) demonstrates that the SBC (5) is a disjunctive inequality. The algebraic details of the proof can be found in [24].

◇

### 3.2 Multiplier Strengthening

The analogy between the strong branching inequality (5) and disjunctive inequality (13) leads immediately to simple ideas for strengthening the strong branching inequality using Lagrange multiplier information. Specifically, a different choice of multipliers for the disjunctive cut (13) leads immediately to a stronger inequality.

**Theorem 2** Let  $(\hat{\eta}^0, \hat{x}^0)$  be the optimal (primal) solution to  $(NLP_i^0)$  with associated Lagrange multipliers  $(\hat{\lambda}^0, \hat{\mu}^0, \hat{\phi}^0, \hat{\theta}^0)$ . Likewise, let  $(\hat{\eta}^1, \hat{x}^1)$  be the optimal (primal) solution to  $(NLP_i^1)$  with associated Lagrange multipliers  $(\hat{\lambda}^1, \hat{\mu}^1, \hat{\phi}^1, \hat{\theta}^1)$ . Define  $\hat{\mu}^* = \min\{\hat{\mu}^0, \hat{\mu}^1\}$ , and  $\phi^* = \min\{\phi^0, \phi^1\}$ . If  $(NLP_i^0)$  and  $(NLP_i^1)$  both satisfy a constraint qualification, then the strengthened strong branching cut (SSBC)

$$\hat{\eta}^0 + (b - Ax)^T \hat{\mu}^* + \hat{\phi}^* x + (\hat{\eta}^1 - \hat{\eta}^0)x_i \leq \eta \quad (16)$$

is a disjunctive inequality (13).

**Proof.** We substitute the multipliers  $\lambda^h = \hat{\lambda}^h$ ,  $\mu^h = \hat{\mu}^h - \hat{\mu}^*$ ,  $h \in \{0, 1\}$ ,  $\sigma = 1$ ,  $\theta^0 = \hat{\theta}^0 - \hat{\eta}^0 + \hat{\eta}^1$ ,  $\theta^1 = \hat{\theta}^1 + \hat{\eta}^0 - \hat{\eta}^1$  into (14) in Theorem 1. Simplifying the resulting expressions using the KKT conditions (15) demonstrates the result. Details of the algebraic steps required are given in the Ph.D. thesis of Kılınç [24].

◇

### 3.3 Strong Branching CGLP

In Theorem 1, we gave necessary conditions for the validity of a disjunctive inequality for the set  $\text{conv}(\mathcal{X}_i^0 \cup \mathcal{X}_i^1)$ . A most violated disjunctive inequality can be found by solving *Cut Generating Linear Program* (CGLP) that maximizes the violation of the resulting cut with respect to a given point  $(\hat{\eta}, \hat{x})$ :

$$\begin{aligned} & \text{maximize} && \beta - \alpha \hat{x} + \sigma \hat{\eta} \\ & \text{subject to} && \alpha \leq \sigma c^0 + \lambda^0 D^0 + \mu^0 A + \theta^0 e_i, \\ & && \alpha \leq \sigma c^1 + \lambda^1 D^1 + \mu^1 A - \theta^1 e_i, \\ & && \beta \geq \sigma b^0 + \lambda^0 d^0 + \mu^0 b, \\ & && \beta \geq \sigma b^1 + \lambda^1 d^1 + \mu^1 b - \theta^1. \\ & && \lambda^0, \mu^0, \theta^0, \lambda^1, \mu^1, \theta^1, \sigma \geq 0 \end{aligned} \quad (17)$$

A feasible solution to (17) with a positive objective function corresponds to a disjunctive inequality violated at  $(\hat{\eta}, \hat{x})$ . However, the set of feasible solutions to CGLP is a cone and needs to be truncated to produce a bounded optimal solution value in case a violated cut exists. The choice of the normalization constraint used to truncate the cone can be a crucial factor in the effectiveness of disjunctive cutting planes. One normalization constraint studied in [8, 9] is the  $\alpha$ -normalization:

$$\sum_{i=1}^n |\alpha_i| + \sigma = 1. \quad (\alpha\text{NORM})$$

The most widely used normalization constraint was proposed by [6] and is called the *Standard Normalization Condition* (SNC)[18]:

$$\sum_{h \in \{0,1\}} \left( \sum_{j=1}^{|J|} \lambda_j^h + \sum_{j=1}^m \mu_j^h + \theta^h + \sigma \right) = 1. \quad (\text{SNC})$$

The SNC normalization is criticized by Fischetti, Lodi, and Tramontani [18] for its dependence on the relative scaling of the constraints. To overcome this drawback, they proposed the *Euclidean Normalization*

$$\sum_{h \in \{0,1\}} \left( \sum_{j=1}^{|J|} \|d_j^h\| \lambda_j^h + \sum_{j=1}^m \|a_j^h\| \mu_j^h + \theta^h + \|c^h\| \sigma \right) = 1. \quad (\text{EN})$$

instead of (SNC). We refer reader to [18] for further discussion on normalization constraints and their impact on the effectiveness of disjunctive inequalities. In Section 4.8 we will report on the effect of different normalization constraints on our disjunctive inequalities.

### 3.4 Monoidal Strengthening

Disjunctive cuts can be further strengthened by exploiting integrality requirements of variables. This method was introduced by Balas and Jeroslow, where they call it *monoidal strengthening* [10]. In any disjunctive cut (13) the coefficient of  $x_k$ ,  $k \in I \setminus \{i\}$  can be strengthened to take the value

$$\tilde{\alpha}_k = \max\{\alpha_k^0 - \theta^0 \lceil \hat{m}_k \rceil, \alpha_k^1 + \theta^1 \lfloor \hat{m}_k \rfloor\}$$

where

$$\begin{aligned} \alpha_k^0 &= \sigma c_k^0 + \lambda^0 D_k^0 + \mu^0 A_k, \\ \alpha_k^1 &= \sigma c_k^1 + \lambda^1 D_k^1 + \mu^1 A_k, \\ \hat{m}_k &= \frac{\alpha_k^0 - \alpha_k^1}{\theta^0 + \theta^1}, \end{aligned}$$

and  $\lambda_k^0, \mu_k^0, \theta^0, \lambda_k^1, \mu_k^1, \theta^1, \sigma$  satisfy the requirements (14) for multipliers in a disjunctive inequality. The notation  $D_k, A_k$  represents the  $k$ th column of the associated matrix.

### 3.5 Lifting

The disjunctive inequality (13) can be lifted to become globally valid if generated at a node of the branch-and-bound tree. Assume that the inequality is generated at a node of the branch-and-bound tree where the variables in the set  $F_0$  are fixed to zero and the variables in the set  $F_1$  are fixed to one. Without loss generality, we can assume that  $F_1$  is empty by complementing all variables before formulating the CGLP (17).

Let  $R$  be the set of unfixed variables and  $(\alpha, \beta, \lambda^0, \mu^0, \lambda^1, \mu^1, \sigma)$  be a solution to (17) in the subspace where the variables in the set  $F_0$  are fixed to zero so that  $\alpha \in \mathbb{R}^{|R|}$ . The lifting coefficient for the fixed variables is given by Balas *et al.* [8, 9] as

$$\gamma_j = \min\{\sigma c_j^0 + \lambda^0 D_j^0 + \mu^0 A_j, \sigma c_j^1 + \lambda^1 D_j^1 + \mu^1 A_j\}.$$

Thus, the inequality

$$\sum_{j \in R} \alpha_j x_j + \sum_{j \in F_0} \gamma_j x_j - \sigma \eta \leq \beta$$

is valid for the MINLP problem (2).

## 4 Computational Experience

In this section, we report on a collection of experiments designed to test the ideas presented in Sections 2 and 3 with the end goal of deducing how to most effectively exploit information obtained when solving NLP subproblems in a strong-branching scheme. Our implementation is done using the FilMINT solver for convex MINLP problems.

### 4.1 FilMINT

FilMINT is an implementation of the LP/NLP-Based Branch-and-Bound algorithm of Quesada and Grossmann [30], which uses the outerapproximation master problem (4). In our experiments, all strong branching inequalities are added directly to (4). FilMINT uses MINTO [28] to enforce integrality of the master problem via branching and filterSQP [19] for solving nonlinear subproblems that are both necessary for convergence of the method and used in this work to obtain NLP-based strong branching information. In our experiments, FilMINT used the CPLEX (v12.2) software to solve linear programs.

FilMINT by default employs nearly all of MINTO’s enhanced MILP features, such as cutting planes, primal heuristics, row management, and enhanced branching and node selection rules. FilMINT uses the best estimate method for node selection [27].

FilMINT uses a reliability branching approach [3], where strong branching based on the current master *linear program* is performed a limited number of times for each variable. The feasible region of the linear master problem (4) may be significantly strengthened by MINTO’s preprocessing and cutting plane mechanisms, and these formulation improvements are extremely difficult to communicate to the nonlinear solver Filter-SQP. Our approach for communicating NLP-based strong branching information to the master problem was implemented in the following manner. For each variable, we perform NLP-based strong branching by solving  $(NLP_i^0)$  and  $(NLP_i^1)$  the first time the variable is fractional in a relaxation solution. Regardless of the inequalities we add to the master problem, we solve  $(NLP_i^0)$  and  $(NLP_i^1)$  only once per variable to limit the computational burden from solving NLP subproblems, which is appropriate in the context of the linearization-based LP/NLP branch-and-bound algorithm that is used in FilMINT. We then simply add the strong branching inequalities under consideration to the master problem and then let FilMINT make its branching decisions using its default mechanism. This affects the bounds in a manner similar to NLP-based strong branching. For example, for a fractional variable  $x_i$ , after adding a simple SBC (5) or linearizations about solutions to  $(NLP_i^0)$  and  $(NLP_i^1)$ , when FilMINT performs LP-based strong branching on  $x_i$ , the bound obtained from fixing  $x_i \leq \lfloor \hat{x}_i \rfloor$  will be at least  $\hat{\eta}_i^0$ , and likewise the bound obtained from fixing  $x_i \geq \lceil \hat{x}_i \rceil$  will be at least  $\hat{\eta}_i^1$ . Note however, that adding inequalities will also likely affect the value of the relaxation, so the pseudocosts, which measure the *rate of change* of the objective function per unit change in variable bound, may also be affected.

### 4.2 Computational Setup

Our test suite consists of convex MINLPs collected from the MacMINLP collection [25], the GAMS MINLP World [13], the collection on the website of the IBM-CMU research group [33],

and instances that we created ourselves. The test suite consists of 40 convex instances covering a wide range of practical applications such as multi-product batch plant design problems [31, 35], layout design problems [32, 14], synthesis design problems [16, 34], retrofit planning [32], stochastic service system design problems [17], cutting stock problems [23], uncapacitated facility location problems [21] and network design problems [12]. Characteristics of the instances are given in Table 1, which lists whether or not the instance has a nonlinear objective function, the total number of variables, the number of integer variables, the number of constraints, how many of the constraints are nonlinear, and the number of GUB constraints. We chose the instances so that no one family of instances is overrepresented in the group and so that each of the instances is not “too easy” or “too hard.” To accomplish this, we chose instances so that the default version of FilMINT is able to solve each of these instances using CPU time in the range of 30 seconds to 3 hours.

The computational experiments have been run on a cluster of identical 64-bit Intel Core2 Duo microprocessors clocked at 3.00 GHz, each with 2 GB RAM. In order to concisely display the relative performance of different solution techniques, we make use of performance profiles (see [15]). A performance profile is a graph of the relative performance of different solvers on a fixed set of instances. In a performance profile graph, the  $x$ -axis is used for the performance factor. The  $y$ -axis gives the fraction of instances for which the performance of that solver is within a factor of  $x$  of the best solver for that instance. In our experiments, we use both the number of nodes in the branch and bound tree and the CPU solution time as performance metrics.

We often use the “extra” gap closed at the root node as a measure to assess the strength of a class of valid inequalities. The extra gap closed measures the relative improvement in lower bound at the root node over the lower bound found without adding the valid inequalities. Specifically, the extra percentage gap closed is

$$100 \left( \frac{z_{CUTS} - z_{MP(\mathcal{K})}}{z_{MINLP} - z_{MP(\mathcal{K})}} \right),$$

where  $z_{CUTS}$  is the value of LP relaxation after adding inequalities,  $z_{MP(\mathcal{K})}$  is the value of LP relaxation of reduced master problem after preprocessing and default set of cuts of MINTO, and  $z_{MINLP}$  is the optimal solution value.

We summarize computational results in small tables that list the (arithmetic) average extra gap closed, the number of nodes, and the CPU solution time. The instances for which at least one solver failed to terminate because of a memory limit are taken out of consideration when reporting aggregate numbers.

Strong branching inequalities are added in rounds. After adding cuts at a node of the branch-and-bound tree, the linear program is resolved, and a new solution to the relaxation of the master problem (4) is obtained. The strong branching subproblem ( $NLP_i^0$ ) and ( $NLP_i^1$ ) are solved for all fractional variables in the new solution that have not yet been initialized, and associated strong branching inequalities are added. If inequalities are generated at a non-root node, they are lifted to make them globally valid, as explained in Section 3.5. Recall that NLP-based strong branching is performed at most once for each variable.

We are primarily interested in the impact of using strong branching information to improve the lower bound of a linearization-based algorithm. Therefore, to eliminate variability in solution

time induced by the effect of finding improved upper bounds during the search, in our experiments we input the optimal solution value to FilMINT as a cutoff value and disable primal heuristics.

### 4.3 Performance of SBC Inequalities and Linearizations

Our first experiment was aimed at comparing *elementary* methods for exploiting information from NLP-based strong branching subproblems. The methods chosen for comparison in this experiment were

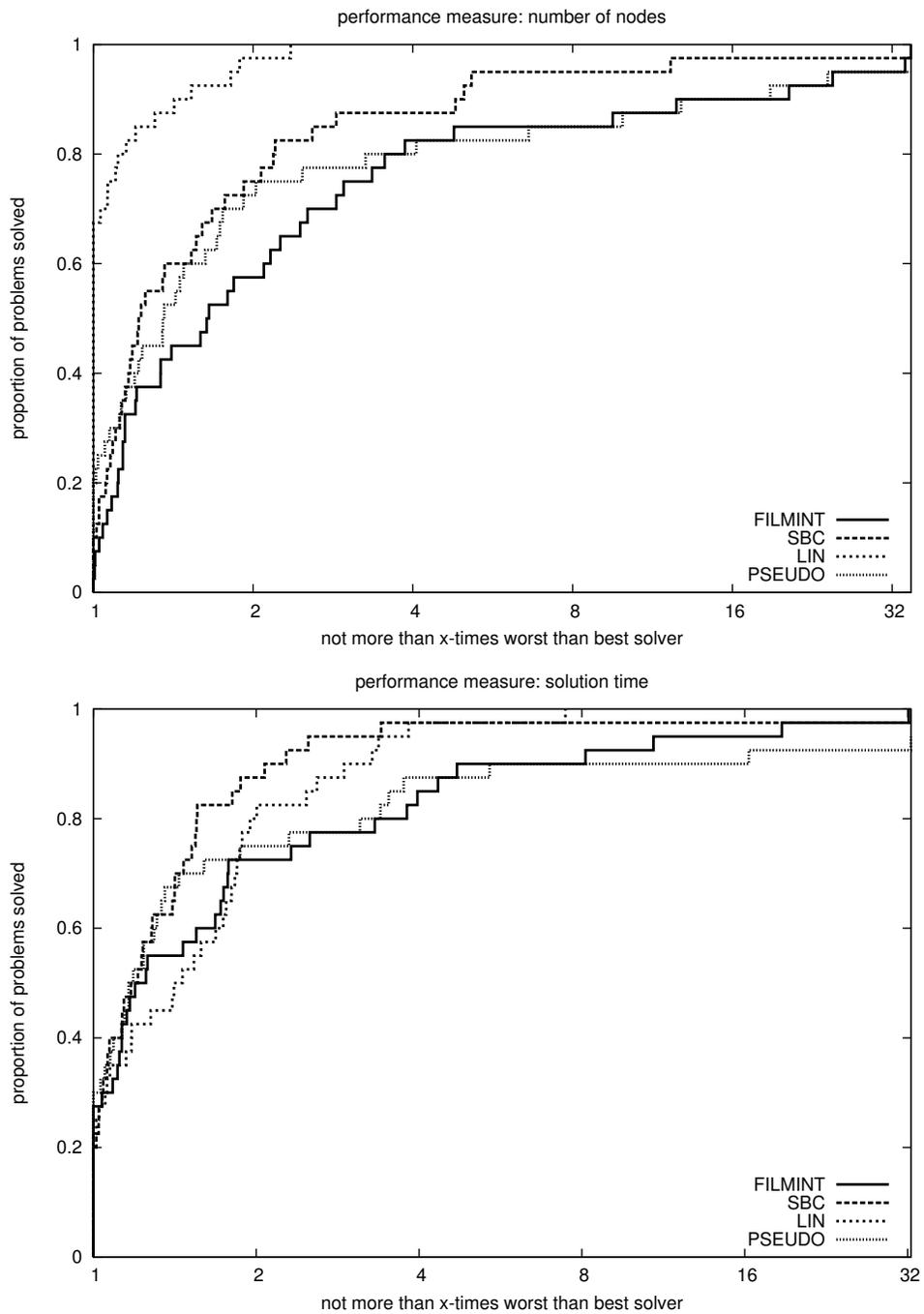
- **FILMINT**: The default version of FilMINT.
- **LIN**: FilMINT, but with the master problem augmented with linearizations from NLP-based branching subproblems, as described in Section 2.1.
- **SBC**: FilMINT, but with the master problem augmented with the simple strong branching cuts (5).
- **PSEUDO**: FilMINT, but with an NLP-based strong branching strategy in which strong branching inequalities *are not added*. Rather, only the pseudocost value are initialized using the NLP-based strong branching information.

We included the method PSEUDO to test whether or not using valid inequalities derived from NLP-based strong branching can yield improvement beyond simply using the information for branching. In PSEUDO, as in the versions that add strong branching inequalities, we perform NLP-based strong branching at most once for each variable. However, rather than adding any strong branching inequalities, we instead simply initialize FilMINT’s pseudocosts based on the optimal values of  $(NLP_i^0)$  and  $(NLP_i^1)$ . These pseudocosts are then updated based on FilMINT’s default update strategy. Thus, although the initial pseudocost information is based directly on the nonlinear programming relaxation, updates to the pseudocosts are dependent on the outer approximation that has been obtained in the master problem.

Tables 7 and 8 in the appendix give the performance of each of these methods on each of the instances in our test suite. The tables are summarized in Figure 1, which consists of two performance profiles. The first profile uses the the number of nodes in the branch and bound tree as the solution metric. This profile indicates that *all* methods that incorporate NLP-based strong branching information are useful for reducing the size of the branch and bound tree, but also that using strong branching information to derive valid inequalities in addition to making branching decisions can further reduce the size. The most effective method in terms of number of nodes is LIN. The second profile uses CPU time as the quality metric. In this measure, SBC is the best method, and all methods are at least as good as FILMINT.

The two profiles together paint the picture that simple strong branching cuts (5) can be an effective mechanism for improving performance of a linearization-based convex MINLP solver. The SBC inequalities are not as strong as adding all linearizations, but this is not a surprising result, as the SBC inequalities aggregate the linearization information into a single inequality. From the results of this experiment, we also conclude that a well-engineered mechanism for incorporating “useful” linearizations from points suggested by NLP-based strong branching, while

Figure 1: Performance Profile of Elementary NLP-based Strong Branching Inequalities



not overwhelming the linear master problem (4) is likely to be the most effective “elementary” mechanism for making use of information from NLP-based strong branching subproblems. We return to this idea in Section 4.6.

#### 4.4 Performance of GUB-SBC Inequalities

A second experiment was designed to test the effectiveness of performing NLP-based strong branching on the GUB disjunction (6) and using the resulting GUBSBC inequality (7). Of primary interest is how the method performs compared to using only the disjunction on the individual binary variables via the simple SBC inequality (5).

In this experiment, if at least one of the variables in a GUB constraint is fractional at a solution to the master problem (4), then strong branching on the GUB constraint is performed, and a GUBSBC inequality (7) is generated. In order to generate a GUBSBC inequality, nonlinear subproblems ( $NLP_i^1$ ) are solved for each of the variables in the GUB constraint, regardless of whether the variable value is fractional. In our implementation, at most one GUBSBC inequality is generated for each GUB, and the GUBSBC inequalities are generated at the root node only. If we encounter a fractional binary variable that is not in any GUB constraint, or we are not at the root node, then a simple SBC inequality (5) is generated for that variable.

In Table 2, we give computational results comparing the relative strength of SBC inequalities (5) and the GUBSBC inequalities (7). The detailed performance of methods on each instance is given in Table 9. The comparison is done for 30 instances from our test set of 40 problems for which there exists at least one GUB constraint in the problem. On average, adding GUBSBC inequalities closed 19.20% of the gap at root node, and adding only SBC inequalities closed 9.66%. It is then somewhat surprising that the number of nodes required for the two methods is approximately equal. One explanation of this phenomenon is that FilmINT chooses worse branching variables when GUBSBC inequalities are introduced to the master problem (4).

While adding GUBSBC inequalities can make a significant positive impact on solving some instances, our primary conclusion from this experiment is that the GUBSBC inequalities do not improve the performance of FilmINT more than the SBC inequalities, thus we focused our remaining computational experiments on evaluating only enhanced versions of SBC inequalities.

#### 4.5 Performance of Mixing Strong Branching Inequalities

We next compared the effectiveness of the mixed strong branching inequalities (MIXSBC) (11) against the unmixed version (5). There may be exponentially many mixed strong branching inequalities, so we use the following strategy for adding them to the master problem. First, as in our of our methods, the NLP subproblems ( $NLP_i^0$ ) and ( $NLP_i^1$ ) are solved for each fractional variable  $x_i$  in the solution to the relaxed master problem (4). The fractional variables for which ( $NLP_i^0$ ) and ( $NLP_i^1$ ) have been solved define the mixing set  $\bar{B}$ . Next, for each variable in the mixing set, we add the *sparsest* MIXSBC inequality for that variable:

$$\eta \geq \underline{\eta} + \sigma_i x_i + (\sigma_h - \sigma_i) x_h \quad \forall i \in \bar{B}, \tag{18}$$

where  $h = \operatorname{argmax}_{i \in \bar{B}} \sigma_i$ . Note that the sparsest MIXSBC inequality (18) already dominates the SBC inequality (5). Finally, after obtaining a fractional solution from the relaxation of the

master problem, (after adding the inequalities (18)), the two most violated mixing inequalities are added and the relaxation is resolved. The MIXSBC inequalities are added in rounds until none are violated or until the inequalities do not change the relaxation solution by a sufficient amount. Specifically, if  $\sum_{i \in B} |x'_i - x''_i| < 0.1$  for consecutive relaxation solutions  $x', x''$ , no further MIXSBC inequalities are added.

In Table 3, we summarize computational results comparing the effect of adding MIXSBC inequalities (11) with adding only SBC inequalities (5). The detailed performance of each method on each instance is given in Table 10. The MIXSBC inequalities are significantly stronger than the SBC inequalities. On average, MIXSBC closed 19.06% of the optimality gap at root node, and SBC closed only 7.77% of the gap on our test set. Despite this, MIXSBC inequalities perform *worse* than SBC in terms of average number of nodes and solution time. An explanation for this counterintuitive behavior is that the addition of the mixed strong branching inequalities (11) results in MINTO (and hence FilmINT) performing “poor” updates on the pseudocost values for integer variables. That is, in subsequent branches, the pseudocosts do not accurately reflect the true change in objective value if a variable is branched on. Therefore, MIXSBC makes poor branching decisions, which in turn leads to a larger search tree. For example, MIXSBC closed 62.3% of the gap at the root node for the instance *SLay09M* and SBC closed only 23.6%. However, 92 seconds and 13,651 nodes are required to prove optimality when using MIXSBC, compared to only 29 seconds and 6,035 nodes for SBC alone.

## 4.6 Linearization Strategies

In our computational experiments we were not able to significantly improve performance of strong-branching inequalities by exploiting GUB disjunctions or by mixing them. We therefore conclude that, among the strategies for obtaining strong branching inequalities with minimal additional computational effort, adding linearizations from NLP strong branching subproblems has the most potential as a computational technique. The performance profiles in Figure 1 indicate that linearizations are very effective in reducing the number of nodes, but often lead to unacceptable solution times due to the large number of linearizations added to the master problem. Two simple ideas to improve the performance of linearizations are to add only violated linearizations and to quickly remove linearizations that are not binding in the solution of the LP relaxation of the master problem.

In Table 4, we summarize computational results comparing our original linearization scheme LIN with an improved version denoted by BESTLIN. In BESTLIN, only linearization inequalities that are violated by the current relaxation solution are added, and if the dual variable for a linearization inequality has value zero for five consecutive relaxation solutions, the inequality is removed for the master problem. Full results of the performance of the two methods on each instance can be found in Table 11. The results show that without degrading the performance of LIN in terms of the number of nodes, BESTLIN can improve the average solution time from 1,568.3 seconds to 1,078.9 seconds.

## 4.7 Performance of Multiplier-Strengthened SBC Inequalities

Initial computational experience with the multiplier-strengthened cuts SSBC (16), introduced in Section 3.2 suggested that the inequalities in general were far too dense to be effectively used in a linearization-based scheme. Very quickly, the LP relaxation of the master problem (4) became prohibitively expensive to solve. Our remedy for these dense cuts was to strengthen only the coefficients of integer variables. Additionally, after the inequality was generated, the monoidal strengthening step described in Section 3.4 was performed on the inequalities.

An experiment was done to compare the performance of using the SBC inequalities against the SSBC inequalities on instances in our test set, and a summary of the results are given in Table 5. The computational results indicate a slight improvement in both the number of nodes and the solution time by strengthening the SBC inequalities using multipliers of the NLP-strong branching subproblems. Full results of the performance of the two methods on each instance can be found in Table 12.

## 4.8 Normalizations for CGLP

In Section 3.3, we described three different normalization constraints that are commonly used for the CGLP (17). We performed a small experiment to compare the relative effectiveness of each in our context. The performance profiles of Figure 2 summarize the results of comparing disjunctive inequalities generated by solving the CGLP (17) with the normalization constraints ( $\alpha$ NORM) denoted by SBCGLP-INF, (SNC) denoted by SBCGLP-SNC, and (EN) denoted by SBCGLP-EN. The performance profiles show that both the standard normalization condition (SNC) and the Euclidean normalization (EN) perform significantly better than the  $\alpha$ -normalization. The results are consistent with the literature. The performance of (SNC) and (EN) are comparable with each other, suggesting that the constraints of the instances in our test suite are well-scaled. The detailed performance of methods on each instance is given in Table 13.

## 4.9 Comparison of All Methods

We make a final comparison of the methods introduced in the paper. In this experiment, we compare the methods that performed best in earlier experiments with the default version of FilMINT. The methods we compare are the following:

- **FilMINT**: The default version of FilMINT.
- **BESTLIN**: FilMINT, with the master problem augmented with linearizations from NLP-based branching subproblems, as described in Section 2.1. The linearization management strategy introduced in Section 4.6 is employed.
- **SSBC**: FilMINT, with the master problem augmented with the multiplier-strengthened strong branching cuts (SSBC) (16).
- **SBCGLP-SNC**: FilMINT, adding disjunctive inequalities based on solving the CGLP (17) using the standard normalization condition (SNC).

Figure 2: Performance Profile of CGLP with Different Normalization Constraints

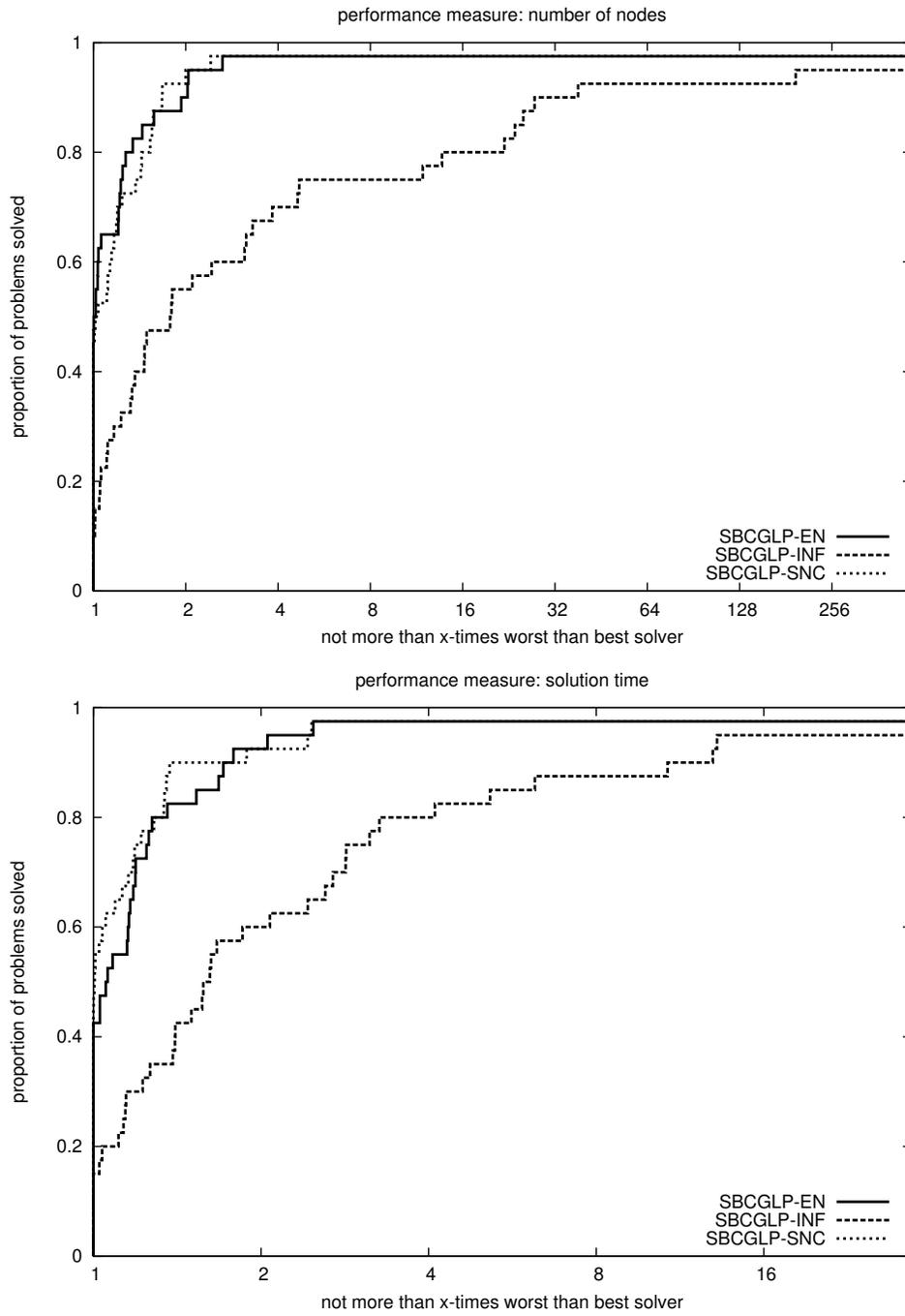
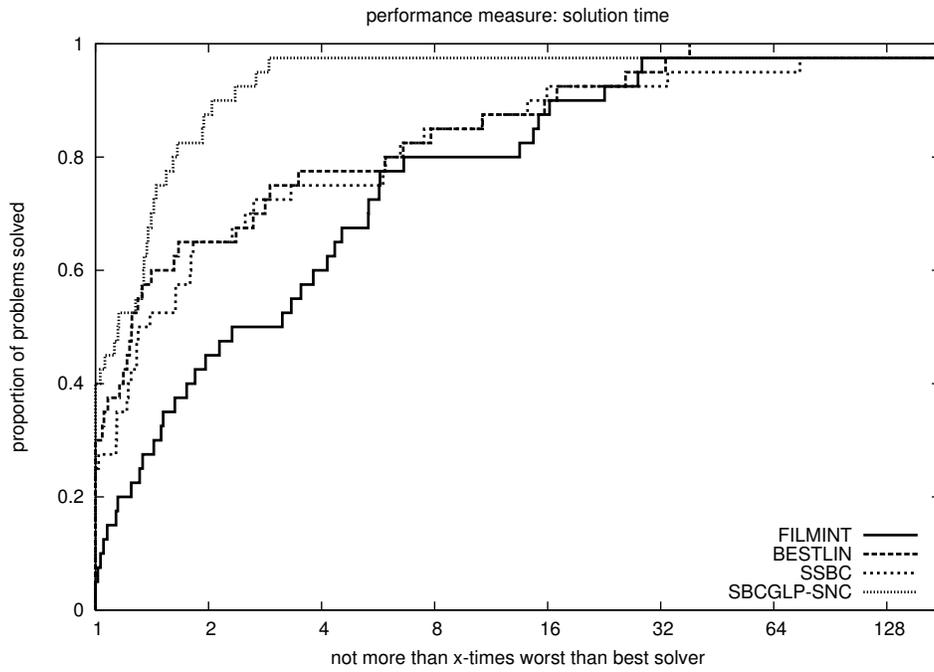
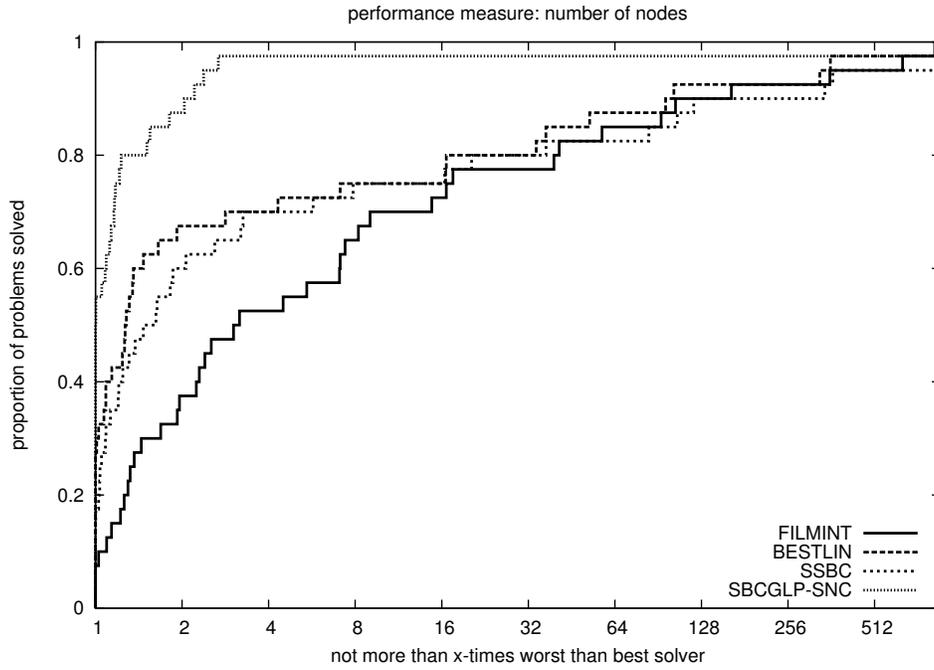


Figure 3: Performance Profile of Best Methods and FILMINT



The monoidal strengthening step described in Section 3.4 was applied to the inequalities generated by methods SSBC and SBCGLP-SNC. In Table 6, we list the average number of nodes and solution time for 38 instances in our test set. (The instances *RSyn0820M02M* and *Safety3* are taken out of consideration when reporting aggregate numbers in Table 6 since SSBC and SBCGLP-SNC failed to terminate for these instances due to a memory limit). The performance profiles in Figure 3 show that creating disjunctive inequalities by solving the CGLP (17) with the standard normalization condition significantly outperforms the other methods. Creating disjunctive inequalities by an extra solve of (17) pays dividends both in terms of number of nodes and solution time. We experienced similar positive effects with other normalization constraints introduced in Section 3.3 as well. Additionally, both linearizations and SSBC inequalities improve the performance of default FilMINT substantially. The detailed performance of methods on each instance is given in Table 14 and Table 15.

## 5 Conclusions

In this work, we demonstrate how to use “discarded” information generated from NLP-based strong branching to strengthen relaxations of MINLP problems. We first introduced strong branching cuts, and we demonstrated the relation of strong branching cuts we derive with other well-known disjunctive inequalities in the literature. We improved these basic cuts by using Lagrange multipliers and the integrality of variables. We combined strong branching cuts via mixing. We demonstrated that simple disjunctive inequalities can be improved by additional linearizations generated from strong branching subproblems. Finally, the methods explained in this paper significantly improve the performance of FilMINT, justifying the use of strong branching based on nonlinear subproblems for solving convex MINLP problems.

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## Appendix

For Tables 7-15, if the method could not provide the optimal solution because the solver hit a memory limit we state the reason with the letter m.

Table 7: Number of Nodes for FILMINT, SBC, LIN and PSEUDO.  
Experiment is described in Section 4.3

| <b>Problem</b>         | <b>FILMINT</b> | <b>SBC</b> | <b>LIN</b> | <b>PSEUDO</b> |
|------------------------|----------------|------------|------------|---------------|
| BatchS151208M          | 5391           | 3071       | 1395       | 1609          |
| Continued on next page |                |            |            |               |

Table 7 – continued from previous page

| Problem                      | FILMINT | SBC     | LIN     | PSEUDO  |
|------------------------------|---------|---------|---------|---------|
| BatchS201210M                | 3447    | 2909    | 1405    | 1911    |
| Batch_Storage10_BM_10_10_6_4 | 15917   | 10351   | 7597    | 11059   |
| Batch_Storage_BM_10_10_6_4   | 21849   | 11817   | 9707    | 15781   |
| CLay0305H                    | 26449   | 11017   | 15979   | 10445   |
| FLay05H                      | 102207  | 99691   | 102791  | 101697  |
| FLay05M                      | 91579   | 83289   | 82173   | 83129   |
| fo7_2                        | 109811  | 105729  | 78265   | 111777  |
| fo7                          | 276547  | 331499  | 293427  | 206665  |
| m7                           | 621027  | 320671  | 395459  | 209605  |
| nd-12                        | 20873   | 12179   | 17125   | 7275    |
| nd-13                        | 24391   | 8023    | 7587    | 6891    |
| nd-14                        | 101813  | 105107  | 95321   | 89597   |
| o7_2                         | 2028675 | 1567755 | 1417959 | 1274499 |
| o7                           | 2675019 | 2741915 | 3213077 | 4634451 |
| RSyn0810M02M                 | 117015  | 114369  | 102065  | 126109  |
| RSyn0810M03M                 | 88593   | 93341   | 87963   | 98585   |
| RSyn0810M04M                 | 156835  | 136943  | 87607   | 92047   |
| RSyn0815M02M                 | 493283  | 486495  | 457321  | 430125  |
| RSyn0820M02M                 | 2157455 | m       | 2032033 | m       |
| RSyn0830M02M                 | 825939  | 903237  | 763039  | 1129401 |
| Safety3                      | 3087205 | 1431767 | 1869475 | 1936385 |
| safety_no_rotation_CH        | 18327   | 5525    | 1925    | 4769    |
| SLay07H                      | 4505    | 635     | 359     | 689     |
| SLay08H                      | 15203   | 1921    | 743     | 1505    |
| SLay09H                      | 82525   | 12257   | 2377    | 15709   |
| SLay09M                      | 16689   | 6035    | 493     | 4899    |
| sssd-16-8-3                  | 2092313 | 2793721 | 1279401 | m       |
| sssd-17-7-3                  | 1287961 | 1297181 | 269459  | 3452263 |
| sssd-18-7-3                  | 265695  | 396017  | 79281   | 1918463 |
| sssd-20-8-3                  | 3573363 | 144505  | 262273  | 2725971 |
| Syn20M04M                    | 90393   | 89875   | 103651  | 153573  |
| Syn30M03M                    | 49731   | 50247   | 43761   | 177719  |
| Syn30M04M                    | 237847  | 245963  | 228451  | 744635  |
| Syn40M02M                    | 145331  | 85947   | 79033   | 89121   |
| trimloss4                    | 25535   | 40863   | 21279   | 25885   |
| uffquad-15-60                | 2523    | 2303    | 2091    | 2501    |
| uffquad-15-80                | 3689    | 3401    | 3319    | 3559    |
| uffquad-20-40                | 3591    | 3359    | 2681    | 3619    |
| uffquad-25-40                | 6973    | 5115    | 4221    | 7399    |

Table 8: Solution Time for FILMINT, SBC, LIN and PSEUDO.  
Experiment is described in Section 4.3

| <b>Problem</b>               | <b>FILMINT</b> | <b>SBC</b> | <b>LIN</b> | <b>PSEUDO</b> |
|------------------------------|----------------|------------|------------|---------------|
| BatchS151208M                | 52.0           | 32.0       | 30.6       | 29.9          |
| BatchS201210M                | 43.3           | 53.3       | 43.9       | 34.4          |
| Batch_Storage10_BM_10_10_6_4 | 112.1          | 92.6       | 63.0       | 81.7          |
| Batch_Storage_BM_10_10_6_4   | 150.8          | 99.9       | 85.1       | 111.6         |
| CLay0305H                    | 108.7          | 28.6       | 96.4       | 32.3          |
| FLay05H                      | 1702.0         | 1486.6     | 1995.6     | 1425.5        |
| FLay05M                      | 696.6          | 695.9      | 726.3      | 617.1         |
| fo7_2                        | 177.0          | 163.0      | 174.9      | 189.4         |
| fo7                          | 355.6          | 442.4      | 535.6      | 284.2         |
| m7                           | 796.8          | 336.8      | 787.8      | 240.4         |
| nd-12                        | 1499.0         | 681.8      | 2814.8     | 377.4         |
| nd-13                        | 497.8          | 509.1      | 1233.2     | 542.4         |
| nd-14                        | 6222.7         | 7502.8     | 10516.8    | 3618.7        |
| o7_2                         | 3384.4         | 2580.0     | 3548.3     | 2015.1        |
| o7                           | 3797.3         | 4292.8     | 6936.7     | 8728.7        |
| RSyn0810M02M                 | 272.7          | 279.0      | 431.5      | 281.0         |
| RSyn0810M03M                 | 326.5          | 461.8      | 654.4      | 436.3         |
| RSyn0810M04M                 | 746.3          | 821.7      | 900.7      | 639.2         |
| RSyn0815M02M                 | 1126.5         | 1136.7     | 1944.3     | 998.1         |
| RSyn0820M02M                 | 5770.4         | m          | 8433.4     | m             |
| RSyn0830M02M                 | 3017.0         | 3204.5     | 5623.0     | 3738.6        |
| Safety3                      | 9063.5         | 3607.8     | 6271.4     | 5779.5        |
| safety_no_rotation_CH        | 63.4           | 20.5       | 13.5       | 18.3          |
| SLay07H                      | 38.4           | 8.9        | 9.4        | 9.3           |
| SLay08H                      | 197.0          | 29.8       | 24.3       | 28.7          |
| SLay09H                      | 1458.6         | 194.2      | 77.8       | 264.1         |
| SLay09M                      | 93.1           | 29.2       | 8.6        | 32.2          |
| sssd-16-8-3                  | 1106.5         | 1563.5     | 2039.2     | m             |
| sssd-17-7-3                  | 614.7          | 602.1      | 265.0      | 1431.0        |
| sssd-18-7-3                  | 110.5          | 185.1      | 98.9       | 1609.8        |
| sssd-20-8-3                  | 3317.5         | 103.2      | 394.9      | 3348.6        |
| Syn20M04M                    | 153.1          | 188.9      | 396.5      | 286.0         |
| Syn30M03M                    | 147.0          | 177.7      | 264.6      | 516.7         |
| Syn30M04M                    | 1127.7         | 1141.5     | 1730.2     | 3507.4        |
| Syn40M02M                    | 317.8          | 213.1      | 345.3      | 205.1         |
| trimloss4                    | 36.9           | 57.0       | 38.8       | 42.2          |
| uflquad-15-60                | 437.4          | 379.5      | 446.0      | 470.0         |
| uflquad-15-80                | 1220.9         | 1177.7     | 1354.1     | 1265.2        |

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Table 8 – continued from previous page

| Problem       | FILMINT | SBC   | LIN   | PSEUDO |
|---------------|---------|-------|-------|--------|
| uflquad-20-40 | 463.4   | 443.3 | 418.1 | 485.8  |
| uflquad-25-40 | 1389.4  | 948.9 | 960.5 | 1365.7 |

Table 9: Computational Results Comparing SBC and GUBSBC Inequalities. Experiment is described in Section 4.4

| Problem                      | SBC     |        | GUBSBC  |        |
|------------------------------|---------|--------|---------|--------|
|                              | Node    | Time   | Node    | Time   |
| BatchS151208M                | 3071    | 32.0   | 2869    | 43.0   |
| BatchS201210M                | 2909    | 53.3   | 2591    | 50.9   |
| Batch_Storage10_BM_10_10_6_4 | 10351   | 92.6   | 8497    | 66.6   |
| Batch_Storage_BM_10_10_6_4   | 11817   | 99.9   | 12643   | 91.5   |
| CLay0305H                    | 11017   | 28.6   | 19173   | 44.1   |
| FLay05H                      | 99691   | 1486.6 | 114785  | 1791.7 |
| FLay05M                      | 83289   | 695.9  | 85747   | 757.0  |
| nd-12                        | 12179   | 681.8  | 11675   | 700.8  |
| nd-13                        | 8023    | 509.1  | 7683    | 736.8  |
| nd-14                        | 105107  | 7502.8 | 105997  | 4642.1 |
| RSyn0810M02M                 | 114369  | 279.0  | 116249  | 312.0  |
| RSyn0810M03M                 | 93341   | 461.8  | 95047   | 632.4  |
| RSyn0810M04M                 | 136943  | 821.7  | 91385   | 826.4  |
| RSyn0815M02M                 | 486495  | 1136.7 | 441117  | 1162.7 |
| RSyn0820M02M                 | m       | m      | 1983993 | 5824.7 |
| RSyn0830M02M                 | 903237  | 3204.5 | 1013379 | 3507.7 |
| Safety3                      | 1431767 | 3607.8 | 1621317 | 4299.8 |
| safety_no_rotation_CH        | 5525    | 20.5   | 5515    | 20.0   |
| SLay07H                      | 635     | 8.9    | 1167    | 11.0   |
| SLay08H                      | 1921    | 29.8   | 3649    | 45.1   |
| SLay09H                      | 12257   | 194.2  | 16239   | 289.7  |
| SLay09M                      | 6035    | 29.2   | 7409    | 50.0   |
| sssd-16-8-3                  | 2793721 | 1563.5 | 2134397 | 1175.7 |
| sssd-17-7-3                  | 1297181 | 602.1  | 318079  | 158.4  |
| sssd-18-7-3                  | 396017  | 185.1  | 358449  | 178.0  |
| sssd-20-8-3                  | 144505  | 103.2  | 696655  | 375.0  |
| Syn20M04M                    | 89875   | 188.9  | 89227   | 261.3  |
| Syn30M03M                    | 50247   | 177.7  | 51149   | 226.9  |
| Syn30M04M                    | 245963  | 1141.5 | 235895  | 1309.7 |
| Syn40M02M                    | 85947   | 213.1  | 141867  | 396.9  |

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Table 9 – continued from previous page

| Problem   | SBC   |      | GUBSBC |      |
|-----------|-------|------|--------|------|
|           | Node  | Time | Node   | Time |
| trimloss4 | 40863 | 57.0 | 29579  | 48.7 |

Table 10: Computational Results Comparing SBC versus MIXSBC. Experiment is described in Section 4.5

| Problem                      | SBC     |        | MIXSBC  |         |
|------------------------------|---------|--------|---------|---------|
|                              | Node    | Time   | Node    | Time    |
| BatchS151208M                | 3071    | 32.0   | 3243    | 74.0    |
| BatchS201210M                | 2909    | 53.3   | 3091    | 71.5    |
| Batch_Storage10_BM_10_10_6_4 | 10351   | 92.6   | 13375   | 108.5   |
| Batch_Storage_BM_10_10_6_4   | 11817   | 99.9   | 10833   | 90.7    |
| CLay0305H                    | 11017   | 28.6   | 15937   | 54.7    |
| FLay05H                      | 99691   | 1486.6 | 104233  | 1569.8  |
| FLay05M                      | 83289   | 695.9  | 82225   | 657.5   |
| fo7_2                        | 105729  | 163.0  | 105729  | 163.0   |
| fo7                          | 331499  | 442.4  | 331499  | 442.0   |
| m7                           | 320671  | 336.8  | 320671  | 336.6   |
| nd-12                        | 12179   | 681.8  | 9715    | 606.5   |
| nd-13                        | 8023    | 509.1  | 7403    | 541.6   |
| nd-14                        | 105107  | 7502.8 | 94813   | 6208.2  |
| o7_2                         | 1567755 | 2580.0 | 1567755 | 2586.1  |
| o7                           | 2741915 | 4292.8 | 2741915 | 4297.7  |
| RSyn0810M02M                 | 114369  | 279.0  | 115679  | 329.8   |
| RSyn0810M03M                 | 93341   | 461.8  | 93097   | 474.6   |
| RSyn0810M04M                 | 136943  | 821.7  | 91211   | 639.1   |
| RSyn0815M02M                 | 486495  | 1136.7 | 464247  | 1197.6  |
| RSyn0820M02M                 | m       | m      | 2001271 | 5427.3  |
| RSyn0830M02M                 | 903237  | 3204.5 | 903237  | 3200.8  |
| Safety3                      | 1431767 | 3607.8 | 3267119 | 10082.1 |
| safety_no_rotation_CH        | 5525    | 20.5   | 5227    | 21.3    |
| SLay07H                      | 635     | 8.9    | 1351    | 13.3    |
| SLay08H                      | 1921    | 29.8   | 2559    | 42.5    |
| SLay09H                      | 12257   | 194.2  | 10699   | 174.2   |
| SLay09M                      | 6035    | 29.2   | 13651   | 92.4    |
| sssd-16-8-3                  | 2793721 | 1563.5 | 3966469 | 1835.8  |
| sssd-17-7-3                  | 1297181 | 602.1  | 846487  | 381.1   |
| sssd-18-7-3                  | 396017  | 185.1  | 389081  | 181.8   |

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Table 10 – continued from previous page

| Problem       | SBC    |        | MIXSBC  |        |
|---------------|--------|--------|---------|--------|
|               | Node   | Time   | Node    | Time   |
| sssd-20-8-3   | 144505 | 103.2  | 1461383 | 761.0  |
| Syn20M04M     | 89875  | 188.9  | 90757   | 184.3  |
| Syn30M03M     | 50247  | 177.7  | 50247   | 177.5  |
| Syn30M04M     | 245963 | 1141.5 | 245963  | 1139.9 |
| Syn40M02M     | 85947  | 213.1  | 83179   | 223.6  |
| trimloss4     | 40863  | 57.0   | 27273   | 56.3   |
| uflquad-15-60 | 2303   | 379.5  | 2431    | 475.8  |
| uflquad-15-80 | 3401   | 1177.7 | 3595    | 1428.9 |
| uflquad-20-40 | 3359   | 443.3  | 4063    | 624.9  |
| uflquad-25-40 | 5115   | 948.9  | 6697    | 1651.8 |

Table 11: Computational Results Comparing LIN versus BESTLIN. Experiment is described in Section 4.6

| Problem                      | LIN     |          | BESTLIN |         |
|------------------------------|---------|----------|---------|---------|
|                              | Node    | Time     | Node    | Time    |
| BatchS151208M                | 1395    | 30.6     | 2347    | 37.91   |
| BatchS201210M                | 1405    | 43.87    | 1791    | 41.99   |
| Batch_Storage10_BM_10_10_6_4 | 7597    | 63.04    | 7101    | 57.09   |
| Batch_Storage_BM_10_10_6_4   | 9707    | 85.12    | 8643    | 65.3    |
| CLay0305H                    | 15979   | 96.38    | 21131   | 83.32   |
| FLay05H                      | 102791  | 1995.57  | 106617  | 1600.88 |
| FLay05M                      | 82173   | 726.34   | 87447   | 647.02  |
| fo7_2                        | 78265   | 174.92   | 85429   | 151.63  |
| fo7                          | 293427  | 535.6    | 300365  | 370.69  |
| m7                           | 395459  | 787.84   | 130141  | 171.17  |
| nd-12                        | 17125   | 2814.81  | 9673    | 771.29  |
| nd-13                        | 7587    | 1233.23  | 8283    | 691.18  |
| nd-14                        | 95321   | 10516.79 | 115987  | 6062.51 |
| o7_2                         | 1417959 | 3548.33  | 1564357 | 3438.74 |
| o7                           | 3213077 | 6936.68  | 3376079 | 4737.24 |
| RSyn0810M02M                 | 102065  | 431.46   | 116957  | 301.3   |
| RSyn0810M03M                 | 87963   | 654.44   | 85217   | 466.33  |
| RSyn0810M04M                 | 87607   | 900.72   | 63177   | 510.04  |
| RSyn0815M02M                 | 457321  | 1944.27  | 462711  | 1206.16 |
| RSyn0820M02M                 | 2032033 | 8433.37  | 1954165 | 5218.89 |
| RSyn0830M02M                 | 763039  | 5622.96  | 856223  | 3572.13 |

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Table 11 – continued from previous page

| Problem               | LIN     |         | BESTLIN |         |
|-----------------------|---------|---------|---------|---------|
|                       | Node    | Time    | Node    | Time    |
| Safety3               | 1869475 | 6271.37 | 2604659 | 7053.02 |
| safety_no_rotation_CH | 1925    | 13.48   | 4573    | 17.98   |
| SLay07H               | 359     | 9.38    | 813     | 11.06   |
| SLay08H               | 743     | 24.25   | 2349    | 36.69   |
| SLay09H               | 2377    | 77.8    | 6345    | 125.43  |
| SLay09M               | 493     | 8.58    | 2295    | 16.35   |
| sssd-16-8-3           | 1279401 | 2039.23 | 1317759 | 789.18  |
| sssd-17-7-3           | 269459  | 264.96  | 787457  | 372.9   |
| sssd-18-7-3           | 79281   | 98.92   | 201431  | 105.19  |
| sssd-20-8-3           | 262273  | 394.88  | 242025  | 146.23  |
| Syn20M04M             | 103651  | 396.46  | 89153   | 264.4   |
| Syn30M03M             | 43761   | 264.62  | 45875   | 182     |
| Syn30M04M             | 228451  | 1730.23 | 249021  | 1261.79 |
| Syn40M02M             | 79033   | 345.33  | 81585   | 210.75  |
| trimloss4             | 21279   | 38.83   | 23583   | 34.36   |
| uflquad-15-60         | 2091    | 445.99  | 1851    | 292.52  |
| uflquad-15-80         | 3319    | 1354.05 | 3019    | 914.67  |
| uflquad-20-40         | 2681    | 418.12  | 2853    | 324.03  |
| uflquad-25-40         | 4221    | 960.54  | 4831    | 794.65  |

Table 12: Computational Results Comparing SBC versus SSBC. Experiment is described in Section 4.7

| Problem                      | SBC    |         | SSBC   |         |
|------------------------------|--------|---------|--------|---------|
|                              | Node   | Time    | Node   | Time    |
| BatchS151208M                | 3071   | 31.97   | 3071   | 31.95   |
| BatchS201210M                | 2909   | 53.33   | 3259   | 51.32   |
| Batch_Storage10_BM_10_10_6_4 | 10351  | 92.55   | 14651  | 131.98  |
| Batch_Storage_BM_10_10_6_4   | 11817  | 99.87   | 11879  | 106.69  |
| CLay0305H                    | 11017  | 28.62   | 11017  | 28.6    |
| FLay05H                      | 99691  | 1486.59 | 99691  | 1484.55 |
| FLay05M                      | 83289  | 695.89  | 83289  | 696.52  |
| fo7_2                        | 105729 | 163.01  | 105729 | 163.26  |
| fo7                          | 331499 | 442.44  | 331499 | 442.27  |
| m7                           | 320671 | 336.79  | 320671 | 336.7   |
| nd-12                        | 12179  | 681.76  | 6587   | 476.7   |
| nd-13                        | 8023   | 509.07  | 8471   | 490.8   |

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Table 12 – continued from previous page

| Problem               | SBC     |         | SSBC    |         |
|-----------------------|---------|---------|---------|---------|
|                       | Node    | Time    | Node    | Time    |
| nd-14                 | 105107  | 7502.78 | 110687  | 4997.81 |
| o7_2                  | 1567755 | 2579.99 | 1567755 | 2581.51 |
| o7                    | 2741915 | 4292.82 | 2741915 | 4327.49 |
| RSyn0810M02M          | 114369  | 279.01  | 115459  | 302.48  |
| RSyn0810M03M          | 93341   | 461.84  | 94485   | 467.8   |
| RSyn0810M04M          | 136943  | 821.72  | 77739   | 597.46  |
| RSyn0815M02M          | 486495  | 1136.73 | 463365  | 1086.75 |
| RSyn0820M02M          | m       | m       | m       | m       |
| RSyn0830M02M          | 903237  | 3204.45 | 1071975 | 3619.43 |
| Safety3               | 1431767 | 3607.77 | 1576035 | 4239.98 |
| safety_no_rotation_CH | 5525    | 20.5    | 5525    | 20.47   |
| SLay07H               | 635     | 8.85    | 635     | 8.85    |
| SLay08H               | 1921    | 29.78   | 1921    | 29.77   |
| SLay09H               | 12257   | 194.23  | 12257   | 194.42  |
| SLay09M               | 6035    | 29.21   | 6035    | 29.22   |
| sssd-16-8-3           | 2793721 | 1563.47 | 1491441 | 834.42  |
| sssd-17-7-3           | 1297181 | 602.06  | 1043811 | 805.61  |
| sssd-18-7-3           | 396017  | 185.12  | 374615  | 191.52  |
| sssd-20-8-3           | 144505  | 103.19  | 354987  | 238.97  |
| Syn20M04M             | 89875   | 188.87  | 91683   | 197.25  |
| Syn30M03M             | 50247   | 177.72  | 47779   | 178.85  |
| Syn30M04M             | 245963  | 1141.51 | 248575  | 1186.51 |
| Syn40M02M             | 85947   | 213.07  | 83171   | 210.24  |
| trimloss4             | 40863   | 57.03   | 23353   | 35.02   |
| uflquad-15-60         | 2303    | 379.45  | 2303    | 379.84  |
| uflquad-15-80         | 3401    | 1177.67 | 3401    | 1177.76 |
| uflquad-20-40         | 3359    | 443.29  | 3435    | 417.04  |
| uflquad-25-40         | 5115    | 948.88  | 5233    | 963.24  |

Table 13: Computational Results Comparing Normalization Conditions  $\alpha$ NORM, *SNC* and *EN*. Experiment is described in Section 4.8

| Problem       | SBCGLP-EN |       | SBCGLP-SNC |       | SBCGLP-INF |       |
|---------------|-----------|-------|------------|-------|------------|-------|
|               | Node      | Time  | Node       | Time  | Node       | Time  |
| BatchS151208M | 4101      | 67.31 | 2843       | 43.77 | 3499       | 65.92 |
| BatchS201210M | 3113      | 70.78 | 3649       | 81.59 | 2555       | 80.17 |

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Table 13 – continued from previous page

| Problem               | SBCGLP-EN |          | SBCGLP-SNC |          | SBCGLP-INF |         |
|-----------------------|-----------|----------|------------|----------|------------|---------|
|                       | Node      | Time     | Node       | Time     | Node       | Time    |
| BatchS10BM101064      | 10601     | 99.1     | 8323       | 72.97    | 14965      | 118.21  |
| BatchSBM101064        | 8943      | 93.01    | 10615      | 93.32    | 16149      | 151.33  |
| CLay0305H             | 14595     | 56.67    | 17037      | 67.33    | 21429      | 64.85   |
| FLay05H               | 106611    | 1767.95  | 104777     | 1667.28  | 106173     | 1902.9  |
| FLay05M               | 83085     | 781.69   | 80525      | 614.33   | 84149      | 680.5   |
| fo7_2                 | 81119     | 139.27   | 65135      | 116.93   | 88959      | 148.01  |
| fo7                   | 240737    | 467.54   | 277537     | 515.96   | 231859     | 375.35  |
| m7                    | 2461      | 3.67     | 3817       | 4.48     | 11401      | 11.5    |
| nd-12                 | 10221     | 662.61   | 9937       | 658.01   | 6487       | 266.88  |
| nd-13                 | 8561      | 649.7    | 8079       | 566.05   | 8491       | 563.07  |
| nd-14                 | 97085     | 6645.54  | 109993     | 7035.41  | 107957     | 5285.1  |
| o7_2                  | 1412825   | 2643.59  | 2826683    | 4971.34  | 1428113    | 2709.14 |
| o7                    | 5710371   | 10136.23 | 6357337    | 10162.08 | m          | m       |
| RSyn0810M02M          | 7173      | 51.15    | 7051       | 51.1     | 83641      | 317.42  |
| RSyn0810M03M          | 12047     | 176.39   | 12007      | 177.23   | 56303      | 500.6   |
| RSyn0810M04M          | 2443      | 173.21   | 3827       | 180.11   | 67095      | 893.06  |
| RSyn0815M02M          | 15143     | 88.17    | 12565      | 76.86    | 274925     | 1011.29 |
| RSyn0820M02M          | 33773     | 185.04   | 37359      | 202.34   | 1285209    | 5446.63 |
| RSyn0830M02M          | 10773     | 126.15   | 8909       | 108.35   | 210943     | 1411.48 |
| Safety3               | m         | m        | m          | m        | 2025337    | 5561.19 |
| safety_no_rotation_CH | 2461      | 14.11    | 3375       | 19.04    | 8139       | 34.24   |
| SLay07H               | 495       | 8.98     | 711        | 11.9     | 1559       | 16.64   |
| SLay08H               | 1617      | 29.2     | 1853       | 34.15    | 1787       | 35.81   |
| SLay09H               | 9121      | 185.26   | 4717       | 108.22   | 8387       | 170.49  |
| SLay09M               | 3777      | 28.4     | 1851       | 16.83    | 5761       | 35.08   |
| sssd-16-8-3           | 1228207   | 686.96   | 465701     | 333.11   | 1131861    | 557.1   |
| sssd-17-7-3           | 370313    | 192.37   | 182611     | 107.42   | 699239     | 306.61  |
| sssd-18-7-3           | 546293    | 251.77   | 444957     | 214.8    | 594687     | 338.96  |
| sssd-20-8-3           | 268233    | 175.43   | 647019     | 423.73   | 563567     | 457.56  |
| Syn20M04M             | 1167      | 35.77    | 869        | 33.81    | 11903      | 110.92  |
| Syn30M03M             | 83        | 27.24    | 139        | 27.59    | 16197      | 111.86  |
| Syn30M04M             | 171       | 66.76    | 287        | 74.52    | 79373      | 717.48  |
| Syn40M02M             | 219       | 20.38    | 227        | 19.64    | 5523       | 53.42   |
| trimloss4             | 21925     | 41.8     | 27185      | 53.02    | 32695      | 58.56   |
| uflquad-15-60         | 2131      | 483.34   | 2147       | 470.82   | 2485       | 487.71  |
| uflquad-15-80         | 3375      | 1708.49  | 3421       | 1510.98  | 3575       | 1435.12 |
| uflquad-20-40         | 2957      | 471.5    | 2861       | 436.28   | 4187       | 611.01  |
| uflquad-25-40         | 5267      | 1242.44  | 5263       | 1054.66  | 6951       | 1463.38 |

Table 14: Number of Nodes for FILMINT, BESTLIN, SSBC, and SBCGLP-SNC. Experiment is described in Section 4.9

| <b>Problem</b>               | <b>FILMINT</b> | <b>BESTLIN</b> | <b>SSBC</b> | <b>SBCGLP-SNC</b> |
|------------------------------|----------------|----------------|-------------|-------------------|
| BatchS151208M                | 5391           | 2347           | 3071        | 2843              |
| BatchS201210M                | 3447           | 1791           | 3259        | 3649              |
| Batch_Storage10_BM_10_10_6_4 | 15917          | 7101           | 14651       | 8323              |
| Batch_Storage_BM_10_10_6_4   | 21849          | 8643           | 11879       | 10615             |
| CLay0305H                    | 26449          | 21131          | 11017       | 17037             |
| FLay05H                      | 102207         | 106617         | 99691       | 104777            |
| FLay05M                      | 91579          | 87447          | 83289       | 80525             |
| fo7_2                        | 109811         | 85429          | 105729      | 65135             |
| fo7                          | 276547         | 300365         | 331499      | 277537            |
| m7                           | 621027         | 130141         | 320671      | 3817              |
| nd-12                        | 20873          | 9673           | 6587        | 9937              |
| nd-13                        | 24391          | 8283           | 8471        | 8079              |
| nd-14                        | 101813         | 115987         | 110687      | 109993            |
| o7_2                         | 2028675        | 1564357        | 1567755     | 2826683           |
| o7                           | 2675019        | 3376079        | 2741915     | 6357337           |
| RSyn0810M02M                 | 117015         | 116957         | 115459      | 7051              |
| RSyn0810M03M                 | 88593          | 85217          | 94485       | 12007             |
| RSyn0810M04M                 | 156835         | 63177          | 77739       | 3827              |
| RSyn0815M02M                 | 493283         | 462711         | 463365      | 12565             |
| RSyn0820M02M                 | 2157455        | 1954165        | m           | 37359             |
| RSyn0830M02M                 | 825939         | 856223         | 1071975     | 8909              |
| Safety3                      | 3087205        | 2604659        | 1576035     | m                 |
| safety_no_rotation_CH        | 18327          | 4573           | 5525        | 3375              |
| SLay07H                      | 4505           | 813            | 635         | 711               |
| SLay08H                      | 15203          | 2349           | 1921        | 1853              |
| SLay09H                      | 82525          | 6345           | 12257       | 4717              |
| SLay09M                      | 16689          | 2295           | 6035        | 1851              |
| sssd-16-8-3                  | 2092313        | 1317759        | 1491441     | 465701            |
| sssd-17-7-3                  | 1287961        | 787457         | 1043811     | 182611            |
| sssd-18-7-3                  | 265695         | 201431         | 374615      | 444957            |
| sssd-20-8-3                  | 3573363        | 242025         | 354987      | 647019            |
| Syn20M04M                    | 90393          | 89153          | 91683       | 869               |
| Syn30M03M                    | 49731          | 45875          | 47779       | 139               |
| Syn30M04M                    | 237847         | 249021         | 248575      | 287               |
| Syn40M02M                    | 145331         | 81585          | 83171       | 227               |
| trimloss4                    | 25535          | 23583          | 23353       | 27185             |
| uflquad-15-60                | 2523           | 1851           | 2303        | 2147              |
| uflquad-15-80                | 3689           | 3019           | 3401        | 3421              |

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Table 14 – continued from previous page

| Problem       | FILMINT | BESTLIN | SSBC | SBCGLP-SNC |
|---------------|---------|---------|------|------------|
| uflquad-20-40 | 3591    | 2853    | 3435 | 2861       |
| uflquad-25-40 | 6973    | 4831    | 5233 | 5263       |

Table 15: Solution Time for FILMINT, BESTLIN, SSBC, and SBCGLP-SNC. Experiment is described in Section 4.9

| Problem                      | FILMINT | BESTLIN | SSBC   | SBCGLP-SNC |
|------------------------------|---------|---------|--------|------------|
| BatchS151208M                | 52.0    | 37.9    | 32.0   | 43.8       |
| BatchS201210M                | 43.3    | 42.0    | 51.3   | 81.6       |
| Batch_Storage10_BM_10_10_6_4 | 112.1   | 57.1    | 132.0  | 73.0       |
| Batch_Storage_BM_10_10_6_4   | 150.8   | 65.3    | 106.7  | 93.3       |
| CLay0305H                    | 108.7   | 83.3    | 28.6   | 67.3       |
| FLay05H                      | 1702.0  | 1600.9  | 1484.6 | 1667.3     |
| FLay05M                      | 696.6   | 647.0   | 696.5  | 614.3      |
| fo7_2                        | 177.0   | 151.6   | 163.3  | 116.9      |
| fo7                          | 355.6   | 370.7   | 442.3  | 516.0      |
| m7                           | 796.8   | 171.2   | 336.7  | 4.5        |
| nd-12                        | 1499.0  | 771.3   | 476.7  | 658.0      |
| nd-13                        | 497.8   | 691.2   | 490.8  | 566.1      |
| nd-14                        | 6222.7  | 6062.5  | 4997.8 | 7035.4     |
| o7_2                         | 3384.4  | 3438.7  | 2581.5 | 4971.3     |
| o7                           | 3797.3  | 4737.2  | 4327.5 | 10162.1    |
| RSyn0810M02M                 | 272.7   | 301.3   | 302.5  | 51.1       |
| RSyn0810M03M                 | 326.5   | 466.3   | 467.8  | 177.2      |
| RSyn0810M04M                 | 746.3   | 510.0   | 597.5  | 180.1      |
| RSyn0815M02M                 | 1126.5  | 1206.2  | 1086.8 | 76.9       |
| RSyn0820M02M                 | 5770.4  | 5218.9  | m      | 202.3      |
| RSyn0830M02M                 | 3017.0  | 3572.1  | 3619.4 | 108.4      |
| Safety3                      | 9063.5  | 7053.0  | 4240.0 | m          |
| safety_no_rotation_CH        | 63.4    | 18.0    | 20.5   | 19.0       |
| SLay07H                      | 38.4    | 11.1    | 8.9    | 11.9       |
| SLay08H                      | 197.0   | 36.7    | 29.8   | 34.2       |
| SLay09H                      | 1458.6  | 125.4   | 194.4  | 108.2      |
| SLay09M                      | 93.1    | 16.4    | 29.2   | 16.8       |
| sssd-16-8-3                  | 1106.5  | 789.2   | 834.4  | 333.1      |
| sssd-17-7-3                  | 614.7   | 372.9   | 805.6  | 107.4      |
| sssd-18-7-3                  | 110.5   | 105.2   | 191.5  | 214.8      |
| sssd-20-8-3                  | 3317.5  | 146.2   | 239.0  | 423.7      |

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Table 15 – continued from previous page

| <b>Problem</b> | <b>FILMINT</b> | <b>BESTLIN</b> | <b>SSBC</b> | <b>SBCGLP-SNC</b> |
|----------------|----------------|----------------|-------------|-------------------|
| Syn20M04M      | 153.1          | 264.4          | 197.3       | 33.8              |
| Syn30M03M      | 147.0          | 182.0          | 178.9       | 27.6              |
| Syn30M04M      | 1127.7         | 1261.8         | 1186.5      | 74.5              |
| Syn40M02M      | 317.8          | 210.8          | 210.2       | 19.6              |
| trimloss4      | 36.9           | 34.4           | 35.0        | 53.0              |
| uflquad-15-60  | 437.4          | 292.5          | 379.8       | 470.8             |
| uflquad-15-80  | 1220.9         | 914.7          | 1177.8      | 1511.0            |
| uflquad-20-40  | 463.4          | 324.0          | 417.0       | 436.3             |
| uflquad-25-40  | 1389.4         | 794.7          | 963.2       | 1054.7            |

Table 1: Test Set Statistics

| Problem                | NL Obj | Vars | Ints | Cons | NL Cons | GUBs |
|------------------------|--------|------|------|------|---------|------|
| BatchS151208M          | ✓      | 446  | 203  | 1780 | 1       | 24   |
| BatchS201210M          | ✓      | 559  | 251  | 2326 | 1       | 24   |
| BatchStorage10BM101064 | ✓      | 239  | 89   | 798  | 1       | 20   |
| BatchStorageBM101064   | ✓      | 239  | 89   | 798  | 1       | 20   |
| CLay0305H              |        | 276  | 55   | 336  | 60      | 15   |
| FLay05H                |        | 383  | 40   | 461  | 5       | 10   |
| FLay05M                |        | 63   | 40   | 61   | 5       | 10   |
| fo7_2                  |        | 115  | 42   | 198  | 14      | 0    |
| fo7                    |        | 115  | 42   | 198  | 14      | 0    |
| m7                     |        | 115  | 42   | 198  | 14      | 0    |
| nd-12                  |        | 601  | 40   | 290  | 40      | 4    |
| nd-13                  |        | 641  | 40   | 317  | 40      | 2    |
| nd-14                  |        | 817  | 48   | 370  | 48      | 2    |
| o7_2                   |        | 115  | 42   | 198  | 14      | 0    |
| o7                     |        | 115  | 42   | 198  | 14      | 0    |
| RSyn0810M02M           |        | 411  | 168  | 855  | 12      | 200  |
| RSyn0810M03M           |        | 616  | 252  | 1435 | 18      | 435  |
| RSyn0810M04M           |        | 821  | 336  | 2117 | 24      | 772  |
| RSyn0815M02M           |        | 471  | 188  | 960  | 22      | 236  |
| RSyn0820M02M           |        | 511  | 208  | 1047 | 28      | 266  |
| RSyn0830M02M           |        | 621  | 248  | 1233 | 40      | 322  |
| Safety3                | ✓      | 260  | 98   | 294  | 0       | 28   |
| safety_no_rotation_CH  | ✓      | 409  | 60   | 456  | 6       | 15   |
| SLay07H                | ✓      | 477  | 84   | 609  | 0       | 21   |
| SLay08H                | ✓      | 633  | 112  | 812  | 0       | 28   |
| SLay09H                | ✓      | 811  | 144  | 1044 | 0       | 36   |
| SLay09M                | ✓      | 235  | 144  | 324  | 0       | 36   |
| sssd-16-8-3            |        | 185  | 152  | 57   | 24      | 24   |
| sssd-17-7-3            |        | 169  | 140  | 53   | 21      | 24   |
| sssd-18-7-3            |        | 176  | 147  | 54   | 21      | 25   |
| sssd-20-8-3            |        | 217  | 184  | 61   | 24      | 28   |
| Syn20M04M              |        | 421  | 160  | 997  | 56      | 462  |
| Syn30M03M              |        | 481  | 180  | 982  | 60      | 386  |
| Syn30M04M              |        | 641  | 240  | 1489 | 80      | 684  |
| Syn40M02M              |        | 421  | 160  | 757  | 56      | 244  |
| trimloss4              |        | 106  | 85   | 61   | 4       | 20   |
| uflquad-15-60          | ✓      | 916  | 15   | 960  | 0       | 0    |
| uflquad-15-80          | ✓      | 1216 | 15   | 1280 | 0       | 0    |
| uflquad-20-40          | ✓      | 821  | 20   | 840  | 0       | 0    |
| uflquad-25-40          | ✓      | 1026 | 25   | 1040 | 0       | 0    |

Table 2: Solution Statistics Comparing SBC versus GUBSBC inequalities

|        | Extra gap closed (%) | Average # of nodes |           | Average time |           |
|--------|----------------------|--------------------|-----------|--------------|-----------|
|        |                      | Arithmetic         | Geometric | Arithmetic   | Geometric |
| SBC    | 9.66                 | 289476.6           | 48531.6   | 840.3        | 244.3     |
| GUBSBC | 19.20                | 261314.3           | 51506.5   | 807.1        | 274.7     |

Table 3: Solution Statistics Comparing Mixing SBC versus SBC

|        | Extra gap closed (%) | Average # of nodes |           | Average time |           |
|--------|----------------------|--------------------|-----------|--------------|-----------|
|        |                      | Arithmetic         | Geometric | Arithmetic   | Geometric |
| SBC    | 7.77                 | 352,975.5          | 50,396.9  | 922.4        | 313.3     |
| MIXSBC | 19.06                | 450,208.7          | 56,829.5  | 1107.7       | 379.4     |

Table 4: Solution Statistics Comparing LIN versus BESTLIN

|         | Average # of nodes |           | Average time |           |
|---------|--------------------|-----------|--------------|-----------|
|         | Arithmetic         | Geometric | Arithmetic   | Geometric |
| LIN     | 338,089.2          | 39,330.3  | 1,568.3      | 407.3     |
| BESTLIN | 375,883.0          | 47,346.7  | 1,078.9      | 312.3     |

Table 5: Solution Statistics Comparing Strengthened SBC versus SBC

|      | Extra gap closed (%) | Average # of nodes |           | Average time |           |
|------|----------------------|--------------------|-----------|--------------|-----------|
|      |                      | Arithmetic         | Geometric | Arithmetic   | Geometric |
| SBC  | 7.77                 | 352,975.5          | 50,396.9  | 922.4        | 313.3     |
| SSBC | 7.86                 | 323,574.1          | 49,200.5  | 865.7        | 310.7     |

Table 6: Solution Statistics Comparing Best Methods and FILMINT

|            | Average # of nodes |           | Average time |           |
|------------|--------------------|-----------|--------------|-----------|
|            | Arithmetic         | Geometric | Arithmetic   | Geometric |
| FILMINT    | 413,917.2          | 71,059.5  | 983.6        | 485.8     |
| BESTLIN    | 275,697.2          | 38,634.1  | 812.7        | 267.1     |
| SSBC       | 290,614.6          | 44,910.6  | 776.9        | 290.1     |
| SBCGLP-SNC | 308,574.7          | 12,797.4  | 847.8        | 161.9     |