Preconditioning for Regular Elliptic Systems

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PRECONDITIONING FOR REGULAR ELLIPTIC SYSTEMS

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Abstract. In this paper we examine preconditioning operators for regular elliptic systems of partial differential operators. We obtain general conditions under which the preconditioned systems are bounded. We also provide some useful guidelines for choosing left and right preconditioning operators for regular elliptic systems. The condition numbers of the discrete operators arising from these preconditioned operators are shown to be bounded independent of grid spacing. Several examples of the two-dimensional regular elliptic systems are discussed, including scalar elliptic operators and the Stokes operator with several different boundary conditions. Several preconditioners for these regular elliptic systems are presented and used in numerical experiments illustrating the theoretical results.

Key word. preconditioning, elliptic systems, numerical methods

AMS subject classifications. 65N06, 65N22

1. Introduction.

Elliptic systems of partial differential equations are important in the study of many physical processes, including those involving steady incompressible viscous flow and elasticity. The solution of elliptic systems by finite difference, finite element, or other numerical method requires the solution of large systems of equations. For the linear systems that arise in this way, iterative solution methods are used and a wide variety of these are available. These iterative procedures can be accelerated by choosing good preconditioners for the linear systems.

In this paper we extend the idea of Manteullel and Parter [9] that examining preconditioning operators for the differential system provides insight into choosing good preconditioners. Whereas Manteullel and Parter [9] consider only a single elliptic equation, we consider elliptic systems. We also extend their results to consider more general boundary conditions, especially allowing for non-homogeneous boundary conditions.

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Regular elliptic systems, defined by Douglis and Nirenberg [3], and Agmon, Douglis and Nirenberg [1], have an order inside the domain and on the boundary given by a sequence of integers. The general idea for choosing a preconditioner for a regular elliptic system is to choose some regular elliptic system with the same order which is "easily invertible". By using the theory of Fredholm operators, we provide some useful guidelines for choosing good left and right preconditioners for regular elliptic systems. We show that the condition numbers of such discrete preconditioned operators arising from above preconditioned operators are bounded independent of the grid size.

Manteullel and Parter [9] discussed the preconditioning and boundary conditions for scalar elliptic operators in the plane. They obtained necessary and sufficient conditions for the condition numbers of preconditioned operators to be bounded in the L_2 and H_1 norms. With the ideas of Bramble and Pasciak [2], they extended the results to discrete preconditioned operators arising from the preconditioned differential operators. They showed that the condition numbers of these discrete preconditioned operators are bounded independent of grid spacing. We extend their ideas to regular elliptic systems. The difficulties of this extension have to do with the non-uniqueness and non-existence for solutions of elliptic systems. Fortunately, regular elliptic operators are Fredholm operators. Using the theory of Fredholm operators, we can modify these operators so that the solutions of the modified operators exist and are unique. By using the regularity estimates of Agmon, Douglis and Nirenberg [1]we can extend the ideas of Manteullel and Parter to regular elliptic systems in any dimension with rather general boundary conditions. We show the condition numbers of left and right preconditioned operators are bounded with respect to more general norms. The main idea is to use norms that are natural for the boundary operators. Moreover, we can use the results of Martin [10] to extend the above results to finite difference schemes such that if the condition numbers of the preconditioned operators are bounded, then the condition numbers of the discrete preconditioned operators arising from the finite difference schemes are also bounded independent of grid spacing. Therefore, we only need to consider the preconditioning problems on partial differential equations, not on finite difference schemes.

Several examples are discussed in some detail, these include two-dimensional regular scalar elliptic operators and the Stokes operator with different boundary operators. Also, good preconditioners for these examples are provided and the numerical experiments using these examples are presented to illustrate the theoretical results.

The structure of this papers is as follows. We first state some definitions and assumptions in Section 2. In Section 3, we discuss the preconditioning method for regular elliptic systems. In Section 4, we will show that the results of Section 3 can be extended to finite difference schemes. We discuss several regular elliptic systems of the Stokes operator with several different kinds of boundary operators in Section 5 and present the preconditioners for these elliptic systems. Numerical experiments are presented in Section 6 and the conclusions of this paper are presented in Section 7.

2. Preliminaries.

Let $\Omega \subset R^n$ be an open bounded domain with smooth boundary $\partial\Omega$. We begin by introducing regular elliptic systems as defined by Douglis, and Nirenberg [3].

Definition 2.1. *A system of partial differential equations L given by*

$$\sum_{j=1}^{k} l_{ij}(x,D)\,u_j(x) = f_i(x), \quad i=1,\ldots,k \quad \text{and} \quad x\in\Omega, \tag{2.1}$$

where $D=(-i\partial_{x_1},\ldots,-i\partial_{x_n})$, *is an elliptic system if there are integers* $(\sigma_i)_{i=1}^{k}$ *and* $(\tau_j)_{j=1}^{k}$ *such that*

(1) $\deg l_{ij}(x,D)\le\sigma_i+\tau_j$;

(2) denoting by $l_{ij}^0(x,D)$ *the sum of terms in* $l_{ij}(x,D)$ *which are exactly of the order* $\sigma_i+\tau_j$, *we have*

$$\chi(x,\xi)=\det\|\,[l_{ij}^0(x,\xi)]\,\|\neq 0 \tag{2.2}$$

for all $x\in\overline{\Omega}$ *and all* $\xi\in R^n$ *and* $\xi\neq 0$.

The determinant $\chi(x,\xi)$ defined in (2.2) is a homogeneous polynomial in ξ, and since it does not vanish for non-zero ξ, its degree must be an even integer, say $2p$. It is easy to see from the determinant that

$$\sum_{i=1}^{m}(\sigma_i+\tau_i)=2p.$$

Definition 2.2. *A system of partial differential equations L defined by Definition 2.1 is said to be a properly elliptic system if for all* $x\in\overline{\Omega}$ *and every pair of linearly independent vectors* $\xi,\xi'\in R^n$, *the equation* $\chi(x,\xi+\eta\xi')=0$ *in* η *has* p *roots with positive imaginary parts.*

A regular elliptic system requires p boundary conditions, specified by an operator B:

$$\sum_{j=1}^{k} b_{ij}(x,D)u_j(x)=\phi_i(x), \quad i=1,\ldots,p, \quad x\in\partial\Omega$$

with integers ρ_i, $i=1,\ldots,p$, such that

$$\deg b_{ij}(x,D)\le\rho_i+\tau_j.$$

Since $\partial\Omega$ may not be simply connected and the order of boundary conditions may be different on different components of $\partial\Omega$, the value of the ρ_i may be different on different components of $\partial\Omega$. To keep the notation simpler, we will not include the possible differences in the ρ_i on the boundary components in our notation.

Denote by $b_{ij}^0(x,D)$ the sum of terms in $b_{ij}(x,D)$ which are exactly of order $\rho_i+\tau_j$, and let $L^0(x,D)=[l_{ij}^0(x,D)]$ and $B^0(x,D)=[b_{ij}^0(x,D)]$ then we define the complementing condition as follows.

Definition 2.3. *The system*

$$L(x,D), B(x,D), \quad x\in\partial\Omega$$

is said to fulfill the complementing condition if for all $x \in \partial\Omega$, $\xi' \in T_x^*(\partial\Omega)$ (dual of the tangent space at x), $\xi' \neq 0$, and if t is the coordinate in the normal direction at x, the initial value problem

$$L^0\left(x, \left(\xi', \frac{1}{i}\frac{d}{dt}\right)\right) w(t) = 0, \quad t > 0,$$

$$B^0\left(x, \left(\xi', \frac{1}{i}\frac{d}{dt}\right)\right) w(t) = 0, \quad t = 0,$$

with the restriction

$$\lim_{t\to\infty} w(t) = 0$$

has the unique solution $w(t) = 0$.

Definition 2.4. *The system*

$$\begin{aligned} LU &= F \quad \text{in } \Omega \\ BU &= \Phi \quad \text{on } \partial\Omega, \end{aligned} \tag{2.3}$$

is a regular elliptic system if
(1) L is properly elliptic in $\overline{\Omega}$ and has infinitely differentiable coefficients in $\overline{\Omega}$;
(2) the coefficients of B are infinitely differentiable on $\partial\Omega$;
(3) the system satisfies the complementing condition.

In practice, we don't need the requirement that the coefficients of L and B are infinitely differentiable. For convenience in the discussion, we assume that the coefficients of L and B are smooth.

We define the Sobolev norms for $\tau \in Z_+^k$ as follows:

$$H^\tau(\Omega) = \prod_{j=1}^{k} H^{\tau_j}(\Omega),$$

$$\|U\|_\tau^2 = \sum_{j=1}^{k} \sum_{|\alpha| \le \tau_j} \|D^\alpha u_j\|_0^2,$$

where $H^{\tau_j}(\Omega)$ is the usual Sobolev space, $U = (u_1, \dots, u_k) \in H^\tau$ and $\|\cdot\|_0$ is the usual L^2 norm in Ω. On the other hand, we define the Sobolev norm on $\partial\Omega$ specified by

$$|BU|_\rho^2 = \sum_{i=1}^{p} |B_i U|_{\rho_i}^2$$

where $|B_i U|_{\rho_i}$ is the trace function space Sobolev norm on $\partial\Omega$ as defined in Lions and Magenes [8, Chapter 1]. Note that if $\partial\Omega$ is not connected then the value of ρ_i maybe different on different boundary components as discussed after Definition 2.2.

A regular elliptic system (2.3) satisfies the well-known Agmon, Douglis, and Nirenberg regularity estimates [1]

$$\|U\|_{H^{\tau+l}(\Omega)} \le C\left(\|LU\|_{H^{l-\sigma}(\Omega)} + |BU|_{H^{l-\rho-1/2}(\partial\Omega)} + \|U\|_0\right), \tag{2.4}$$

where l is an integer and $l \geq l_1 = \max(0, \rho_i + 1)$.

In this paper, we call the regularity estimates (2.4) the ADN estimates.

We now introduce the important definitions of Fredholm operators and index which will play important role throughout this paper.

Definition 2.5. *Let X and Y be Banach spaces. A bounded linear operator $P{:}X \to Y$ is a Fredholm operator if*
(1) kernel(P) *is finite dimensional;*
(2) range(P) *is closed;*
(3) the co-dimension of range(P) *is finite.*

The index of the Fredholm operator P is

$$\text{index}(P) = \dim(\text{kernel}(P)) - \text{codim}(\text{range}(P)). \tag{2.5}$$

We now give these important results regarding regular elliptic systems and Fredholm operators as following:

(1) A regular elliptic operator of system (2.3) is a Fredholm operator (see Lions and Magenes [8], Peetre [11], and Wloka, Rowley, and Lawruk [18]).
(2) Regular scalar elliptic operators with Dirichlet boundary conditions have index zero (see Lions and Magenes [8], pp. 198-199).
(3) For regular scalar elliptic operators, the value l in the ADN estimates in (2.4) can be a real number (see Lions and Magenes [8]).

3. Preconditioning for Regular Elliptic Systems.

In this section we present our main result about preconditioners for regular elliptic systems of partial differential equations. Let Ω be an open bounded domain with smooth boundary $\partial\Omega$. Let $E = \{L, B\}$ be the regular elliptic operator of system (2.3) with the order (τ, σ, ρ) such that

$$E : H^{\tau+l}(\Omega) \to H^{l-\sigma}(\Omega) \times H^{l-\rho-1/2}(\partial\Omega),$$

where l is an integer and $l \geq l_1 = \max(0, \rho_i + 1)$. We assume, at first, that E is one-to-one and onto. The general properties for choosing a good preconditioner P are the following:

(1) P is a regular elliptic operator with the same order (τ, σ, ρ) as E such that

$$P : H^{\tau+l}(\Omega) \to H^{l-\sigma}(\Omega) \times H^{l-\rho-1/2}(\partial\Omega).$$

(2) P is one-to-one and onto.
(3) P is easily invertible.

Condition (3) is not precise for the differential system, but for the numerical approximation, it means that the numerical approximation can be easily inverted.

By the above properties, the preconditioned operators $P^{-1}E$ and EP^{-1} are one-to-one and onto. From the ADN estimates, $P^{-1}E$ and EP^{-1} are bounded operators (we show this later). Therefore, iteration methods such as GMRES(m) [12] will be efficient for the preconditioned operators $P^{-1}E$ and EP^{-1}.

Notice, if P has different order τ', σ', or ρ', then the preconditioned operators $P^{-1}E$ or EP^{-1} are not bounded.

In general, regular elliptic operators are not one-to-one and onto. Fortunately, regular elliptic operators are Fredholm operators. Because the kernel and co-range of Fredholm operators are finite dimensional, we can easily modify regular elliptic operators so that the modified operators are one-to-one and onto.

We begin by discussing the modification of regular elliptic operators and then present the main theorem of this paper. First, we introduce a useful lemma that will be used to proved the main theorem.

Lemma 3.1. *Let X, Y, and X_0 be Banach spaces and we denote their norms by $\|\cdot\|_X$, $\|\cdot\|_Y$, and $\|\cdot\|_{X_0}$. Suppose that $X \subset X_0$ and the natural injection $u \to u$ of X into X_0 is compact. Let Ψ be a continuous linear mapping from X into Y that satisfies the inequality*

$$\|u\|_X \leq c_1 \left(\|\Psi u\|_Y + \|u\|_{X_0}\right), \tag{3.1}$$

where c is a constant. If Ψ is one-to-one on X, then we can improve the inequality (3.1) to

$$\|u\|_X \leq c_2 \, \|\Psi u\|_Y. \tag{3.2}$$

Proof. If Ψ satisfies (3.1) and the natural injection of X into X_0 is compact, then Ψ has closed range (Peetre [11]). Thus by the Open Mapping Theorem, Ψ^{-1} is bounded on the range of Ψ. This proves (3.2).

The ideas for the modification of E are the following: Considering

$$E : H^{\tau+l}(\Omega)/K(E) \ \oplus \ K(E) \to R(E) \ \oplus \ R^{\perp}(E),$$

where $K(E)$ and $R(E)$ are the kernel and range of E, we have that the elliptic operator $E : H^{\tau+l}(\Omega)/K(E) \to R(E)$ is one-to-one and onto, and the dimensions of $K(E)$ and $R^{\perp}(E)$ are finite.

Define the norm $\|\cdot\|_{H^{\tau+l}/K(E)}$ on $H^{\tau+l}/K(E)$ as follows:

$$\|U\|_{H^{\tau+l}/K(E)} = \ \inf \left\{\|U'\|_{H^{\tau+l}}, \quad U' \sim U, \quad \text{for } U, U' \in H^{\tau+l}\right\}$$

where $U' \sim U$ if $U' - U \in K(E)$. By the definition of $\|\cdot\|_{H^{\tau+l}/K(E)}$, we obtain

$$\|U\|_{H^{\tau+l}/K(E)} \leq \|U\|_{H^{\tau+l}}. \tag{3.3}$$

We modify E as follows, such that the modified operator $\hat{E}$ is one-to-one and onto.

Case 1: index$(E) = d \geq 0$.

We modify E to $\hat{E}$ such that

$$\hat{E} : H^{\tau+l}(\Omega)/K(E) \ \oplus \ K(E) \to R(E) \ \oplus \ R^{\perp}(E) \ \oplus \ R^{d}$$

where $\hat{E} = \left(E|_{H^{\tau+l}/K(E)}, T\right)$ and T is a finite dimensional bijection from $K(E)$ to $R^{\perp}(E)\oplus R^d$. Then $\hat{E}$ is one-to-one and onto.

Notice that T is not unique. If we take a different bijection T', we will get a different modified operator $\hat{E}'$. But if the data of $\hat{E}$ are only contained in $R(E)$, T does not affect the solution of the original equation.

Since E is a regular elliptic operator, the ADN estimates hold

$$\|U\|_{H^{\tau+l}} \leq c\left(\|LU\|_{H^{l-\sigma}} + |BU|_{H^{l-\rho-1/2}} + \|U\|_0\right).$$

By (3.3), we have

$$\|U\|_{H^{\tau+l}/K(E)} \leq c\left(\|LU\|_{H^{l-\sigma}} + |BU|_{H^{l-\rho-1/2}} + \|U\|_0\right). \tag{3.4}$$

Since $E|_{H^{\tau+l}/K(E)}$ is one-to-one and (3.4) holds, by Lemma 3.1, we have

$$\|U\|_{H^{\tau+l}/K(E)} \leq c\left(\|LU\|_{H^{l-\sigma}} + |BU|_{H^{l-\rho-1/2}}\right),$$

for $U \in H^{\tau+l}/K(E)$. This implies

$$\|u\|_{K(E)} + \|U\|_{H^{\tau+l}/K(E)} \leq c\left(\|LU\|_{H^{l-\sigma}} + |BU|_{H^{l-\rho-1/2}} + \|Tu\|_{R^{d'}}\right) \tag{3.5}$$

where $d' = \dim(K(E))$ and $u \in K(E)$.

Define the norms $\|\cdot\|_D$ and $\|\cdot\|_R$ as follows:

$$\|(U,u)\|_D = \|U\|_{H^{\tau+l}/K(E)} + \|u\|_{K(E)}$$

and

$$\|\hat{E}(U,u)\|_R = \|LU\|_{H^{l-\sigma}} + |BU|_{H^{l-\rho-1/2}} + \|Tu\|_{R^{d'}},$$

we have by (3.5)

$$\|(U,u)\|_D \leq \ c\ \|\hat{E}(U,u)\|_R.$$

Hence $\hat{E}^{-1}$ is bounded. On the other hand, $\hat{E}$ is naturally bounded under the norms $\|\cdot\|_D$ and $\|\cdot\|_R$, i.e.,

$$\|\hat{E}(U,u)\|_R \leq \ c\ \|(U,u)\|_D.$$

completing this case for the modification of E.

Examples of how to modify the Stokes operator with several boundary conditions are given in section 5.

Case 2: $\text{index}(E) = -d < 0$.

We modify E to $\hat{E}$ such that

$$\hat{E} : H^{\tau+l}(\Omega)/K(E) \ \oplus\ K(E) \ \oplus\ R^d \rightarrow R(E) \ \oplus\ R^{\perp}(E)$$

where $\hat{E} = \left(E|_{H^{\tau+l}/K(E)}, T\right)$ and T is a finite dimensional bijection from $K(E)\oplus R^d$ to $R^{\perp}(E)$. Then $\hat{E}$ is one-to-one and onto. With the same idea as Case 1, $E|_{H^{\tau+l}/K(E)}$ satisfies (3.4) and $E|_{H^{\tau+l}/K(E)}$ is one-to-one, by Lemma 3.1,

$$\|U\|_{H^{\tau+l}/K(E)} \leq c\left(\|LU\|_{H^{l-\sigma}} + |BU|_{H^{l-\rho-1/2}}\right),$$

for $U \in H^{\tau+l}/K(E)$. This implies

$$\|U\|_{H^{\tau+l}/K(E)} + \|u\|_{K(E)\oplus R^d} \leq c\,(\|LU\|_{H^{l-\sigma}} + |BU|_{H^{l-\rho-1/2}} + \|Tu\|_{R^{d'}})\,.$$

where $d' = \dim(K(E) \oplus R^d)$ and $u \in K(E) \oplus R^d$.

Define the norms $\|\cdot\|_D$ and $\|\cdot\|_R$ as follows:

$$\|(U,u)\|_D = \|U\|_{H^{\tau+l}/K(E)} + \|u\|_{K(E)\oplus R^d}$$

and

$$\|\hat{E}(U,u)\|_R = \|LU\|_{H^{l-\sigma}} + |BU|_{H^{l-\rho-1/2}} + \|Tu\|_{R^{d'}},$$

we have

$$\|(U,u)\|_D \leq \ c\ \|\hat{E}(U,u)\|_R.$$

Hence $\hat{E}^{-1}$ is bounded. On the other hand, $\hat{E}$ is naturally bounded under the norms $\|\cdot\|_D$ and $\|\cdot\|_R$, i.e.,

$$\|\hat{E}(U,u)\|_R \leq \ c\ \|(U,u)\|_D.$$

completing the modification of E.

By the above discussion, $\hat{E}$ is one-to-one and onto, and so $\hat{E}$ and $\hat{E}^{-1}$ are bounded. If we have a regular elliptic system P with the same (σ, τ) in Ω and (τ, ρ) on $\partial\Omega$ as E, then we can modify P to $\hat{P}$ such that $\hat{P}$ is one-to-one and onto, and $\hat{P}$, $\hat{P}^{-1}$ are bounded. Also, if $\text{index}(P) = \text{index}(E)$, then $\hat{E}$ and $\hat{P}$ have the same domain and same range. We also can easily show that the condition numbers of preconditioned operators $\hat{P}^{-1}\hat{E}$ and $\hat{E}\hat{P}^{-1}$ are bounded.

Now we come to the main theorem of this paper.

Theorem 3.2. *Let $E = \{L_1, B_1\}$ and $P = \{L_2, B_2\}$ be two regular elliptic operators defined on the same space. Suppose E and P have the same order boundary conditions on the same boundary components and* $\text{index}(E) = \text{index}(P)$. *Then, for some constant c*

$$\|\hat{P}^{-1}\hat{E}\|_D < c \quad \textit{and} \quad \|\hat{E}^{-1}\hat{P}\|_D < c$$

$$\|\hat{E}\hat{P}^{-1}\|_R < c \quad \textit{and} \quad \|\hat{P}\hat{E}^{-1}\|_R < c.$$

Hence, the condition numbers of the left and right preconditioned operators are bounded,

$$C_D(\hat{P}^{-1}\hat{E}) = \|\hat{P}^{-1}\hat{E}\|_D\ \|\hat{E}^{-1}\hat{P}\|_D < c^2$$

$$C_R(\hat{E}\hat{P}^{-1}) = \|\hat{E}\hat{P}^{-1}\|_R\ \|\hat{P}\hat{E}^{-1}\|_R < c^2.$$

Proof. From the above modification, we have that $\hat{E}$ and $\hat{P}$ are one-to-one and onto from the same domain to the same range. Also, $\hat{E}$, $\hat{E}^{-1}$, $\hat{P}$, and $\hat{P}^{-1}$ are bounded operators. Hence, $\hat{E}\hat{P}^{-1}$, $\hat{P}\hat{E}^{-1}$, $\hat{P}^{-1}\hat{E}$, and $\hat{E}^{-1}\hat{P}$ are well-defined, one-to-one, onto, and bounded. Therefore, the results follow immediately.

From Theorem 3.2, we only need the same order boundary conditions on the same boundary components for E and P. This result lessens the restrictions on boundary conditions discussed in Manteuffel and Parter [9]. On the other hand, it is hard to imagine that we can find weaker conditions such that $\hat{P}^{-1}\hat{E}$, $\hat{E}^{-1}\hat{P}$, $\hat{E}\hat{P}^{-1}$, and $\hat{P}\hat{E}^{-1}$ are well-defined, one-to-one, onto, and bounded.

Also, Theorem 3.2 shows that the modification of regular elliptic operators not only ensures the existence and uniqueness of the solution but also provides some useful guidelines for choosing good preconditioners. The relations between the modification operator and original regular operator are described in the following important theorem.

Theorem 3.3. *If the data of $\hat{E}$ is contained in $R(E)$, then the solution of the system of equations with respect to $\hat{E}$ is the same as the solution of the system of equations with respect to E within the additive finite dimensional kernel. The choice of the finite dimensional operator T does not affect the solution of the system.*

Proof. The results follow immediately from the above two cases and the theory of Fredholm operators.

4. Preconditioning for Regular Difference Schemes.

In this section we show how to extend the results of the previous section to finite difference operators. Let $\Omega \in R^n$ be an open bounded domain with smooth boundary. We consider only boundary-fitted grids, i.e., those in which the boundary is a coordinate surface. This excludes the grid systems which the boundary curve is not parallel to a coordinate line. Boundary-fitted grids are in common use in computational fluid dynamics. We define Ω_h as the set of grid points inside Ω and $\partial\Omega_h$ as grid points on $\partial\Omega$ (for the details of definitions of Ω_h, $\partial\Omega_h$, and boundary-fitted grids see Strikwerda, Wade and Bube [14]). Consider a finite difference system (L_h, B_h) approximating the regular elliptic system (2.3) defined by:

$$\sum_{j=1}^{k} L_{h,i,j} v_{h,j}(x) = f_{h,i}(x), \quad i = 1, \ldots, n \quad x \in \Omega_h \tag{4.1}$$

$$\sum_{j=1}^{k} B_{1,h,i,j} v_{h,j}(x) = \phi_{1,h,i}(x), \quad i = 1, \ldots, p \quad x \in \partial\Omega_h \tag{4.2}$$

$$\sum_{j=1}^{k} B_{2,h,i,j} v_{h,j}(x) = \phi_{2,h,i}(x), \quad i = p+1, \ldots, q \quad x \in \partial\Omega_h \tag{4.3}$$

Here the boundary operators $B_{1,h,i,j}$ approximate the differential operators b_{ij}, for $i = 1, \ldots, p$, while $B_{2,h,i,j}$ are numerical boundary operators which must arise if the stencil of L_h goes "outside" of $\partial\Omega_h$.

Under regularity constraints on the finite difference operator, Strikwerda, Wade, and Bube [14] obtained the discrete Sobolev space regularity estimates

$$\|v_h\|_{H_h^{r+l}} + |v_h|_{H_h^{r+l-1/2}}$$

$$\leq c\left(|B_{1,h}v_h|_{H_h^{l-\rho-1/2}} + |h^{\rho-t+1/2}B_{2,h}v_h|_{H_h^{l-t}} + \|L_hv_h\|_{H_h^{l-\sigma}} + \|v_h\|_0\right), \qquad (4.4)$$

with $\overline{\rho} \leq l < \rho^*$ and $t = \overline{\rho} + \frac{1}{2}[2(l-\overline{\rho})]$ where

$$\overline{\rho} = \max_{1\leq i\leq p} (\rho_i + 1, 0),$$

and

$$\rho^* = \begin{cases} \min_{i\geq q}(\rho_i) + 1, & \text{if } p < q; \\ \infty, & \text{if } p = q. \end{cases}$$

The discrete Sobolev norms above are defined by:

$$H_h^\tau(\Omega_h) = \prod_{i=1}^{k} H_h^{\tau_i}(\Omega_h),$$

$$\|v_h\|_{H_h^\tau}^2 = \sum_{i=1}^{k} \sum_{|\alpha|\leq\tau_i} \sum_{x\in\Omega_h} |\delta_+^\alpha v_h(x)|^2 h^n,$$

$$|v_h|_{H_h^\rho}^2 = \sum_{i=1}^{p} \sum_{|\alpha|\leq\rho_i} \sum_{x\in\partial\Omega_h} |\delta_+^\alpha v_h(x)|^2 h^{n-1}.$$

Let $E = \{L, B\}$ be a regular elliptic operator defined in (2.3) and let

$$Q : H^{\tau+l} \to K(E)$$

is a projection on the kernel of E in $H^{\tau+l}$. We first state the theorem which was proved by Martin [10].

Theorem 4.1. *If $E_h = \{L_h, B_{1,h}, B_{2,h}\}$ is a regular approximation to a regular elliptic operator $E = \{L, B\}$ and the system with the operator E satisfies*

$$\|U\|_{H^{\tau+l}} \leq c_1 \left(\|LU\|_{H^{l-\sigma}} + |BU|_{H^{l-\rho-1/2}} + \|QU\|_0\right), \qquad (4.5)$$

then the finite difference solutions v_h satisfy

$$\|v_h\|_{H_h^{\tau+l}} + |v_h|_{H_h^{\tau+l-1/2}}$$

$$\leq c_2\left(|B_{1,h}v_h|_{H_h^{l-\rho-1/2}} + |h^{\rho-t+1/2}B_{2,h}v_h|_{H_h^{l-t}} + \|L_hv_h\|_{H_h^{l-\sigma}} + \|Q_hv_h\|_0\right) \qquad (4.6)$$

for h sufficient small.

In order to use Theorem 4.1 to extend the results of Section 3 to finite difference schemes, we need to modify Theorem 4.1. Using ideas similar to that of Lemma 3.1, we can easily modify Theorem 4.1 as follows.

Theorem 4.2. *If $E_h = \{L_h, B_{1,h}, B_{2,h}\}$ is a regular approximation to a regular elliptic operator $E = \{L, B\}$ and the system with the operator E satisfies*

$$\|U\|_{H^{\tau+l}/K(E)} \leq c_1 \left(\|LU\|_{H^{l-\sigma}} + |BU|_{H^{l-\rho-1/2}}\right),$$

then the finite difference solutions v_h satisfy

$$\|v_h\|_{H_h^{\tau+l}/K(E_h)} + |v_h|_{H_h^{\tau+l-1/2}/K(E_h)}$$

$$\leq c_2 \left(|B_{1,h}v_h|_{H_h^{l-\rho-1/2}} + |h^{\rho-t+1/2}B_{2,h}v_h|_{H_h^{l-t}} + \|L_h v_h\|_{H_h^{l-\sigma}}\right)$$

for h sufficient small.

Using Theorem 4.2 and ideas analogous to those of Section 3 we obtain

Theorem 4.3. *Let $E = \{L_1, B_1\}$ and $P = \{L_2, B_2\}$ be two regular elliptic operators defined on the same space. Suppose E and P have the same order boundary conditions on the same boundary components and* index(E)=index(P). *Let E_h=$\{L_{1,h}, B_{1,1,h}, B_{1,2,h}\}$ and P_h=$\{L_{2,h}, B_{2,1,h}, B_{2,2,h}\}$ be regular approximations to E and P, repectively. Then there exists a constant c, independent of h such that*

$$\|\hat{P}_h^{-1}\hat{E}_h\|_D < c \quad \textit{and} \quad \|\hat{E}_h^{-1}\hat{P}_h\|_D < c$$

$$\|\hat{E}_h\hat{P}_h^{-1}\|_R < c \quad \textit{and} \quad \|\hat{P}_h\hat{E}_h^{-1}\|_R < c.$$

Hence, the condition numbers of left and right discrete preconditioned operators are bounded

$$C_D(\hat{P}_h^{-1}\hat{E}_h) = \|\hat{P}_h^{-1}\hat{E}_h\|_D \;\|\hat{E}_h^{-1}\hat{P}_h\|_D < c^2$$

$$C_R(\hat{E}_h\hat{P}_h^{-1}) = \|\hat{E}_h\hat{P}_h^{-1}\|_R \;\|\hat{P}_h\hat{E}_h^{-1}\|_R < c^2.$$

5. Examples.

In this section, we discuss three regular elliptic systems. We also provide good preconditioners for these elliptic systems. Numerical experiments for some of these examples are presented in Section 6.

Example 5.1 (The Stokes Operator with the Dirichlet Boundary Condition.)

Let Ω be an open bounded domain with smooth boundary $\partial\Omega$ in R^2. The Stokes operator $S : H^\tau(\Omega) \to H^{-\sigma}(\Omega)$ is defined by

$$SU = \begin{pmatrix} \nabla^2 & 0 & -\partial_x \\ 0 & \nabla^2 & -\partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ g \end{pmatrix} \tag{5.1}$$

with the values $\tau = (\tau_1, \tau_2, \tau_3) = (2, 2, 1)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3) = (0, 0, -1)$. We consider the Dirichlet boundary operator $B : H^{\tau-1/2}(\partial\Omega) \rightarrow H^{-\rho-1/2}(\partial\Omega)$ with $\rho = (\rho_1, \rho_2) = (-2, -2)$ such that

$$BU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{5.2}$$

Let $E = \{S, B\}$ be the regular elliptic operator. It is well known that the solution of this system is unique to within an additive constant for pressure (see Temam [16]). Hence, $K(E) = < (0, 0, 1)^T >$. Also, the solution exists under the constraint

$$\int_\Omega g = \int_\Omega (u_x + v_y) = \int_\Omega \text{div}(u, v) = \int_{\partial\Omega} (u, v) \cdot \vec{n} = \int_{\partial\Omega} (b_1, b_2) \cdot \vec{n}$$

on the range. Hence the co-dimension of the range is 1. Therefore, index$(E) = 0$. It is easy to check that a basis of the complementary space of $R(E)$ is $(0, 0, 1)^T \times (0, 0)^T$, where $(0, 0, 1)^T$ is in Ω and $(0, 0)^T$ on $\partial\Omega$.

The modification $\hat{E}$ of E that we use is the following:

$$\hat{S}U = \begin{pmatrix} \nabla^2 & 0 & -\partial_x \\ 0 & \nabla^2 & -\partial_y \\ \partial_x & \partial_y & \Lambda \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} \quad \text{in} \quad \Omega,$$

and

$$\hat{B}U = BU \quad \text{on} \quad \partial\Omega,$$

where

$$\Lambda p = \bar{p} = \frac{1}{|\Omega|} \int_\Omega p. \tag{5.3}$$

Note that $\hat{E}$ maps $(0, 0, 1)^T$, a basis of $K(E)$, to $(0, 0, 1)^T \times (0, 0)^T$, a basis of the complementary space of $R(E)$. Hence $\hat{E}$ is a bijection on $H^\tau(\Omega)$.

We provide three preconditioning operators $P^{(i)} = \{L^{(i)}, B\}$, $i = 1, 2, 3$, for E using the following:

$$L^{(1)} = \begin{pmatrix} \nabla^2 & 0 & 0 \\ 0 & \nabla^2 & 0 \\ \partial_x & \partial_y & 1 \end{pmatrix}, \tag{5.4}$$

$$L^{(2)} = \begin{pmatrix} \nabla^2 & 0 & 0 \\ 0 & \nabla^2 & 0 \\ -\partial_x & -\partial_y & 1 \end{pmatrix}, \tag{5.5}$$

and

$$L^{(3)} = \begin{pmatrix} \nabla^2 & 0 & 0 \\ 0 & \nabla^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.6}$$

in Ω with the same σ and τ as the Stokes operator and the same Dirichlet boundary operator B.

We now show that $P^{(1)}$ consisting of

$$L^{(1)}U = \begin{pmatrix} \nabla^2 & 0 & 0 \\ 0 & \nabla^2 & 0 \\ \partial_x & \partial_y & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ g \end{pmatrix} \quad \text{in} \quad \Omega$$

and the boundary operator B from (5.2) has index zero. Given data f_1, f_2 in $L_2(\Omega)$ and b_1, b_2 in $H^{3/2}(\partial\Omega)$, it is well known that the solutions u and v for the equations

$$\begin{cases} \nabla^2 u = f_1 & \text{in } \Omega \\ u = b_1 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \nabla^2 v = f_2 & \text{in } \Omega \\ v = b_2 & \text{on } \partial\Omega \end{cases}$$

exist and are unique on $H^2(\Omega)$, see Gilbarg and Trudinger [4]. We can obtain a unique solution $p \in H^1(\Omega)$ from the equation

$$u_x + v_y + p = g$$

for any $g \in H^1(\Omega)$. Therefore $P^{(1)}$ is one-to-one and onto. Similarly, $P^{(2)}$ and $P^{(3)}$ are one-to-one and onto. Since $P^{(i)}$ is one-to-one and onto, we don't need to modify $P^{(i)}$, i.e., $\hat{P}^{(i)} = P^{(i)}$.

Grisvard [6] has shown that the above properties of Stokes equations are still true on convex polygonal domains. Also, the properties of above preconditioners are valid on convex polygonal domains (see Grisvard [7]). Therefore, the above discussion is still valid on convex polygonal domains.

Example 5.2 (The Stokes Operator with the Stress Boundary Condition.)

Let Ω be an open bounded domain with smooth boundary $\partial\Omega$ in R^2. The Stokes operator S was defined in Example 5.1. Also, define the stress boundary operator $B_1 : H^{\tau-1/2}(\partial\Omega) \to H^{-\rho-1/2}(\partial\Omega)$ with $\rho = (\rho_1, \rho_2) = (-1, -1)$ such that

$$B_1 U = \begin{pmatrix} -\partial_n & 0 & n_x \\ 0 & -\partial_n & n_y \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{5.7}$$

where $\vec{n} = (n_x, n_y)$ is the outer normal along the boundary.

Let $U = (u, v, p)^T$ and $V = (\alpha, \beta, \gamma)^T$. By integration by parts, we have

$$(SU, V) + \int_{\partial\Omega} \alpha \cdot (p \cdot n_x - \frac{\partial u}{\partial n}) + \int_{\partial\Omega} \beta \cdot (p \cdot n_y - \frac{\partial v}{\partial n}) =$$

$$(U, SV) + \int_{\partial\Omega} u \cdot (\gamma \cdot n_x - \frac{\partial \alpha}{\partial n}) + \int_{\partial\Omega} v \cdot (\gamma \cdot n_y - \frac{\partial \beta}{\partial n}), \tag{5.8}$$

where $(\cdot,\ \cdot)$ is the inner product operator. Therefore, the operator $E = \{S, B_1\}$ is self-adjoint. On the other hand, E is a Fredholm operator, so we have

$$\text{codim}(\text{range}\, E) = \dim(\text{kernel}\, E^*),$$

where E^* is the adjoint of E (see Wloka, Rowley, Lawruk [18] pp. 367). Since E is self-adjoint, we have

$$\dim(\text{kernel}\, E) = \dim(\text{kernel}\, E^*).$$

Therefore,

$$\text{codim}(\text{range}\, E) = \dim(\text{kernel}\, E),$$

and hence, $\text{index}(E) = 0$.

It is easy to see that the system

$$SU = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad B_1 U = 0 \quad \text{on} \quad \partial\Omega$$

has the nontrivial solutions $(u_0,\ 0,\ 0)^T$ and $(0,\ v_0,\ 0)^T$ for any constants u_0 and v_0, hence, $\dim(\text{kernel}(E)) = 2$. On the other hand, we know $\text{codim}(\text{range}(E)) = 2$ from the above discussion. If we put $(\alpha_0\ ,\ \beta_0,\ 0)^T$ with α_0 and β_0 any constants into (5.8), then we obtain two constraints for the range of E:

$$\int_\Omega f_1 + \int_{\partial\Omega} b_1 = 0 \quad \text{and} \quad \int_\Omega f_2 + \int_{\partial\Omega} b_2 = 0.$$

It is easy to check that a basis of the complementary space of $R(E)$ is $(1,0,0)^T \times (0,0)^T$ and $(0,1,0)^T \times (0,0)^T$.

The modification operator $\hat{E}$ of E that we use is

$$\hat{S}U = \begin{pmatrix} \nabla^2 + \Lambda & 0 & -\partial_x \\ 0 & \nabla^2 + \Lambda & -\partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} \quad \text{in} \quad \Omega$$

and

$$\hat{B}_1 U = B_1 U$$

where Λ is the average operator defined in (5.3). $\hat{E}$ is easily seen to be a bijection on $H^\tau(\Omega)$.

We provide three preconditioning operators $P^{(i)} = \{L^{(i)}, B_2\}$ where $i = 1, 2, 3$, and $L^{(i)}$ were defined in (5.4) to (5.6), with the Neumann boundary operator:

$$B_2 = \begin{pmatrix} -\partial_n & 0 & 0 \\ 0 & -\partial_n & 0 \end{pmatrix} \quad \text{on} \quad \partial\Omega. \tag{5.9}$$

We now show that $P^{(1)} = (L^{(1)}, B_2)$ has the index zero. It is well known that the solutions u and v of equations

$$\begin{cases} \nabla^2 u = f_1 & \text{in } \Omega \\ -\dfrac{\partial u}{\partial n} = b_1 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \nabla^2 v = f_2 & \text{in } \Omega \\ -\dfrac{\partial v}{\partial n} = b_2 & \text{on } \partial\Omega. \end{cases}$$

exist on $H^2(\Omega)$ for f_1, $f_2 \in L_2(\Omega)$ and b_1, $b_2 \in H^{3/2}(\partial\Omega)$ if they satisfy the two constraints

$$\int_\Omega f_1 + \int_{\partial\Omega} b_1 = 0 \quad \text{and} \quad \int_\Omega f_2 + \int_{\partial\Omega} b_2 = 0,$$

and the solutions u and v are then unique to within an additive constant. Therefore, we can obtain a unique solution $p \in H^1(\Omega)$ from

$$u_x + v_y + p = g$$

for $g \in H^1(\Omega)$. Also,

$$K(P^{(1)}) =< (1,0,0)^T, (0,1,0)^T > .$$

It is easy to check that a basis of the complementary space of $R(P)$ is $(1,0,0)^T \times (0,0)^T$ and $(0,1,0)^T \times (0,0)^T$.

Similarly, we have

$$\dim(\text{kernel}\, P^{(2)}) = \text{codim}(\text{range}\, P^{(2)}) = 2$$

and

$$\dim(\text{kernel}\, P^{(3)}) = \text{codim}(\text{range}\, P^{(3)}) = 2.$$

Therefore,

$$\text{index}(P^{(1)}) = \text{index}(P^{(2)}) = \text{index}(P^{(3)}) = 0.$$

By using the same idea for the modification of E, we can modify $P^{(i)}$ to $\hat{P}^{(i)}$ as the following:

$$\hat{L}^{(1)} = \begin{pmatrix} \nabla^2 + \Lambda & 0 & 0 \\ 0 & \nabla^2 + \Lambda & 0 \\ \partial_x & \partial_y & 1 \end{pmatrix},$$

$$\hat{L}^{(2)} = \begin{pmatrix} \nabla^2 + \Lambda & 0 & 0 \\ 0 & \nabla^2 + \Lambda & 0 \\ -\partial_x & -\partial_y & 1 \end{pmatrix},$$

and

$$\hat{L}^{(3)} = \begin{pmatrix} \nabla^2 + \Lambda & 0 & 0 \\ 0 & \nabla^2 + \Lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in Ω, and

$$\hat{B}_2 = B_2$$

where Λ is the average operator defined in (5.3). $\hat{P}^{(i)}$ is easily seen to be a bijection on $H^\tau(\Omega)$.

Grisvard [5] has shown that the traction (Neumann) problem of linear elasticity is a regular elliptic system on open bounded convex polygonal domains. It is likely that the Stokes operator with the stress boundary condition is a regular elliptic system on any open bounded convex polygonal domains. The numerical experiment on a square domain is presented in Experiment 6.3.

Example 5.3 (The Stokes Operator with Mixed Boundary Conditions.)

Let Ω be an open bounded domain with smooth $\partial\Omega = \Gamma_0 \bigcup \Gamma_1$ where $\overline{\Gamma}_0 \bigcap \overline{\Gamma}_1 = \emptyset$. The Stokes operator S was defined in Example 5.1. Define the mixed boundary operator B_1 consistent with (5.2) on Γ_0 and consistent with (5.7) on Γ_1.

Proposition 5.1. *The solution of the above system exists and is unique.*

Proof. Let $E = \{S, B_1\}$ be the regular elliptic operator. We first show that the solution of system is unique. Consider

$$SU = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad B_1 U = 0 \quad \text{on} \quad \partial\Omega.$$

Using integration by parts, we have

$$0 = \int_\Omega u(\nabla^2 u - p_x) + v(\nabla^2 v - p_y)$$

$$= \int_{\partial\Omega} u(\frac{\partial u}{\partial n} - pn_x) + \int_{\partial\Omega} v(\frac{\partial v}{\partial n} - pn_y) - \int_\Omega [(\nabla u)^2 + (\nabla v)^2]$$

$$= -\int_\Omega [(\nabla u)^2 + (\nabla v)^2].$$

This implies

$$\nabla u = \nabla v = 0,$$

i.e., u and v are constant. But $u = v = 0$ on Γ_0, so we have $u \equiv v \equiv 0$. On the other hand,

$$p_x = \nabla^2 u = 0 \quad \text{and} \quad p_y = \nabla^2 v = 0.$$

This implies p is constant. But $p = 0$ on Γ_1, so we have $p \equiv 0$. Therefore, the solution is unique.

We show that there is a solution $(u, v, p)^T \in (H^2(\Omega), H^2(\Omega), H^1(\Omega))^T$ satisfying

$$SU = F \quad \text{in} \quad \Omega \quad \text{and} \quad B_1 U = \Phi \quad \text{on} \quad \partial\Omega \tag{5.10}$$

for any data

$$F = (f_1, f_2, g)^T \in (L_2(\Omega), L_2(\Omega), H^1(\Omega))^T,$$

$$\Phi = (b_1, b_2)^T \in (H^{3/2}(\Gamma_0), H^{3/2}(\Gamma_0))^T,$$

$$\text{and} \quad \Phi = (b_3, b_4)^T \in (H^{1/2}(\Gamma_1), H^{1/2}(\Gamma_1))^T.$$

It is well known that the solutions of $\tilde{u}$ and $\tilde{v}$ of equations

$$\begin{cases} \nabla^2 \tilde{u} = f_1 & \text{in } \Omega \\ \tilde{u} = b_1 & \text{on } \Gamma_0 \\ -\dfrac{\partial \tilde{u}}{\partial n} = b_3 & \text{on } \Gamma_1 \end{cases}$$

and

$$\begin{cases} \nabla^2 \tilde{v} = f_2 & \text{in } \Omega \\ \tilde{v} = b_2 & \text{on } \Gamma_0 \\ -\dfrac{\partial \tilde{v}}{\partial n} = b_4 & \text{on } \Gamma_1 \end{cases}$$

exist and are unique on $H^2(\Omega)$. Let

$$\begin{cases} u = u' + \tilde{u} \\ v = v' + \tilde{v} \end{cases} \tag{5.11}$$

and substitute these expressions for u and v in system (5.10), we have

$$SU' = \tilde{F} \quad \text{in} \quad \Omega \quad B_1 U' = 0 \quad \text{on} \quad \partial\Omega \tag{5.12}$$

where $\tilde{F} = (0, 0, g')^T$ and $g' = g - (\tilde{u}_x + \tilde{v}_y) \in H^1(\Omega)$. Consider

$$SU' = F' \quad \text{in} \quad \Omega \quad B_1 U' = \Phi' \quad \text{on} \quad \partial\Omega \tag{5.13}$$

where

$$\begin{aligned} U' &= (u', v', p)^T \in (H^2_{\Gamma_0}(\Omega), H^2_{\Gamma_0}(\Omega), H^1(\Omega))^T, \\ H^2_{\Gamma_0}(\Omega) &= \{u \in H^2(\Omega) | u = 0 \quad \text{on} \quad \Gamma_0\}, \\ F' &= (f_1', f_2', g)^T, \\ \Phi' &= (0, 0)^T \quad \text{on} \quad \Gamma_0 \\ \text{and} \quad \Phi' &= (b_3', b_4')^T \quad \text{on} \quad \Gamma_1. \end{aligned}$$

Let $V' = (\alpha', \beta', \gamma)^T \in (H^2_{\Gamma_0}(\Omega), H^2_{\Gamma_0}(\Omega), H^1(\Omega))^T$, by integration by parts, we have

$$(SU', V') + \int_{\Gamma_1} \alpha' \cdot (p \cdot n_x - \frac{\partial u'}{\partial n}) + \int_{\Gamma_1} \beta' \cdot (p \cdot n_y - \frac{\partial v'}{\partial n}) + \int_{\Gamma_0} (u'\alpha' + v'\beta') =$$

$$(U', SV') + \int_{\Gamma_1} u' \cdot (\gamma \cdot n_x - \frac{\partial \alpha'}{\partial n}) + \int_{\Gamma_1} v' \cdot (\gamma \cdot n_y - \frac{\partial \beta'}{\partial n}) + \int_{\Gamma_0} (u'\alpha' + v'\beta'),$$

i.e., $E = (S, B_1)$ is self-adjoint on $(H^2_{\Gamma_0}(\Omega), H^2_{\Gamma_0}(\Omega), H^1(\Omega))^T$. By the same computation as the original system, the solution of (5.13) is unique in $(H^2_{\Gamma_0}(\Omega), H^2_{\Gamma_0}(\Omega), H^1(\Omega))^T$. Thus, $\dim(\mathrm{K}(E')) = 0$. On the other hand, E' is a self-adjoint operator and Fredholm operator. Then E' is onto. Hence, for any $g' \in H^1(\Omega)$, system (5.12) has a unique solution $(u', v', p)^T$

$\in (H^2_{\Gamma_0}(\Omega), H^2_{\Gamma_0}(\Omega), H^1(\Omega))^T$. We can find a solution $(u, v, p)^T \in (H^2(\Omega), H^2(\Omega), H^1(\Omega))^T$ for system (5.10). This proves Proposition 5.1.

Consider the preconditioner $P_1 = (L^{(1)}, B_2)$ where $L^{(1)}$ was defined in (5.4) and B_2 was defined by (5.2) on Γ_0 and (5.9) on Γ_1. We show that P_1 is one-to-one and onto on (H^2, H^2, H^1). It is well known that the solutions u and v for the following equations

$$\begin{cases} \nabla^2 u = f_1 & \text{in} \quad \Omega \\ -u_n = b_1 & \text{on} \quad \Gamma_1 \\ u = b_2 & \text{on} \quad \Gamma_0 \end{cases} \quad \text{and} \quad \begin{cases} \nabla^2 v = f_2 & \text{in} \quad \Omega \\ -v_n = b_3 & \text{on} \quad \Gamma_1 \\ v = b_4 & \text{on} \quad \Gamma_0 \end{cases}$$

exist and are unique in $H^2(\Omega)$ for $f_1, f_2 \in L^2(\Omega)$, $b_1, b_3 \in H^{1/2}(\Gamma_1)$, and $b_2, b_4 \in H^{3/2}(\Gamma_0)$. We can get a unique solution $p \in H^1$ from the equation

$$u_x + v_y + p = g.$$

Therefore, P_1 is one-to-one and onto on (H^2, H^2, H^1).

Grisvard [6] has shown that the Laplacian operator with mixed boundary conditions is a regular elliptic operator on open bounded convex polygonal domains with all corners having interior angle less than or equal $\pi/2$ and Γ_0 and Γ_1 meet at the corners. It is likely that the Stokes operator with mixed boundary conditions is a regular elliptic system on such domains. The numerical experiment on a square domain is presented in Experiment 6.4.

6. Numerical Experiments.

In this section, we describe computational results for scalar elliptic operators and the Stokes operator with several different kinds of boundary operators on the annulus and square. The experiments use a second-order difference scheme for the scalar elliptic operators and regularized central difference schemes for the Stokes operator with $\alpha = 0.001$. The regularized central difference scheme was developed by Strikwerda [13]. The initial condition was that the initial iterate was zero inside the domain. We use the GMRES(m) [12] method for the iteration method, and take $m = 7$ for all experiments. Each solution procedure was terminated when the minimized residual was less than 10^{-6}. The inversions of the Laplacian operator for preconditioning are accomplished by the Fourier method for polar coordinates and the Fast Poisson Solver of [15] for Cartesian coordinates. Both of methods solve the discrete Laplacian operator directly by taking the Fourier transformation to reduce the original problem to a tridiagonal system, and using the Thomas algorithm to solve this system.

Experiment 6.1

The first experiment is with a scalar elliptic operator with mixed boundary conditions on the unit square. Let $\Omega = \{(x, y) | 0 \le x, y \le 1\}$ and E be the scalar elliptic operator

$$E = \begin{cases} \nabla^2 u + u_x + u_y & \text{in} \quad \Omega \\ u & \text{on} \quad x = 0 \\ u & \text{on} \quad y = 0 \\ u_x + u & \text{on} \quad x = 1 \\ u_y + u & \text{on} \quad y = 1 \end{cases}$$

Take the domain of E as $H^2(\Omega)$. By standard results, E is one-to-one and onto on $H^2(\Omega)$ (see Grisvard [5]). Hence, we don't need to modify E.

Consider the following three preconditioning operators:

$$P_1 = \begin{cases} \nabla^2 u & \text{in} \quad \Omega \\ u & \text{on} \quad x = 0 \\ u & \text{on} \quad y = 0 \\ u_x & \text{on} \quad x = 1 \\ u_y & \text{on} \quad y = 1 \end{cases}, \quad P_2 = \begin{cases} \nabla^2 u & \text{in} \quad \Omega \\ u & \text{on} \quad x = 0 \\ u & \text{on} \quad y = 0 \\ u_x & \text{on} \quad x = 1 \\ u & \text{on} \quad y = 1 \end{cases}, \quad P_3 = \begin{cases} \nabla^2 u & \text{in} \quad \Omega \\ u_x & \text{on} \quad x = 0 \\ u & \text{on} \quad y = 0 \\ u_x & \text{on} \quad x = 1 \\ u & \text{on} \quad y = 1 \end{cases}.$$

Similarly, P_1, P_2, and P_3 are one-to-one and onto on $H^2(\Omega)$. Therefore, we don't need to modify these three preconditioners.

The preconditioner P_1 has the same order of boundary conditions on the same boundary components as E and $\text{index}(E) = \text{index}(P_1)$. By Theorem 3.2, P_1 should be a good preconditioner to E.

We take the data so that the exact solution is

$$u = ye^x. \tag{6.1}$$

The computational results are shown in Table 6.1.

	Left-Preconditioning		Right-Preconditioning	
h	Iterations	Error	Iterations	Error
1/10	2	$1.49E{-3}$	2	$1.49E{-3}$
1/20	1	$3.66E{-4}$	2	$3.66E{-4}$
1/40	1	$9.05E{-5}$	2	$9.05E{-5}$
1/80	1	$2.24E{-5}$	2	$2.25E{-5}$

Table 6.1

From Table 6.1, we can see the iteration numbers are very small and almost constant, showing that this is a very good preconditioner. The order of the errors are $O(h^2)$.

On one of the sides of the square the boundary condition for P_2 has an order different from E. Therefore, Theorem 3.2 does not apply to P_2, but because P_1 does satisfy Theorem 3.2, we can predict preconditioner P_1 will be better than P_2. The results of using P_2 are shown in Table 6.2. Also, Table 6.2 shows the order of errors are $O(h^2)$. Notice that the number of iterations grows as h decreases.

The preconditioner P_3 has two different order of boundary conditions from E. We can predict the computational results will be poor and the results show in Table 6.3 confirm this.

	Left-Preconditioning		Right-Preconditioning	
h	Iterations	Error	Iterations	Error
1/10	5	$1.49E{-3}$	6	$1.49E{-3}$
1/20	6	$3.66E{-4}$	8	$3.66E{-4}$
1/40	9	$9.11E{-5}$	13	$9.03E{-5}$
1/80	18	$2.18E{-5}$	83	$2.25E{-5}$

Table 6.2

	Left-Preconditioning		Right-Preconditioning	
h	Iterations	Error	Iterations	Error
1/10	29	$1.49E{-3}$	45	$1.49E{-3}$
1/20	$\star$	$\star$	129	$3.66E{-4}$
1/40	$\star$	$\star$	445	$9.07E{-5}$
1/80	$\star$	$\star$	$\star$	$\star$

Table 6.3

In the table, $\star$ indicates the solution didn't converge in 500 iterations or the GMRES(m) method stalled.

Experiment 6.2

The second experiment was with the Stokes operator with the Dirichlet boundary condition on the annulus. Let Ω be the annulus centered at the origin with interior radius equal 1 and exterior radius equal 2. Let $E = \{S, B\}$ be the elliptic operator with S defined in (5.1) and B defined in (5.2). By the discussion of Example 5.1, we know the dimension of kernel(E) and co-dimension of range(E) are equal to one. For convenience of computation, we use the notion of factor spaces rather than the modification ideas in Section 3. Consider E mapping from factor space $H^{\tau+l}/K(E)$ to $H^{l-\rho}/R^{\perp}(E)$. For the modified operator $\hat{E}$, we have

$$\hat{E}|_{H^{\tau+l}/K(E)} = E|_{H^{\tau+l}/K(E)}.$$

Therefore, the idea of factor spaces is equivalent to the modification ideas in Section 3. In this experiment, we consider E on $(H^2(\Omega), H^2(\Omega), H^1(\Omega)/R)^T$ where R is the real line. The algorithm for computing the averages and norms is due to West [17], see also Strikwerda [13].

Consider the following preconditioner $P = (L^{(1)}, B)$ with $L^{(1)}$ defined in (5.4) and B defined in (5.2). From the discussion of Example 5.1, P is one-to-one and onto on $(H^2(\Omega), H^2(\Omega), H^1(\Omega))^T$. It is also one-to-one and onto on $(H^2(\Omega), H^2(\Omega), H^1(\Omega)/R)^T$.

Since P has the same order of boundary conditions on the same boundary components as E and both index are the same, by Theorem 3.2, P is a good preconditioner.

We take the data so that the solution is

$$(u, v, p)^T = (\sin(y), \cos(x), \cos(x) + \cos(y))^T \tag{6.2}$$

for the exact solution. The computational results are shown in Table 6.4.

	Left-Preconditioning		Right-Preconditioning	
$\Delta r, \quad \Delta\theta$	Iterations	Error	Iterations	Error
$1/10, \pi/10$	6	$7.67E{-}02$	8	$7.67E{-}02$
$1/20, \pi/10$	7	$2.00E{-}02$	10	$2.00E{-}02$
$1/40, \pi/10$	7	$7.26E{-}03$	8	$7.25E{-}03$
$1/80, \pi/10$	7	$5.54E{-}03$	7	$5.56E{-}03$

Table 6.4

As we expected, the iteration numbers are essentially constant for all grid spacings and the order of errors are approximately $O(h^2)$.

Experiment 6.3

The third experiment was with the Stokes operator with the stress boundary condition on the unit square. Let $\Omega = \{(x, y)|0 \leq x, y \leq 1\}$ and $E = \{S, B\}$ be the elliptic operator with S defined in (5.1) and B defined in (5.7). From Example 5.2, we know the $\dim(\text{kernel}(E)) = \text{codim}(\text{range}(E)) = 2$. For convenience of computation, we use the ideas of factor space which are discussed above to modify E on $(H^2(\Omega)/R, H^2(\Omega)/R, H^1(\Omega))^T$ such that E is one-to-one and onto.

We use two preconditioners for this experiment. The first preconditioning operator is $P_1 = (L^{(1)}, B_2)$ with $L^{(1)}$ defined in (5.4) and B_2 defined in (5.9) and the second preconditioning operator is $P_2 = (L^{(1)}, B)$ where B is the Dirichlet boundary operator in (5.2).

From Example 5.2, we know that the dimension of kernel(P_1) and the co-dimension of range(P_1) are 2. With the ideas of factor space, P_1 on $(H^2(\Omega)/R, H^2(\Omega)/R, H^1(\Omega))^T$ is one-to-one and onto.

Since P_1 has the same order of boundary conditions on the same boundary components as E and both index are the same, Theorem 3.2 is satisfied. Therefore, P_1 is a good preconditioner.

We take the data so that the exact solution is

$$(u, v, p)^T = (x^2, -y^2, x + 2y)^T \tag{6.3}$$

	Left-Preconditioning		Right-Preconditioning	
h	Iterations	Error	Iterations	Error
1/10	3	$3.79E{-}06$	3	$5.98E{-}07$
1/20	2	$3.48E{-}06$	3	$7.35E{-}07$
1/40	2	$1.04E{-}06$	3	$2.42E{-}07$
1/80	2	$5.31E{-}07$	3	$9.08E{-}08$

Table 6.5

and we obtain the computational results in Table 6.5.

From Example 5.1, P_2 is one-to-one and onto on $(H^2(\Omega), H^2(\Omega), H^1(\Omega))^T$. But the order of the boundary conditions for P_2 are totally different from Those of E. Even though the values of index of E and P_2 are the same, we can predict that P_2 is not a good preconditioner. For the experiments with preconditioner P_2, GMRES(7) stalled for grid size less than 1/10. These results are not displayed.

Experiment 6.4

The fourth experiment was with the Stokes operator and mixed boundary conditions on the unit square. Let $\Omega = \{(x,y)|0 \le x, y \le 1\}$, $\Gamma_0 = \{y = 0 \quad \text{or} \quad y = 1\}$, and $\Gamma_1 = \{x = 0 \quad \text{or} \quad x = 1\}$. E is the Stokes operator with mixed boundary operators that we discussed in Example 5.3. By Example 5.3, we know E is one-to-one and onto on $(H^2(\Omega), H^2(\Omega), H^1(\Omega))^T$ and its index is zero.

We use two preconditioners for this experiment. First preconditioning operator $P_1 = (L^{(1)}, B_1)$ with $L^{(1)}$ defined by (5.4) and B_1 defined by (5.2) on Γ_0 and (5.9) on Γ_1. Second preconditioning operator $P_2 = (L^{(1)}, B_2)$ with the Dirichlet boundary operator B_2 defined in (5.2).

By Example 5.3, P_1 is one-to-one and onto on $(H^2(\Omega), H^2(\Omega), H^1(\Omega))^T$. Since P_1 has the same order of boundary conditions on the same boundary components as E and both indices are the same, by Theorem 3.2, P_1 is a good preconditioner.

We take the data so that the exact solution is

$$(u, v, p)^T = (\sin(y), \sin(x), \cos(x) + \cos(y))^T \tag{6.4}$$

and get the computational results in Table 6.6. In Table 6.6, we see that the l_2 errors are $O(h^2)$.

	Left-Preconditioning		Right-Preconditioning	
h	Iterations	Error	Iterations	Error
1/10	4	$1.45E{-}03$	6	$1.45E{-}03$
1/20	4	$3.42E{-}04$	6	$3.42E{-}04$
1/40	3	$8.29E{-}05$	7	$8.16E{-}05$
1/80	3	$2.05E{-}05$	7	$1.97E{-}05$

Table 6.6

From Example 5.1, P_2 is one-to-one and onto on $(H^2(\Omega), H^2(\Omega), H^1(\Omega))^T$ and so we don't need to modify it. But since P_2 has two boundary conditions of different order than E, Theorem 3.2 is not satisfied. We can predict that preconditioner P_1 will be better than P_2. With the exact solution (6.4) we can see the results of P_2 in Table 6.7. Notice that the number of iterations increases as h decreases which is a consequence of $P_2^{-1}E$ being unbounded.

	Left-Preconditioning		Right-Preconditioning	
h	Iterations	Error	Iterations	Error
1/10	31	$1.45E{-}03$	50	$1.45E{-}03$
1/20	41	$3.44E{-}04$	$\star$	$\star$
1/40	78	$8.54E{-}05$	$\star$	$\star$
1/80	151	$2.23E{-}05$	$\star$	$\star$

Table 6.7

In the table, $\star$ indicates the solution either didn't converge in 500 iterations or that the GMRES(m) method stalled.

7. Conclusions.

We have presented methods for analysis of preconditioners for regular elliptic systems and provided some useful guidelines for choosing good left and right preconditioners.

Some good preconditioners for the scalar elliptic operators and the Stokes operator with several different kinds of boundary conditions are given in Section 5. Several computational results illustrating the theoretical results and good preconditioners that we provided are presented in Section 6.

The methods given in this paper lead to a great improvement in the efficiency of numerical solutions of regular elliptic systems. Although our examples have all used finite difference methods, the basic ideas can be applied with other numerical methods.

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