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NEW ERROR BOUNDS FOR THE LINEAR COMPLEMENTARITY PROBLEM

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Abstract

New local and global error bounds are given for both nonmonotone and monotone linear complementarity problems. Comparisons of various residuals used in these error bounds are given. A possible candidate for a "best" error bound emerges from our comparison as the sum of two natural residuals.

1 Introduction

We consider the classical linear complementarity problem ([Mur88] [CoPS92]) of finding an x in the n-dimensional real space $I\!R^n$ such that

$$Mx + q \geq 0, \quad x \geq 0, \quad x(Mx+q) = 0, \tag{1}$$

where $M \in I\!R^{n\times n}$ and $q \in I\!R^n$. We denote this problem by LCP(M, q) for short. Let the solution set be

$$\bar{X} := \{x \mid Mx + q \geq 0,\ x \geq 0,\ x(Mx+q) = 0\}. \tag{2}$$

We assume that $\bar{X}$ is nonempty. Define two *natural* residuals: ([Pang86], [MaSh86], [LuT92])

$$r(x) := \|x - (x - Mx - q)_+\| \tag{3}$$

and

$$s(x) := \|(-Mx - q,\ -x,\ x(Mx+q))_+\|, \tag{4}$$

where $\|\cdot\|$ is some norm. Note that $x \in \bar{X}$ if and only if $r(x) = 0$ or $s(x) = 0$. These two residuals will play a major role in defining our error bounds.

A principal result of this paper is to show that $r(x) + s(x)$ bounds the distance to the solution set of the monotone LCP from any point when multiplied by an appropriate condition constant. This is a better result than that in [MaSh86] because it also serves as a local error bound for the nonmonotone cases and in addition it does not contain a square root term. We also show that $r(x) + s(x)$ is a global error bound for classes of LCPs that are larger than the monotone cases. Another interesting result is to show that the *natural* residual $r(x)$ is a global error bound not only when M has nonzero principal minors ([MaPa90] [LMRS92]), but also for the larger class of $M \in R_0$ ([CoPS92]). The proof we give here is much simpler than that in previous weaker

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results ([MaPa90] and [LMRS92]). Other error bounds are derived in this paper which are listed in convenient tabular form in Table 1.

A word about our notation. For a vector x in the n-dimensional space $\mathbb{R}^n$, x_+ will denote the orthogonal projection on the nonnegative orthant $\mathbb{R}^n_+$, that is $(x_+) =: \max\{x_i, 0\}, i = 1, \ldots, n$. A norm $\|\cdot\|$ is called a monotonic if $\|x\| = \| \, |x| \, \|$. The scalar product of two vectors x and y in $\mathbb{R}^n$ is denoted by xy. An LCP(M, q) is monotone if M is positive semi-definite, that is $xMx \geq 0$, for all $x \in \mathbb{R}^n$.

2 New Local and Global Error Bounds for the LCP

We begin with the following definitions of a residual and error bound for the LCP.

Definition 2.1 *Let* $e : \mathbb{R}^n \to \mathbb{R}$.

1. $e(x)$ *is a* ***residual*** *for the LCP(M, q) if* $e(x) \geq 0$, *for all* $x \in \mathbb{R}^n$, *and* $e(x) = 0$ *if* x *solves LCP(M,q).*

2. $e(x)$ *is a global (local) error bound for the LCP(M, q) if it is a residual such that there exists some constant* $\tau > 0$ *(and* $\epsilon > 0$*) such that for each* $x \in \mathbb{R}^n$ *(when* $e(x) \leq \epsilon$*)*

$$\|x - \bar{x}(x)\| \leq \tau e(x)$$

where $\bar{x}(x)$ *is a closest solution of LCP(M, q) to* x *under the norm* $\|\cdot\|$.

The first global error bound employing $r(x)$ was introduced by Pang ([Pang86]) for an LCP(M, q) with a positive definite M. Recently, it was found that $r(x)$ provides a global error bound for larger classes of matrices M. For example, $r(x)$ is a global error bound for P-matrices M ([MaPa90]) and more generally, for matrices M which have nonzero principal minors ([LMRS92]). In this work, we extend further these results to a larger class of matrices. Specifically, we prove that $r(x)$ is a global error bound for R_0-matrices. This class contains all the other three classes ([CoPS92]). We give a completely different and simpler proof by using a local result ([Rob81] [LuT92]). First we define the class of R_0-matrices ([CoPS92]).

Definition 2.2 *A matrix* $M \in \mathbb{R}^{n\times n}$ *is called an* R_0*-matrix if the LCP(M, 0) has zero as its unique solution.*

Theorem 2.1 *Let* $M \in \mathbb{R}^{n\times n}$ *be an* R_0*-matrix. Then there exists* $\tau > 0$ *such that for each* $x \in \mathbb{R}^{n\times n}$

$$\|x - \bar{x}(x)\| \leq \tau r(x), \tag{5}$$

where $\bar{x}(x)$ *is a closest solution of LCP(M, q) to* x *under the norm* $\|\cdot\|$.

Proof. Assume that the theorem is false. Then for each integer k, there exists an x^k such that (5) is violated, i.e.

$$\|x^k - \bar{x}(x^k)\| > kr(x^k),$$

where $\bar{x}(x^k)$ is a closest solution of LCP(M, q) to x^k under the norm $\|\cdot\|$. In particular choose a fixed solution $\bar{x}$ such that

$$\|x^k - \bar{x}\| \geq \|x^k - \bar{x}(x^k)\| > kr(x^k). \tag{6}$$

Since $r(x)$ is a local error bound [Rob81] [LuT92], it follows that there exist $K > 0$ and $\epsilon > 0$ such that for all $k > K$, $r(x^k) > \epsilon$. Otherwise, we would have for all $K > 0$, $\epsilon > 0$, there exists some $k > K$ such that $r(x^k) \leq \epsilon$. This implies, because $r(x)$ is a local error bound, that

$$\frac{\tau}{k}\|x - \bar{x}(x^k)\| > \tau r(x^k) \geq \|x - \bar{x}(x^k)\|,$$

where the first inequality follows from (6). This leads to the contradiction $\frac{\tau}{k} \geq 1$ as $k \to \infty$. Since $r(x^k) > \epsilon$, the right hand side of (6) goes to infinity as k goes to infinity and so does the left hand side of (6) since it is bigger. Therefore, $\|x^k\|$ goes to infinity. Without loss of generality, let

$$\lim_{k\to\infty} \frac{x^k}{\|x^k\|} = s$$

Note that $\|s\| = 1$. Divide both sides of (6) by $\|x^k\|$ and let k go to infinity to obtain

$$\begin{aligned} 1 &= \lim_{k\to\infty} \frac{\|x^k - \bar{x}\|}{\|x^k\|} \\ &\geq \lim_{k\to\infty} k \cdot \frac{r(x^k)}{\|x^k\|} \\ &= \lim_{k\to\infty} k \cdot \|\min\{\frac{x^k}{\|x^k\|}, \frac{Mx^k + q}{\|x^k\|}\}\| \\ &= \lim_{k\to\infty} k \cdot \|\min\{s, Ms\}\|. \end{aligned}$$

Therefore $\min\{s, Ms\} = 0$. This is equivalent to the LCP(M, 0) having a nonzero solution s. This contradicts the assumption that $M \in R_0$. **Q.E.D.**

Previous results of [Pang86], [MaPa90], [LMRS92] follow as a corollaries of Theorem 2.1.

Corollary 2.1 *Let $M \in I\!\!R^{n\times n}$ have nonzero principal minors. Then there exists a $\tau > 0$ such that for any $x \in I\!\!R^n$*

$$\|x - \bar{x}(x)\| \leq \tau r(x),$$

where $\bar{x}(x)$ is a closest solution of LCP(M, q) to x under the norm $\|\cdot\|$.

Proof. Since the matrix M is an R_0-matrix ([CoPS92]), the proof follows from Theorem 2.1. **Q.E.D.**

Remark 2.1 The residual $r(x)$ fails to bound the distance to the solution set of a monotone LCP from any point in general [MaSh86]. However, we do not know whether Theorem 2.1 holds for M beyond the class of R_0-matrices. We suspect that the answer is no. That is if $M \notin R_0$, then there exists some $q \in I\!\!R^n$ such that $r(x)$ is not a global error bound. However we are not able to prove it. This is an open question.

As the first part of Remark 2.1 indicates, in order to obtain a global error bound for the monotone LCP, other types of residuals are required. The residual $s(x)$ is one such suitable residual. Mangasarian and Shiau ([MaSh86]) established the first global error bound for the monotone LCP by using $s(x) + s(x)^{\frac{1}{2}}$. In addition Mangasarian has further showed that $s(x)$ itself is a global error bound for a special class of LCPs ([Ma90]). However, there are examples which show that either $s(x)$ or $s(x)^{\frac{1}{2}}$ by itself cannot serve as error bounds for the monotone LCP. Also $r(x)$ itself is not an error bound. Surprisingly, the sum $r(x) + s(x)$ does indeed provide a global error bound for the monotone LCP. We now show that how to get rid of the square rot term $s(x)^{\frac{1}{2}}$ in Mangasarian and Shiau's error bound. In particular we show that it can be replaced by $r(x)$.

Theorem 2.2 *Let the residual $s(x) + s(x)^{\frac{1}{2}}$ be a global error bound for LCP(M, q) for some M, that is, there exists a τ such that*

$$\|x - \bar{x}(x)\| \leq \tau(s(x) + s(x)^{\frac{1}{2}}), \tag{7}$$

where $\bar{x}(x)$ is a closest solution of LCP(M, q) to x under the norm $\|\cdot\|$. Then there exists a constant $\bar{\tau} > 0$ such that

$$\|x - \bar{x}(x)\| \leq \bar{\tau}(r(x) + s(x)), \tag{8}$$

Proof. Since $r(x)$ is a local error bound [Rob81] [LuT92], there exist $\epsilon > 0$ and $\tau_1 > 0$ such that if $r(x) \leq r(x) + s(x) \leq \epsilon$, then

$$\|x - \bar{x}(x)\| \leq \tau_1 r(x) \leq \tau_1(r(x) + s(x)).$$

Suppose now that $r(x) + s(x) \geq \epsilon$, then there are following two cases to consider.
Case 1 $r(x) \geq s(x)$: In this case $r(x) \geq \epsilon/2$. By taking square roots and multiplying we get that $r(x) \geq [(\epsilon/2)s(x)]^{\frac{1}{2}}$. We now have by (7)

$$\begin{aligned} \|x - \bar{x}(x)\| &\leq \tau(s(x) + s(x)^{\frac{1}{2}}) \\ &\leq \tau(s(x) + (2/\epsilon)^{\frac{1}{2}} r(x)) \\ &\leq \tau max\{1, (2/\epsilon)^{\frac{1}{2}}\}(s(x) + r(x)) \\ &\leq \bar{\tau}(s(x) + r(x)), \end{aligned}$$

where $\bar{\tau} = \max\{\tau_1, \tau(1 + (2/\epsilon)^{\frac{1}{2}})\}$. This gives (8).
Case 2 $r(x) \leq s(x)$: It follows that $s(x) \geq \epsilon/2$, and so $s(x) \geq [(\epsilon/2)s(x)]^{\frac{1}{2}}$, hence,

$$\begin{aligned} \|x - \bar{x}(x)\| &\leq \tau(s(x) + s(x)^{\frac{1}{2}}) \\ &\leq \tau(s(x) + (2/\epsilon)^{1/2} s(x)) \\ &\leq \bar{\tau} s(x) \\ &\leq \bar{\tau}(r(x) + s(x)). \end{aligned}$$

Therefore (8) holds for all $x \in I\!R^n$. **Q.E.D.**

Corollary 2.2 *Let LCP(M,q) is monotone. Then $r(x) + s(x)$ is a global error bound for LCP(M, q).*

Proof. Since $s(x) + s(x)^{\frac{1}{2}}$ is a global error bound [MaSh86], then, by Theorem 2.2, $r(x) + s(x)$ is also a global error bound. **Q.E.D.**

In fact, There are LCPs where $r(x) + s(x)$ is a global error bound, but $s(x) + s(x)^{\frac{1}{2}}$ is not.

Theorem 2.3 *The residual $r(x) + s(x)$ is a global error bound for a wider class of LCPs than the residual $s(x) + s(x)^{\frac{1}{2}}$.*

Proof. By Theorem 2.2, it is sufficient to prove that there is an LCP(M, q) for which $r(x) + s(x)$ is a global error bound, but $s(x) + s(x)^{\frac{1}{2}}$ is not. Let

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad q = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solution set $\bar{X}$ of LCP(M, q) is $\{x \mid x_1 = x_3 = 0, x_2 \geq 0, \text{ or } x_2 = x_3 = 0, x_1 \geq 0\}$. First we prove that $r(x) + s(x)$ is a global error bound for this LCP. By computing the $r(x)$ and $\|x - \bar{x}(x)\|$, it follows that

$$\begin{aligned} r^2(x) &= \|\min\{x,\ Mx\}\|_2^2 = (\min\{x_1,\ x_2\})^2 + (-x_2)_+^2 + x_3^2, \\ \|x - \bar{x}(x)\|_2^2 &= \min\{x_1^2 + (-x_2)_+^2 + x_3^2,\ (-x_1)_+^2 + x_2^2 + x_3^2\}. \end{aligned} \tag{9}$$

First consider $x_1 \leq x_2$. Obviously

$$r^2(x) = x_1^2 + (-x_2)_+^2 + x_3^2.$$

Furthermore,

$$\begin{aligned} &\text{If} \quad x_2 \geq x_1 \geq 0: \quad \|x - \bar{x}(x)\|_2^2 = \min\{x_1^2 + x_3^2, x_2^2 + x_3^2\} \leq x_1^2 + x_3^2 = r(x)^2; \\ &\text{If} \quad x_2 \geq 0 \geq x_1: \quad \|x - \bar{x}(x)\|_2^2 = \min\{x_1^2 + x_3^2, x_1^2 + x_2^2 + x_3^2\} \leq x_1^2 + x_3^2 \leq r(x)^2; \\ &\text{If} \quad 0 \geq x_2 \geq x_1: \quad \|x - \bar{x}(x)\|_2^2 = \min\{x_1^2 + x_2^2 + x_3^2, x_1^2 + x_2^2 + x_3^2\} = r(x)^2. \end{aligned}$$

Hence for the these cases above

$$\|x - \bar{x}(x)\|^2 \leq r^2(x).$$

For the case where $x_1 \geq x_2$, (9) gives

$$r^2(x) = x_2^2 + (-x_2)_+^2 + x_3^2$$

and

$$\begin{aligned} &\text{If} \quad x_1 \geq x_2 \geq 0: \quad \|x - \bar{x}(x)\|_2^2 = \min\{x_1^2 + x_3^2, x_2^2 + x_3^2\} \leq x_2^2 + x_3^2 = r(x)^2; \\ &\text{If} \quad x_1 \geq 0 \geq x_2: \quad \|x - \bar{x}(x)\|_2^2 = \min\{x_1^2 + x_2^2 + x_3^2, x_2^2 + x_3^2\} \leq x_2^2 + x_3^2 \leq r(x)^2; \\ &\text{If} \quad 0 \geq x_1 \geq x_2: \quad \|x - \bar{x}(x)\|_2^2 = \min\{x_1^2 + x_2^2 + x_3^2, x_1^2 + x_2^2 + x_3^2\} \leq r(x)^2. \end{aligned}$$

Hence $r(x)+s(x)$ is a global error bound for this LCP. On the other hand, take $x^k = (-k^{-4}, k^2, k^{-1})$, it follows that

$$\begin{aligned} s(x^k) &= \|(-Mx^k - q, -x^k, x^k(Mx^k + q))_+\|_2 = k^{-4} \\ \|x^k - \bar{x}(x^k)\| &= \|k^{-4}, k^{-1}\|_2 > k^{-1}. \end{aligned}$$

Hence $s(x) + s(x)^{\frac{1}{2}}$ is not an error bound for this LCP. **Q.E.D.**

Remark 2.2 Although $r(x) + s(x)$ and $s(x) + s(x)^{\frac{1}{2}}$ are global error bounds for the monotone LCP, yet in general they both fail to be global error bounds for an indefinite LCP. The following example illustrates this even for an LCP(M, q) with a symmetric M.

Let

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad q = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

It is easy to see that the LCP(M, q) has the solution set $\bar{X} = \{(x_1,\ x_2) \mid x_1 = 0,\ x_2 \geq 1\}$. Take the sequence $x^k = (k, 1/k), k = 1, 2, ...,$ then

$$\|x^k - \bar{x}(x^k)\|_2 = \|(k,\ 1/k) - (0, 1)\|_2,$$

$$\begin{aligned} s(x^k) &= \|(-1/k + 1,\ -k,\ -k, -1/k,\ 1 - k + 1)_+\| \\ &= \|(-1/k + 1,\ 2 - k)_+\|, \end{aligned}$$

$$r(x^k) = \|min\{(k, 1/k),\ (1/k - 1,\ k)\}\| = \|(1/k - 1,\ 1/k)\|.$$

It is easy to see that the distance between x^k and the solution set goes to the infinity, but both $s(x^k) + s^{\frac{1}{2}}(x^k)$ and $r(x^k) + s(x^k)$ remain bounded as k goes to the infinity.

By Theorem 2.1, $r(x) + s(x)$ is a global error bound for R_0-matrices. The following theorem proves that this is also true for $s(x) + s(x)^{\frac{1}{2}}$. Thus this theorem gives another class of matrices for which the error bound of Mangasarian and Shiau holds besides the semidefinite case.

Theorem 2.4 *Let $M \in \mathbb{R}^{n\times n}$ be an R_0-matrix. Then there exists a positive σ such that*

$$\|x - \bar{x}(x)\| \leq \sigma(s(x) + s(x)^{\frac{1}{2}}), \tag{10}$$

where $\bar{x}(x)$ is a closest solution to x under the norm $\|\cdot\|$.

Proof. Assume that the theorem is false. Then for each integer k, there exists an x^k such that (10) is violated, i.e.

$$\|x^k - \bar{x}(x^k)\| > k(s(x^k) + s(x^k)^{\frac{1}{2}}),$$

where $\bar{x}(x^k)$ is a closest solution of LCP(M, q) to x^k under the norm $\|\cdot\|$.
Case 1. $\|x^k\|$ is unbounded. Without loss of generality, let

$$\lim_{k\to\infty} \frac{x^k}{\|x^k\|} = s.$$

Note that $\|s\| = 1$. By taking a fixed solution $\bar{x}$ as $\bar{x}(x^k)$, it follows that

$$\|x^k - \bar{x}\| \geq \|x^k - \bar{x}(x^k)\| > k(s(x^k) + s(x^k)^{\frac{1}{2}}). \tag{11}$$

Dividing by $\|x^k\|$ and letting k go to infinity give

$$\begin{aligned} 1 &= \lim_{k\to\infty} \frac{\|x^k - \bar{x}\|}{\|x^k\|} \\ &\geq \lim_{k\to\infty} k\left(\frac{s(x^k) + s^{\frac{1}{2}}(x^k)}{\|x^k\|}\right) \\ &= \lim_{k\to\infty} k(\|(-s, -Ms, \lim_{k\to\infty} \frac{x^k(Mx^k+q)}{\|x^k\|})_+\| + (sMs)_+^{\frac{1}{2}}). \end{aligned}$$

Therefore $s \geq 0,\ Ms \geq 0,\ sMs = 0$. This contradicts the assumption that M is an R_0-matrix.
Case 2. $\|x^k\|$ is bounded. The left hand side of (11) is finite when k goes to infinity, thus $s(x^k)$ goes to zero. Without loss of generality, let $\{x^k\}$ converge to x^* and $s(x^*) = 0$. Therefore x^* is a solution. On the other hand, since $r(x)$ is a local error bound for each LCP(M, q) [Rob81] [LuT92], there exist positive K, ϵ and τ such that when $k > K$, $r(x^k) \leq \epsilon$ holds. Therefore for $k > K$

$$\|x^k - \bar{x}(x^k)\| \leq \tau r(x^k). \tag{12}$$

Let $I = \{x \mid x_i \geq (Mx+q)_i\}$. We now estimate $r(x)$ as follows

$$\begin{aligned} r(x) &= \|x - [x - Mx - q]_+\|_1 \\ &= \sum_{i\in I} |(Mx+q)_i| + \sum_{i\notin I} |x_i| \\ &= \sum_{i\in I, (Mx+q)_i \geq 0} |(Mx+q)_i| + \sum_{i\in I, (Mx+q)_i<0} |(Mx+q)_i| \end{aligned}$$

$$
\begin{aligned}
& + \sum_{i \notin I, x_i \geq 0} |x_i| + \sum_{i \notin I, x_i < 0} |x_i| \\
\leq\ & \sum_{i \in I, (Mx+q)_i \geq 0} |(Mx+q)_i| + \sum_{i \notin I, x_i \geq 0} |x_i| + \|(-x,\ -Mx-q)_+\|_1 \\
\leq\ & n^{\frac{1}{2}}[\sum_{i \in I, (Mx+q)_i \geq 0} (Mx+q)_i^2 + \sum_{i \notin I, x_i \geq 0} x_i^2]^{\frac{1}{2}} + \|(-x,\ -Mx-q)_+\|_1 \\
\leq\ & n^{\frac{1}{2}}[\sum_{i \in I, (Mx+q)_i \geq 0} x_i(Mx+q)_i + \sum_{i \notin I, x_i \geq 0} x_i(Mx+q)_i]^{\frac{1}{2}} \\
& + \|(-x,\ -Mx-q)_+\|_1 \\
\leq\ & n^{\frac{1}{2}}[\sum_{x_i(Mx+q)_i \geq 0}^{n} x_i(Mx+q)_i]^{\frac{1}{2}} + \|(-x,\ -Mx-q)_+\|_1 \\
=\ & n^{\frac{1}{2}}[\sum_{x_i(Mx+q)_i \geq 0} x_i(Mx+q)_i + \sum_{x_i(Mx+q)_i < 0} x_i(Mx+q)_i \\
& - \sum_{x_i(Mx+q)_i < 0} x_i(Mx+q)_i]^{\frac{1}{2}} + \|(-x,\ -Mx-q)_+\|_1 \\
=\ & n^{\frac{1}{2}}[x(Mx+q) - \sum_{x_i(Mx+q)_i < 0} x_i(Mx+q)_i]^{\frac{1}{2}} + \|(-x,\ -Mx-q)_+\|_1 \\
\leq\ & n^{\frac{1}{2}}[(x(Mx+q))_+ - \sum_{x_i(Mx+q)_i < 0} x_i(Mx+q)_i]^{\frac{1}{2}} + \|(-x,\ -Mx-q)_+\|_1 \\
\leq\ & n^{\frac{1}{2}}[(x(Mx+q))_+^{\frac{1}{2}} + (\sum_{x_i(Mx+q)_i < 0} |x_i(Mx+q)_i|)^{\frac{1}{2}}] + \|(-x,\ -Mx-q)_+\|_1 \\
\leq\ & n^{\frac{1}{2}}[(s(x) + s(x)^{\frac{1}{2}}) + (\sum_{x_i(Mx+q)_i < 0} |x_i(Mx+q)_i|)^{\frac{1}{2}}]. \qquad (13)
\end{aligned}
$$

By combining (11), (12) and (13), it follows that

$$
\begin{aligned}
& k(s(x^k) + s(x^k)^{\frac{1}{2}}) < \|x^k - \bar{x}(x^k)\| \leq \tau r(x^k) \\
& \leq\ \tau n^{\frac{1}{2}} \left\{ (s(x^k) + s(x^k)^{\frac{1}{2}}) + (\sum_{x_i^k(Mx^k+q)_i < 0} |x_i^k(Mx^k+q)_i|)^{\frac{1}{2}} \right\}. \qquad (14)
\end{aligned}
$$

But

$$
(k - \tau n^{\frac{1}{2}})\|(-x^k, -Mx^k - q)_+\|_1^{\frac{1}{2}} \leq (k - \tau n^{\frac{1}{2}})(s(x^k) + s(x^k)^{\frac{1}{2}}) \qquad (15)
$$

Combining (14) and (15) gives

$$
(k - \tau n^{\frac{1}{2}})\|(-x^k, -Mx^k - q)_+\|_1^{\frac{1}{2}} < \tau n^{\frac{1}{2}}(\sum_{x_i^k(Mx^k+q)_i < 0} |x_i^k(Mx^k+q)_i|)^{\frac{1}{2}}.
$$

Since x^k and $Mx^k + q$ are bounded, i.e. $|x_i^k| \leq N$ and $|(Mx^k+q)_i| \leq N$, $i = 1, \ldots, n$ for some fixed $N > 0$, take $(k - \tau n^{\frac{1}{2}}) \geq \tau(nN)^{\frac{1}{2}}$ and the above inequality fails to hold. Therefore, we get the contradiction. **Q.E.D.**

3 Comparisons of Different Error Residuals

In this section, we establish relationships among the different residuals. Note that we will not been able to provide a global error bound, for an indefinite matrix M, which is not of the type

$(1+\|x\|)e(x)$ where $e(x)=0$ on the solution set. We exhibit all known error bounds for the LCP in a useful tabular form given in Table 1. First we define some other error residuals.

Definition 3.1 *Let M be in $I\!R^{n\times n}$. Define the following error residuals for LCP(M, q)*

$$t(x)=\|(-Mx-q,\ -x)_+,\sum_{i=1}^{n}(x_i(Mx+q)_i)_+\|,$$

$$v(x)=\|(-Mx-q,\ -x)_+,\ \sum_{i=1}^{n}|x_i(Mx+q)_i|\|$$

It is obvious that $s(x)\le t(x)\le v(x)$ for a monotone norm. In addition we find a relationship between $r(x)$ and $t(x)$ as can be seen from the following proposition.

Lemma 3.1 *Let $\|\cdot\|$ denote the 1-norm in the definition of $r(x)$ and $t(x)$. Then for any $x\in I\!R^n$,*

$$r(x)\le\|(-x,\ -Mx-q)_+\|_1+[n\sum_{i=1}^{n}(x_i(Mx+q)_i)_+]^{\frac{1}{2}}.$$

Consequently,

$$r(x)\le n^{\frac{1}{2}}(t(x)+t(x)^{\frac{1}{2}}) \tag{16}$$

Proof. Let $I=\{i\mid x_i\ge(Mx+q)_i\}$, then

$$\begin{aligned}
r(x) &= \|x-[x-Mx-q]_+\|_1\\
&= \sum_{i\in I}|(Mx+q)_i|+\sum_{i\notin I}|x_i|\\
&= \sum_{i\in I,(Mx+q)_i\ge 0}|(Mx+q)_i|+\sum_{i\in I,(Mx+q)_i<0}|(Mx+q)_i|\\
&\quad+\sum_{i\notin I,x_i\ge 0}|x_i|+\sum_{i\notin I,x_i<0}|x_i|\\
&\le \sum_{i\in I,(Mx+q)_i\ge 0}|(Mx+q)_i|+\sum_{i\notin I,x_i\ge 0}|x_i|+\|(-x,\ -Mx-q)_+\|_1\\
&\le n^{\frac{1}{2}}[\sum_{i\in I,(Mx+q)_i\ge 0}(Mx+q)_i^2+\sum_{i\notin I,x_i\ge 0}x_i^2]^{\frac{1}{2}}+\|(-x,\ -Mx-q)_+\|_1\\
&\le n^{\frac{1}{2}}[\sum_{i\in I,(Mx+q)_i\ge 0}(x_i(Mx+q)_i)_++\sum_{i\notin I,x_i\ge 0}(x_i(Mx+q)_i)_+]^{\frac{1}{2}}\\
&\quad+\|(-x,\ -Mx-q)_+\|_1\\
&\le n^{\frac{1}{2}}[\sum_{i=1}^{n}(x_i(Mx+q)_i)_+]^{\frac{1}{2}}+\|(-x,\ -Mx-q)_+\|_1.
\end{aligned}$$

Q.E.D.

Remark 3.1 The converse of Lemma 3.1 is not true, i.e. the residual $r(x)$ cannot bound the residual $t(x)+t(x)^{\frac{1}{2}}$ as the following example shows. In fact, the following example shows that $r(x)$ cannot bound even the smaller residual $s(x)+s(x)^{\frac{1}{2}}$. Let

$$M=\begin{bmatrix}1&0\\0&1\end{bmatrix},\qquad q=\begin{bmatrix}0\\0\end{bmatrix}.$$

It is easy to see that $\bar{x} = (0,\ 0)$ is the unique solution for the LCP. Now let the sequence $x^k = (-\frac{1}{k},\ 0)$, $k = 1, 2, \ldots$. Then $Mx^k + q = x^k$ and under the 1-norm

$$r(x^k) = \|x^k - (x^k - Mx^k - q)_+\|_1 = \frac{1}{k},$$

$$s(x^k) = \|(-Mx^k - q,\ -x^k,\ x^k(Mx^k + q))_+\|_1 = 2\frac{1}{k} + \frac{1}{k^2}.$$

It follows that

$$\begin{aligned} &\lim_{k\to\infty} \frac{s(x^k) + s(x^k)^{\frac{1}{2}}}{r(x^k)} \\ =& \lim_{k\to\infty} 2 + \frac{1}{k} + (2k+1)^{\frac{1}{2}} \\ =& +\infty \end{aligned}$$

From Lemma 3.1, although $r(x)$ can be bounded by $t(x) + t(x)^{\frac{1}{2}}$, it can not be bounded by the smaller residual $s(x) + s(x)^{\frac{1}{2}}$ as the following example implies. Let

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad q = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solution set $\bar{X}$ of LCP(M, q) is $\{x \mid x_1 = x_3 = 0, x_2 \geq 0 \text{ or } x_2 = x_3 = 0, x_1 \geq 0\}$. Take $x^k = (-k^{-4}, k^2, k^{-1})$, then

$$r(x^k) = \|min\{x^k, Mx^k + q\}\| = \|(-k^{-4}, 0, k^{-1})\|$$

but

$$s(x^k) = \|(-Mx^k - q, -x^k, x^k(Mx^k + q)\| = k^{-4}$$

Therefore there does not exist a constant τ such that

$$r(x^k) \leq \tau(s(x^k) + s(x^k)^{\frac{1}{2}}).$$

This example even gives a further indication that the residual $s(x^k) + s(x^k)^{\frac{1}{2}}$ is not a local error bound for LCP(M, q). This is so, because

$$\|x^k - \bar{x}(x^k)\|_2 = \|(-k^{-4}, 0, k^{-1}\|_2$$

goes to zero slower than $s(x^k) + s(x^k)^{\frac{1}{2}}$. In addition, since $r(x^k)$ goes to zero, we know that even locally $s(x^k) + s(x^k)^{\frac{1}{2}}$ cannot bound $r(x^k)$.

Now we consider another residual defined as $p(x) := \|(-x,\ -Mx - q)_+\| + [n\sum_{i=1}^n (x_i(Mx + q)_i)_+]^{\frac{1}{2}}$. Under the 1-norm it follows from the Lemma 3.1 that

$$r(x) \leq p(x) \leq n^{\frac{1}{2}}(t(x) + t(x)^{\frac{1}{2}}). \tag{17}$$

Hence $p(x)$ is a residual for any LCP(M, q) and it is a local error bound since it bounds $r(x)$. However it does not provide a global error bound for an LCP(M, q) for a positive semi-definite M as can be seen from the following example. Let

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

M is positive semi-definite since $M + M^T = 0$ and the LCP has the unique solution $\bar{x} = (0,0)$. Let $x^k = (0,\ k)$. Under the 1-norm,

$$\begin{aligned} p(x^k) &= \|(-x^k,\ -Mx^k - q)_+\| + [n\sum_{i=1}^{n}(x_i^k(Mx^k+q)_i)_+]^{\frac{1}{2}} \\ &= (n(0,\ k)(k+1,\ 1))^{\frac{1}{2}} \\ &= \sqrt{nk}. \end{aligned}$$

It follows that

$$\begin{aligned} &\lim_{k\to\infty} \frac{\|x^k - \bar{x}\|}{p(x^k)} \\ &= \lim_{k\to\infty} \frac{k}{\sqrt{nk}} \\ &= +\infty \end{aligned}$$

We also point out that we cannot reverse the inequality sign in (17) even if we are permitted to multiply $r(x)$ and $p(x)$ by constants.

So far, we have studied several error residuals such as $p(x), r(x), s(x)$ and $t(x)$. We have explored a number of relationships among them and by combining them, we have obtained new error bounds such as $r(x) + s(x)$ and $t(x) + t(x)^{\frac{1}{2}}$. Now we further ask: What are necessary conditions for some residual to be an error bound? In other words: How big should a residual be to become an error bound? The following simple theorem provides the desired answer.

Theorem 3.1 *Let $M \in I\!R^{n\times n}$ and $c = 2 + \|M\|$, then*

$$r(x) \le c\|x - \bar{x}(x)\|, \tag{18}$$

where $\bar{x}(x)$ is a closest solution to x under norm $\|\cdot\|$.

Proof. By the definition of $r(x)$, it follows that

$$\begin{aligned} r(x) &= \|x - (x - Mx - q)_+\| \\ &= \|x - (x - Mx - q)_+ - \bar{x}(x) + (\bar{x}(x) - M\bar{x}(x) - q)_+\| \\ &\le \|x - \bar{x}(x)\| + \|(x - Mx - q)_+ - (\bar{x}(x) - M\bar{x}(x) - q)_+\| \\ &\le \|x - \bar{x}(x)\| + \|(x - Mx - q) - (\bar{x}(x) - M\bar{x}(x) - q)\| \\ &\le \|x - \bar{x}(x)\| + \|x - \bar{x}(x)\| + \|Mx - M\bar{x}(x)\| \\ &\le c\|x - \bar{x}(x)\| \end{aligned}$$

Q.E.D.

Theorem 3.1 implies that the order of the distance from any point x to the solution set $\bar{X}$ of any LCP(M, q) is at least as big as $r(x)$. Therefore, in order to be an error bound, a residual must bound $r(x)$. In addition, since $r(x)$ is a local bound [Rob81] [LuT92], locally $r(x)$ is equivalent to the distance to the solution set. So, this distance is precisely characterized by $r(x)$ at least locally. This is a very useful result because $r(x)$, which is a computable quantity, can be used as an alterative for the non-computable distance to the solution set. For other types of residuals, Theorem 3.1 unfortunately fails to hold.

We point out that all these error residuals that we have studied are not good enough to globally bound the distance from any point x to the solution set $\bar{X}$ of any LCP(M, q). Consider the following example. Let

$$M = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \qquad q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It is easy to see that the LCP(M, q) has the unique solution $\bar{x} = (0,0)$. Take the sequence $x^k = (k, 1-1/k), k = 1,2,...,$ then

$$\begin{aligned} & \lim_{k\to\infty} \frac{\|x-\bar{x}\|}{v(x)+v(x)^{1/2}} \\ = & \lim_{k\to\infty} \frac{\|(k, 1-\frac{1}{k})\|}{\|(0,0,1+(1-\frac{1}{k})(2-\frac{1}{k}))\| + \|(0,0,1+(1-\frac{1}{k})(2-\frac{1}{k}))\|^{1/2}} \\ = & \infty. \end{aligned}$$

This example shows that $v(x)+v(x)^{\frac{1}{2}}$ fails to be an error bound for LCP(M,q) with an indefinite matrix M, since $v(x)+v(x)^{\frac{1}{2}}$ is the largest error residual that is known to us without multiplying the size of the point under consideration. However, we have no counterexample to the possibility that $v(x)+v(x)^{\frac{1}{2}}$ is a global error bound for an LCP with an indefinite symmetric matrix M.

Although all these error residuals fail to provide a global error bound for an arbitrary LCP, there is a way to globalize *any* local error bound as can be seen from the following theorem.

Theorem 3.2 *Let $M \in \mathbb{R}^{n\times n}$ and $l(x)$ be any local error bound. Then, there exists a positive τ such that*

$$\|x-\bar{x}(x)\| \le \tau(1+\|x\|)l(x), \tag{19}$$

where $\bar{x}(x)$ is a closest solution from x to the solution set of LCP(M, q)

Proof. Assume that the theorem is false. then for each k, there exists an x^k such that (19) is violated, i.e.

$$\|x^k-\bar{x}(x^k)\| \ge k(1+\|x^k\|)l(x^k),$$

where $\bar{x}(x^k)$ is a closest solution from x^k to the solution set of LCP(M, q). Then for a fixed solution $\bar{x}$, we have

$$\|x^k-\bar{x}\| \ge \|x^k-\bar{x}(x^k)\| \ge k(1+\|x^k\|)l(x^k). \tag{20}$$

Since $l(x)$ is a local error bound by [Rob81] [LuT92], there exist $K > 0$ and $\epsilon > 0$ such that $l(x^k) > \epsilon$, for $k > K$ (See proof of Theorem 2.1). Hence the right hand side above goes to infinity as k goes to infinity and so does the left hand side since it is bigger. Therefore, $\|x^k\|$ goes to infinity. Without loss of generality, let

$$\lim_{k\to\infty} \frac{x^k}{\|x^k\|} = s$$

and $s \neq 0$ since $\|s\| = 1$. Divide both sides of (20) by $\|x^k\|$ and let k goes to infinity, then it follows that

$$\begin{aligned} 1 &= \lim_{k\to\infty} \frac{\|x^k-\bar{x}\|}{\|x^k\|} \\ &\ge \lim_{k\to\infty} k\frac{(1+\|x^k\|)l(x^k)}{\|x^k\|} \\ &= \lim_{k\to\infty} kl(x^k) \\ &= \lim_{k\to\infty} k\epsilon. \end{aligned}$$

We get contradiction since the left hand side is finite. Therefore, Theorem 3.2 holds. **Q.E.D.**

Remark 3.2 Note that in (19) if x is far away from the origin, then the error bound value increases, which could have nothing to do with the actual distance between x and the solution set of an LCP. This is a major drawback of this type of global error bounds. There are similar cases in [Ma92].

4 Conclusion

We conclude by summarizing our error bound relationships in Table 1, where the following definitions have been used:

$$\begin{aligned} r(x) &:= \|x-(x-Mx-q)_+\| \\ s(x) &:= \|(-Mx-q,-x,x(Mx+q))_+\| \\ t(x) &:= \|(-Mx-q,-x)_+,\sum_{i=1}^{n}(x_i(Mx+q)_i)_+\| \\ \text{psd} &:= \text{positive semi-definite} \\ \text{pd} &:= \text{positive definite} \\ R_0 &:= \text{Class of matrices } M \text{ such that zero is the only solution for LCP(M, 0)} \end{aligned}$$

We also note that there exist positive τ_1, τ_2 and c such that:

$$\begin{aligned} r(x) &\le c\|x-\bar{x}(x)\| \\ s(x)+s(x)^{\frac{1}{2}} &\le \tau_1(t(x)+t(x)^{\frac{1}{2}}) \\ r(x)+s(x) &\le \tau_2(t(x)+t(x)^{\frac{1}{2}}). \end{aligned}$$

Table 1. Validity of Various Residuals as Local and Global Error Bounds for LCP

$M \in \mathbb{R}^{n\times n}$	$r(x)$	$s(x)+s(x)^{\frac{1}{2}}$	$r(x)+s(x)$	$t(x)+t(x)^{\frac{1}{2}}$
arbitrary	local	not local	local	local
psd	local	global	global	global
R_0	global	global	global	global
pd	global	global	global	global

We note that the residual $r(x)+s(x)$, which can be thought of as the average of $r(x)$ and $s(x)$, covers most cases without recourse to an irrational square root residual. In this sense $r(x)+s(x)$ can be thought of as the best residual.

References

[CoPS92] R. W. Cottle, J.-S. Pang & R. E. Stone: "*The Linear Complementarity Problem*", Academic Press, New York 1992.

[LMRS92] Z.-Q. Luo, O. L. Mangasarian, J. Ren & M. V. Solodov: "New error bounds for the linear complementarity problem", Mathematics of Operations Research, to appear.

[LuT92] Z.-Q. Luo & P. Tseng: "Error bound and the convergence analysis of matrix splitting algorithms for the affine variational inequality problem", SIAM Journal in Optimization 2, 1992, 43-54.

[MaSh86] O. L. Mangasarian & T. -H. Shiau: "Error bounds for monotone linear complementarity problems", Mathematical Programming 36, 1986, 81-89.

[Ma90] O. L. Mangasarian: "Error bounds for nondegenerate monotone linear complementarity problems", Mathematical Programming 48, 1990, 437-445.

[Ma92] O. L. Mangasarian: "Global error bounds for monotone affine variational inequality problems", Linear Algebra and Its Applications 174, 153-163, 1992.

[MaPa90] R. Mathias and J.-S. Pang: "Error bounds for the linear complementarity problem with a P-matrix", Linear Algebra and Applications 132, 1990, 123-136.

[Mur88] K. G. Murty: "*Linear Complementarity, Linear and Nonlinear Programming*", Heldermann Verlag, Berlin 1988.

[Pang86] J.-S. Pang: "Inexact newton methods for the nonlinear complementarity problem", Mathematical Programming 36, 1986, 54-71.

[Rob81] S. M. Robinson: "Some continuity properties of polyhedral multifunctions", Mathematical Programming Study 14, 1981, 206-214.