CENTER FOR
PARALLEL OPTIMIZATION

NEW ERROR BOUNDS FOR THE NONLINEAR COMPLEMENTARITY PROBLEM

by

O. L. Mangasarian & J. Ren

Computer Sciences Technical Report #1202

December 1993

New Error Bounds for the Nonlinear Complementarity Problem

O. L. Mangasarian & J. Ren *

December 1993

Abstract

A natural residual is shown to be globally "equivalent" to the distance between an arbitrary point and the unique solution to a strongly monotone nonlinear complementarity problem, under a Lipschitz continuity assumption. This extends Pang's error bound [10] to the nonlinear case. Under monotonicity alone, we show that this error residual is merely a "lower" error bound. In addition, we extend the linear complementarity error bound given in [7] to a global error bound for a strongly monotone nonlinear complementarity problem, without Lipschitz continuity. Comparisons among several residuals are given.

1 Introduction

We consider the nonlinear complementarity problem (NCP for short) of finding an x in $\mathbb{R}^n$ such that

$$F(x) \geq 0, \quad x \geq 0, \quad xF(x) = 0, \tag{1}$$

where F is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. The linear complementarity problem [1, 9] obtains when $F(x) = Mx + q$. A primary goal of this paper is to develop new error bounds for the nonlinear complementarity problem. We note the following residuals for the NCP (1):

$$\begin{aligned} r(x) &:= \|x - (x - F(x))_+\|, \\ s(x) &:= \|(-x,\ -F(x),\ xF(x))_+\|, \\ t(x) &:= \|(-x,\ -F(x),\ \sum_{i=1}^{n}(x_iF_i(x))_+\|, \\ v(x) &:= \|(-x,\ -F(x),\ \sum_{i=1}^{n}|x_iF_i(x)|\|. \end{aligned} \tag{2}$$

These are natural extensions of corresponding residuals for the linear complementarity problem [6, 7, 8, 10]. Note that $\bar{x}$ is a solution of (1) if and only if any of the residuals above is equal to zero. A number of error bounds have been established for the linear complementarity problem and linear variational inequality problems by using these residuals. Some of the error bounds lead to a linear convergence rate for widely-used algorithms [6]. Here, we establish some corresponding

*Computer Sciences Department, University of Wisconsin, 1210 West Dayton Street, Madison, WI 53706, email: *olvi@cs.wisc.edu, jun@cs.wisc.edu* This material is based on research supported by Air Force Office of Scientific Research Grant F49620-94-1-0036 and National Science Foundation Grant CCR-9101801.

error bounds for the distance between an arbitrary point and a "closest" solution of the nonlinear complementarity problem. In Section 2, we prove that the residual $r(x)$ is both an upper and lower bound for this distance for the strongly monotone complementarity problem under Lipschitz continuity of F. In Section 3 we prove, without Lipschitz continuity, that the residual $s(x) + s(x)^{\frac{1}{2}}$ is also an error bound for the strongly monotone complementarity problem. We also relate various residuals to each other.

A word about our notation. For a vector x in the n-dimensional space $I\!R^n$, x_+ will denote the orthogonal projection on the nonnegative orthant $I\!R^n_+$, that is $(x_+)_i := \max\{x_i, 0\}, i = 1, \ldots, n$. A norm $\|\cdot\|$ is called a monotonic if $\|x\| = \|\,|x|\,\|$. The scalar product of two vectors x and y in $I\!R^n$ is denoted by xy.

The NCP (1) is strongly monotone if there exists a constant $c > 0$ such that for any $x,\ y \in I\!R^n$

$$(F(x) - F(y))(x - y) \geq c\|x - y\|^2 \tag{3}$$

A function $F : I\!R^n \to I\!R^n$ is Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|.$$

Note that for a monotonic norm, $\|x_+ - y_+\| \leq \|x - y\|$ for any $x, y \in I\!R^n$.

2 The Error Bound $r(x)$

We begin by defining the concepts of a residual, lower and upper error bounds, and a distance-equivalent bound. Each of these quantities characterizes, to certain degree, how well a residual estimates the distance from any x to the solution set of an NCP.

Definition 2.1 *Let $e : I\!R^n \to I\!R$ and $dist(x, \bar{X}) = \inf_{\bar{x} \in \bar{X}} \|x - \bar{x}\|$. Assume that the NCP has a nonempty solution set.*

1. *$e(x)$ is a **residual** for the NCP if $e(x) \geq 0$, for all $x \in I\!R^n$, and $e(x) = 0$ if and only if x solves the NCP.*

2. *A residual $e(x)$ is a **lower global error bound** for the NCP if it is a residual for the NCP and there exists some constant $\tau_1 > 0$ such that for each $x \in I\!R^n$ and any solution $\bar{x}$*

$$\tau_1 e(x) \leq \|x - \bar{x}\|$$

3. *A residual $e(x)$ is an (upper) **global error bound** for the NCP if it is a residual for the NCP and there exists some constant $\tau_2 > 0$ such that for each $x \in I\!R^n$*

$$dist(x, \bar{X}) \leq \tau_2 e(x)$$

4. *A residual $e(x)$ is a **distance-equivalent bound** if it is both a lower and (upper) global error bound for the NCP.*

Note that the best estimate for $dist(x, \bar{X})$ is a distance-equivalent residual because this residual is bounded from both below and above by $dist(x, \bar{X})$ itself.

We first show that a Lipschitz continuity is all that is needed for $r(x)$ to be a lower error bound for $dist(x, \bar{X})$ for any NCP.

Theorem 2.1 *Let $F : I\!R^n \to I\!R^n$ be Lipschitz continuous with constant L. Assume that the NCP (1) has a solution. Then for any solution $\bar{x}$ of the NCP (1),*

$$r(x) \leq (2+L)\|x-\bar{x}\|. \tag{1}$$

where $\|\cdot\|$ is a monotonic norm on $I\!R^n$.

Proof. By the definition of $r(x)$, it follows that

$$\begin{aligned} r(x) &\leq \|x-(x-F(x))_+\| \\ &= \|x-(x-F(x))_+ - \bar{x} + (\bar{x}-F(\bar{x}))_+\| \\ &\leq \|x-\bar{x}\| + \|(x-F(x))_+ - (\bar{x}-F(\bar{x})_+\| \\ &\leq \|x-\bar{x}\| + \|x-\bar{x}\| + \|F(x)-F(\bar{x})\| \\ &\leq (2+L)\|x-\bar{x}\|. \end{aligned}$$

Q.E.D.

Hence residual $r(x)$ is a lower error bound for any NCP under merely a Lipschitz continuity assumption on F. However, for $r(x)$ to be an upper error bound, a much stronger condition on the NCP is needed, such as strong monotonicity condition as seen from the following theorem.

Theorem 2.2 *Let the NCP (1) be strongly monotone with constant $c > 0$ and let F be Lipschitz continuous with constant $L > 0$. Then $\forall x \in I\!R^n$*

$$\|x-\bar{x}\| \leq \frac{1+2L}{c}\|r(x)\|, \tag{2}$$

where $\bar{x}$ is the unique solution of the NCP.

Proof. Since the NCP is strongly monotone, it has a unique solution [2, Corollary 3.2], say $\bar{x}$. For a **fixed** x, the point $p(x) := (x-F(x))_+$ is a solution of the related LCP

$$\bar{F}(p) := p - x + F(x) \geq 0, \quad p \geq 0, \quad p\bar{F}(p) = 0.$$

It follows that

$$\begin{aligned} &(p(x)-\bar{x})(\bar{F}(p(x)) - F(\bar{x})) \\ &\quad = p(x)\bar{F}(p(x)) - p(x)F(\bar{x}) - \bar{x}\bar{F}(p(x)) + \bar{x}F(\bar{x}) \\ &\quad = -p(x)F(\bar{x}) - \bar{x}\bar{F}(p(x)) \\ &\quad \leq 0. \end{aligned}$$

From the definition $\bar{F}(p(x)) = p(x) - x + F(x)$ and the inequality above we have that

$$\begin{aligned} 0 &\geq (p(x)-\bar{x})(p(x)-x) + (p(x)-\bar{x})(F(x)-F(\bar{x})) \\ &= (p(x)-\bar{x})(p(x)-x) + (p(x)-x+x-\bar{x})(F(x)-F(\bar{x})) \\ &= (p(x)-\bar{x})(p(x)-x) + (p(x)-x)(F(x)-F(\bar{x})) \\ &\quad +(x-\bar{x})(F(x)-F(\bar{x})) \\ &= (p(x)-x)(p(x)-\bar{x}+F(x)-F(\bar{x})) + (x-\bar{x})(F(x)-F(\bar{x})) \\ &\geq (p(x)-x)(p(x)-\bar{x}+F(x)-F(\bar{x})) + c\|x-\bar{x}\|^2. \end{aligned} \tag{3}$$

Since $\bar{x} = p(\bar{x}), \|x_+ - y_+\| \leq \|x - y\|, \forall x, y \in I\!R^n$ and $\|F(x) - F(\bar{x})\| \leq L\|x - \bar{x}\|$, it follows that

$$\begin{aligned} \|p(x) - \bar{x}\| &= \|p(x) - p(\bar{x})\| \\ &= \|(x - F(x))_+ - (\bar{x} - F(\bar{x}))_+\| \\ &\leq \|x - F(x) - (\bar{x} - F(\bar{x}))\| \\ &\leq \|x - \bar{x}\| + \|F(x) - F(\bar{x}))\| \\ &\leq \|x - \bar{x}\| + L\|x - \bar{x}\| \\ &\leq (1 + L)\|x - \bar{x}\| \end{aligned} \tag{4}$$

By using the Cauchy-Schwartz inequality in (3) and noting that $r(x) = x - p(x)$, we have

$$\begin{aligned} c\|x - \bar{x}\|^2 &\leq \|x - p(x)\|(\|p(x) - \bar{x}\| + \|F(x) - F(\bar{x})\|) \\ &\leq r(x)((1 + L)\|x - \bar{x}\| + L\|x - \bar{x}\|) \quad \text{(By (4))} \\ &\leq (1 + 2L)r(x)\|x - \bar{x}\| \end{aligned}$$

Therefore,

$$\|x - \bar{x}\| \leq \frac{1 + 2L}{c} r(x).$$

Q.E.D.

Corollary 2.1 *Let the NCP (1) be strongly monotone and let F be Lipschitz continuous. Then $r(x)$ is distance-equivalent.*

Proof. The proof is immediate from Theorems 2.1 and 2.2. **Q.E.D.**

Remark. When $r(x)$ is a distance-equivalent bound, it precisely characterizes $dist(x, \bar{X})$. More importantly, it is explicitly computable for any x. Hence it can be used as an accurate measure of the error in an approximate solution to the NCP and thus serves as a useful termination criterion for any algorithm for the NCP. It can also be used to measure the error reduction rate in the convergence proof of such algorithms [4, 5, 6].

3 The Error Bound $s(x) + s(x)^{\frac{1}{2}}$

Mangasarian and Shiau proved that $s(x) + s(x)^{\frac{1}{2}}$ is an (upper) error bound for the monotone linear complementarity problem [7]. Here we prove that $s(x) + s(x)^{\frac{1}{2}}$ is also an (upper) error bound for the strongly monotone **nonlinear** complementarity problem as can be seen from the following theorem. Note that there is no Lipschitz continuity requirement on the NCP.

Theorem 3.1 *Let $F : I\!R^n \to I\!R^n$ be strongly monotone. Then there exists a positive λ such that*

$$\|x - \bar{x}\| \leq \lambda(s(x) + s(x)^{\frac{1}{2}}). \tag{5}$$

where $\bar{x}$ is the unique solution of the NCP (1), and the norm is any monotonic norm.

Proof. By strong monotonicity of the NCP, there exits the unique solution $\bar{x}$ for the NCP [2]. Assume that the theorem is false. Then for each integer k, there exists an $x^k \neq \bar{x}$ such that

$$\|x^k - \bar{x}\| > k(s(x^k) + s(x^k)^{\frac{1}{2}}). \tag{6}$$

Case 1. Let $\{x^k\}$ be unbounded. Then

$$\begin{aligned}
\frac{\|x^k - \bar{x}\|}{\|x^k\|} &\geq \frac{k(s(x^k) + s(x^k)^{\frac{1}{2}})}{\|x^k\|} \\
&\geq k\frac{s(x^k)}{\|x^k\|} \\
&= k\|(-\frac{x^k}{\|x^k\|}, -\frac{F(x^k)}{\|x^k\|}, \frac{x^kF(x^k)}{\|x^k\|})_+\|.
\end{aligned} \tag{7}$$

Since $\lim_{k\to\infty} \frac{\|x^k-\bar{x}\|}{\|x^k\|} = 1$, it follows from the above inequality between the first and last terms that for each $\delta > 0$, there exists a $k(\delta)$ such that

$$-\frac{x^k}{\|x^k\|} \leq \delta e, \quad -\frac{F(x^k)}{\|x^k\|} \leq \delta e, \quad \frac{x^kF(x^k)}{\|x^k\|} \leq \delta e, \quad \forall k \geq k(\delta) \tag{8}$$

Hence for $k \geq k(\delta)$, we have

$$\begin{aligned}
\delta \geq \frac{x^kF(x^k)}{\|x^k\|} &= \frac{x^kF(x^k) + \bar{x}F(\bar{x}) - x^kF(\bar{x}) - \bar{x}F(x^k) + x^kF(\bar{x}) + \bar{x}F(x^k)}{\|x^k\|} \\
&\geq \frac{(F(x^k) - F(\bar{x}))(x^k - \bar{x})}{\|x^k\|} + \frac{x^kF(\bar{x}) + \bar{x}F(x^k)}{\|x^k\|} \\
&\geq \frac{c\|x^k - \bar{x}\|^2}{\|x^k\|} + \delta e(F(\bar{x}) + \bar{x}).
\end{aligned}$$

This however is a contradiction since $\lim_{k\to\infty} \frac{\|x^k-\bar{x}\|^2}{\|x^k\|} = \infty$.

Case 2. Let $\|x^k\|$ be bounded. Without loss of generality, let $\{x^k\}$ converge to $\tilde{x}$. Since the left hand side of (6) is bounded, so $s(x^k)$ goes to zero. Therefore $s(\tilde{x}) = 0$, and hence $\tilde{x}$ is a solution of the NCP. Since the NCP has a unique solution, $\bar{x} = \tilde{x}$. Again from (6),

$$\begin{aligned}
\|x^k - \bar{x}\| &> ks(x^k)^{\frac{1}{2}} \\
&= k\|(-x^k, -F(x^k), x^kF(x^k))_+\|^{\frac{1}{2}} \\
&= k\|(-x^k, -F(x^k), x^kF(x^k) + \bar{x}F(\bar{x}) - \bar{x}F(x^k) - x^kF(\bar{x}) \\
&\quad +\bar{x}F(x^k) + x^kF(\bar{x}))_+\|^{\frac{1}{2}} \\
&= k\|(-x^k, -F(x^k), (F(x^k) - F(\bar{x}))(x^k - \bar{x}) + \bar{x}F(x^k) + x^kF(\bar{x}))_+\|^{\frac{1}{2}} \\
&\geq k\|(-x^k, -F(x^k), c\|x^k - \bar{x}\|^2 + \bar{x}F(x^k) + x^kF(\bar{x}))_+\|^{\frac{1}{2}}
\end{aligned}$$

Hence

$$\|x^k - \bar{x}\| > k\|(-x^k, -F(x^k), c\|x^k - \bar{x}\|^2 + \bar{x}F(x^k) + x^kF(\bar{x}))_+\|^{\frac{1}{2}}. \tag{9}$$

We now show that this inequality cannot hold when k is sufficiently large, and hence we have a contradiction. In fact, if $\bar{x}F(x^k) + x^kF(\bar{x}) < -\frac{c}{2}\|x^k - \bar{x}\|^2$, then

$$\begin{aligned}
\frac{c}{2}\|x^k - \bar{x}\|^2 &< -\bar{x}F(x^k) - x^kF(\bar{x}) \\
&\leq (-x^k)_+F(\bar{x}) + (-F(x^k))_+\bar{x} \\
&\leq \|(-x^k, -F(x^k))_+\|\|\bar{x}, F(\bar{x})\|'.
\end{aligned}$$

where $\|\cdot\|'$ is the dual norm to $\|\cdot\|$. It follows that $\|\bar{x}, F(\bar{x})\|' > 0$. Hence we have from (9) that

$$\begin{aligned} \|x^k - \bar{x}\|^2 &> k^2\|(-x^k, -F(x^k))_+\| \\ &\geq \frac{k^2 c\|x^k - \bar{x}\|^2}{2\|\bar{x}, F(\bar{x})\|}. \end{aligned}$$

Hence

$$1 > \frac{k^2 c}{2\|\bar{x}, F(\bar{x})\|}.$$

This inequality fails to hold for k sufficiently large.
Suppose now that $\bar{x}F(x^k) + x^k F(\bar{x}))_+ \geq -\frac{c}{2}\|x^k - \bar{x}\|^2$, then by (9)

$$\begin{aligned} \|x^k - \bar{x}\| &> k\|(-x^k, -F(x^k), c\|x^k - \bar{x}\|^2 + \bar{x}F(x^k) + x^k F(\bar{x}))_+\|^{\frac{1}{2}} \\ &\geq k\|(c\|x^k - \bar{x}\|^2 + \bar{x}F(x^k) + x^k F(\bar{x}))_+\|^{\frac{1}{2}} \quad \text{(By norm monotonicity)} \\ &\geq k\|(c\|x^k - \bar{x}\|^2 - \frac{c}{2}\|x^k - \bar{x}\|^2)_+\|^{\frac{1}{2}} \\ &= k\sqrt{\frac{c}{2}}\|x^k - \bar{x}\|. \end{aligned}$$

Hence $k \leq \sqrt{\frac{2}{c}}$ which is a contradiction to k being unbounded. **Q.E.D.**

Remark. It seems that there are some fundamental differences between the residuals $r(x)$ and $s(x) + s(x)^{\frac{1}{2}}$. In the case of an LCP $r(x)$ is always a local error bound, but $s(x) + s(x)^{\frac{1}{2}}$ is not. For the monotone LCP, $s(x) + s(x)^{\frac{1}{2}}$ is always an global error bound, but $r(x)$ is not. In the case of strongly monotone NCP, $s(x) + s(x)^{\frac{1}{2}}$ is always a global error bound, however Lipschitz continuity is needed for $r(x)$ to be a global error bound for the same problem. On the other hand, $r(x)$ is always a lower error bound for any NCP under Lipschitz continuity. We are unable to prove that $s(x) + s(x)^{\frac{1}{2}}$ is a lower bound to even the strongly monotone NCP.

In the rest of this paper, we relate some error residuals used. First of all, it is easy to see that $s(x) \leq t(x) \leq v(x)$ for a monotonic norm. The following proposition relates $r(x)$ and $t(x)$.

Proposition 3.1 *Let $\|\cdot\|$ denote the 1-norm in the definition of $r(x)$ and $t(x)$. Then for any $x \in I\!R^n$,*

$$r(x) \leq \|(-x, -F(x))_+\|_1 + [n\sum_{i=1}^{n}(x_i F_i(x))_+]^{\frac{1}{2}}.$$

Consequently,

$$r(x) \leq n^{1/2}(t(x) + t(x)^{1/2}) \tag{10}$$

Proof. Let $I = \{i \mid x_i \geq F_i(x)\}$, then

$$\begin{aligned} r(x) &= \|x - [x - F(x)]_+\|_1 \\ &= \sum_{i\in I}|F_i(x)| + \sum_{i\notin I}|x_i| \\ &= \sum_{i\in I, F_i(x)\geq 0}|F_i(x)| + \sum_{i\in I, F_i(x)<0}|F_i(x)| \\ &\quad + \sum_{i\notin I, x_i\geq 0}|x_i| + \sum_{i\notin I, x_i<0}|x_i| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i\in I, F_i(x)\geq 0} |F_i(x)| + \sum_{i\notin I, x_i\geq 0} |x_i| + \|(-x,\ -F(x))_+\|_1 \\
&\leq n^{\frac{1}{2}}[\sum_{i\in I, F_i(x)\geq 0} F_i^2(x) + \sum_{i\notin I, x_i\geq 0} x_i^2]^{\frac{1}{2}} + \|(-x,\ -F(x))_+\|_1 \\
&\leq n^{\frac{1}{2}}[\sum_{i\in I, F_i(x)\geq 0} (x_iF_i(x))_+ + \sum_{i\notin I, x_i\geq 0} (x_iF_i(x))_+]^{\frac{1}{2}} \\
&\quad +\|(-x,\ -F(x))_+\|_1 \\
&\leq n^{\frac{1}{2}}[\sum_{i=1}^{n}(x_iF_i(x))_+]^{\frac{1}{2}} + \|(-x,\ -F(x))_+\|_1.
\end{aligned}
$$

Q.E.D.

4 Conclusions

We have been able to establish global error bounds for the strongly monotone NCP. In some cases, an error bound is equivalent to the distance $dist(x, \bar{X})$ to the solution set, in which case, a precise estimate of $dist(x, \bar{X})$ is obtained. Hoverer, it is not obvious how to extend these error bounds to more general NCPs. In other words, are there even local error bounds for the monotone NCP? What are the minimal conditions needed for a given residual to be an error bound? These are some open questions for future research.

References

[1] R. W. Cottle, J.-S. Pang & R. E. Stone: "*The Linear Complementarity Problem*", Academic Press, New York 1992.

[2] P. T. Harker & J.-S Pang: "Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications", Mathematical Programming 48 (1990) 161-220.

[3] Z.-Q. Luo, O. L. Mangasarian, J. Ren & M. V. Solodov: "New error bounds for the linear complementarity problem", Mathematics of Operations Research, to appear.

[4] Z.-Q. Luo & P. Tseng: "On the convergence rate of dual ascent methods for linearly constrained convex minimization", Mathematics of Operations Research 18, 1993.

[5] Z.-Q. Luo & P. Tseng: "On the convergence of the coordinate descent method for convex differentiable minimization", J. Optimization Theory & Application 72, 1992, 7-35.

[6] Z.-Q. Luo & P. Tseng: "Error bound and the convergence analysis of matrix splitting algorithms for the affine variational inequality problem", SIAM Journal in Optimization 2, 1992, 43-54.

[7] O. L. Mangasarian & T. -H. Shiau: "Error bounds for monotone linear complementarity problems", Mathematical Programming 36, 1986, 81-89.

[8] O. L. Mangasarian and J. Ren: "New improved error bounds for the linear complementarity problem", Mathematical Programming, to appear.

[9] K. G. Murty: "*Linear Complementarity, Linear and Nonlinear Programming*", Heldermann Verlag, Berlin 1988.

[10] J.-S. Pang: "Inexact newton methods for the nonlinear complementarity problem", Mathematical Programming 36, 1986, 54-71.