Geometric Continuity of Spline Curves and Surfaces

by

Klaus Höllig

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ABSTRACT

We review β-spline theory for curves and show how some of the concepts can be extended to surfaces. Our approach is based on the Bézier form for piecewise polynomials which yields simple geometric characterizations of smoothness constraints and shape parameters. For curves most of the standard "spline calculus" has been developed. We discuss in particular the construction of B-splines, the conversion from B-spline to Bézier representation and interpolation algorithms. A comparable theory for spline surfaces for general meshes does at present not exist. We merely describe how to join triangular and rectangular patches and discuss the corresponding β-spline constraints in terms of the Bézier representation.

AMS (MOS) Subject Classifications: 41A15, 65D10

Key Words: curves, surfaces, β-splines, Bézier form, geometric continuity

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1. Introduction

In these lecture notes we describe the construction and properties of parametric representations of smooth spline curves and surfaces. This is not a straightforward generalization of the well understood theory of spline functions. The reason is, that smoothness of a [spline] curve or surface does not imply smoothness of a particular parametrization. In other words, a curve or surface can be smoother than its parametrization. This observation led to the discovery of **β-splines** which are the main subject of these lecture notes. These splines yield new approximation and design techniques by exploiting the admissible lack of smoothness in parametric representations. In their general form, β-splines have been introduced by Barsky [Ba82] and subsequently a considerable theory has been developed (cf. [BBB85, De85, DM85]). As an illustration of the general results, we describe in this first section the definition of β-splines and the difference between parametric and "geometric" smoothness in the simplest setting, for quadratic spline curves.

Denote the Bézier coefficients (cf. A1 of the Appendix) of a quadratic spline curve by $a_{i,\nu} \in \mathbb{R}^2$, i.e. the i-th curve segment is parametrized by

$$t \mapsto p_i(t) := \sum_{\nu=0}^{2} a_{i,\nu} B_\nu(t), \qquad t \in [0,1], \tag{1}$$

where $B_0(t) := (1-t)^2$, $B_1(t) := 2(1-t)t$ and $B_2(t) := t^2$. Moreover, assume that the parametrization of each segment is regular [Do76], i.e., that the tangent vector $p_i'(t)$ is nonzero for all $t \in [0,1]$.

Parametric C¹-continuity: The parametrization (1) is continuously differentiable if $p_{i-1}(1) = p_i(0)$ and $p'_{i-1}(1) = p'_i(0)$ for all i. In terms of the Bézier coefficients, these conditions become

$$\begin{aligned} a_{i-1,2} &= a_{i,0} \\ a_{i-1,2} - a_{i-1,1} &= a_{i,1} - a_{i,0}. \end{aligned} \tag{P^1}$$

Geometric C¹-continuity: The curve, parametrized by (1), is continuously differentiable if the direction of the tangent vector changes continuously. This means that the tangent vectors of adjacent curve segments are parallel, or, in terms of the Bézier coefficients, that

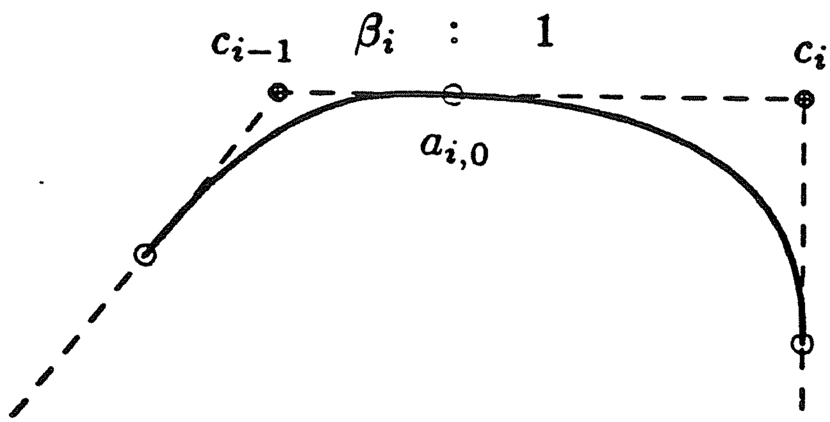
$$\begin{aligned} a_{i-1,2} &= a_{i,0} \\ a_{i-1,2} - a_{i-1,1} &= \beta_i\big(a_{i,1} - a_{i,0}\big) \quad \text{for some } \beta_i > 0. \end{aligned} \tag{G^1}$$

Conditions (G^1) are less restrictive than conditions (P^1) which correspond to the special case $\beta_i = 1$. The additional freedom in selecting the **shape parameters** β_i is useful for design and approximation purposes. For example, while standard quadratic spline interpolation requires the solution of a linear system, using β-splines the problem becomes trivial.

Algorithm 1. For given data d_i construct an interpolating quadratic β-spline curve as follows:
step 1: set $a_{i-1,2} = a_{i,0} := d_i$;
step 2: choose tangent vectors v_i [e.g. as average of the vectors $d_{i+1} - d_i$ and $d_i - d_{i-1}$];
step 3: define $a_{i,1}$ as the intersection of the lines $a_{i,0} + \mathbb{R}v_i$ and $a_{i,2} + \mathbb{R}v_{i+1}$.

A quadratic β-spline curve is uniquely determined by the **control points** $c_i := a_{i,1}$ and the shape parameters β_i (cf. Figure 1), i.e.

$$a_{i-1,2} = a_{i,0} = \frac{1}{1+\beta_i} c_{i-1} + \frac{\beta_i}{1+\beta_i} c_i. \tag{2}$$



⟨ Figure 1 ⟩

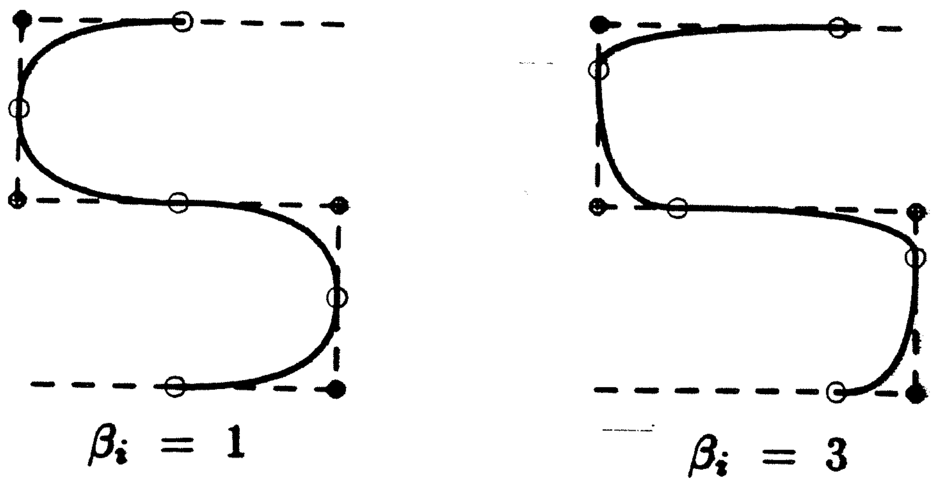
By (2), the curve can be equivalently parametrized by

$$t \mapsto s(t) := \sum_i c_i M_i(t).$$

The functions M_i are generalizations of quadratic **B-splines** and are explicitly defined by

$$M_i(t) = \begin{cases} \frac{\beta_i}{1+\beta_i} B_2(t-i), & \text{if } t-i \in [0,1]; \\ \frac{\beta_i}{1+\beta_i} B_0(t-i-1) + B_1(t-i-1) + \frac{1}{1+\beta_{i+1}} B_2(t-i-1), & \text{if } t-i \in [1,2]; \\ \frac{1}{1+\beta_{i+1}} B_0(t-i-2), & \text{if } t-i \in [2,3]; \\ 0, & \text{otherwise.} \end{cases}$$

Figure 2 shows the effect of varying β; for β_i approaching 0 or ∞, the curve approaches the **"control polygon"** which connects the control points c_i.



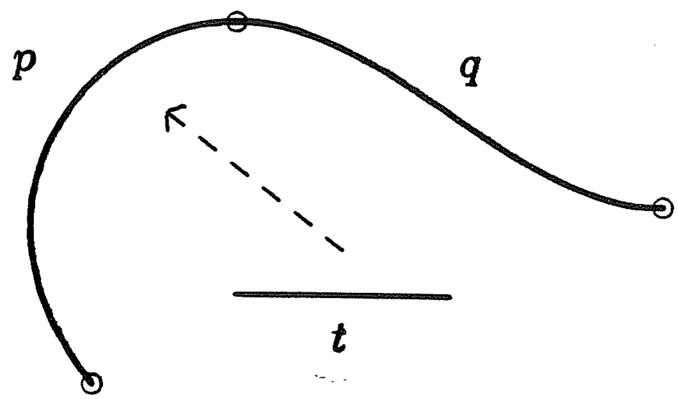
⟨ **Figure 2** ⟩

In the following sections, we describe the generalization of the above ideas to spline curves and surfaces of arbitrary degree and smoothness. To keep the lecture notes self-contained, we review in an appendix some properties of the Bézier form which will be frequently used.

2. Cubic β-spline curves

Cubic β-splines are most frequently used in applications and the discussion of this special case exhibits most of the essential features of the general theory. The "ν-spline", a particular β-spline was introduced by Nielson [N75]. This spline is the piecewise polynomial analogue to the "spline in tension" and arises naturally as solution of a minimization problem. Barsky [Ba81] developed the B-spline calculus for cubic β-splines and started a systematic study of their properties. In this section, we follow the approach of Farin [F82$_2$] and Böhm [Bö85] who described cubic β-splines in terms of their Bézier representation and gave natural geometric interpretations of the smoothness constraints and shape parameters.

We begin by deriving the conditions for geometric C^2-continuity of two cubic arcs in $\mathbb{R}^3$, parametrized by $t \mapsto p(t)$ and $t \mapsto q(t)$ with $t \in [0,1]$, which join at a point $d = p(1) = q(0)$ and have nonzero tangent vectors for all $t \in [0,1]$.



⟨ Figure 3 ⟩

By definition, a curve is twice continuously differentiable if there exists a twice continuously differentiable parametrization. Applying this to the union R of the two cubic arcs, we consider parametrizations of the form

$$t \mapsto \begin{cases} p(t+1), & \text{if } -1 \le t \le 0; \\ q(\varphi(t)), & \text{if } 0 \le t \le 1, \end{cases} \tag{3}$$

where φ is a smooth strictly increasing function mapping $[0, T]$ onto $[0, 1]$. It is easy to see that any other parametrization of the curve R is equivalent to (3) via a smooth change of variables $\tau \mapsto t(\tau)$. Therefore, the conditions for geometric C^2-continuity are (cf. [BD 85] for more details)

$$\begin{aligned} p'(1) &= q'(0)\beta \\ p''(1) &= q''(0)\beta^2 + q'(0)\gamma \end{aligned} \tag{G^2}$$

where $\beta := \varphi'(0)$ and $\gamma := \varphi''(0)$. The numbers $\beta > 0$ and γ can be arbitrarily chosen and, in analogy with the example in the section 1, can be interpreted as shape parameters.

Conditions (G^2) can be equivalently described in terms of the Bézier coefficients p_0, p_1, p_2, $p_3 = d = q_0$, q_1, q_2, q_3 for p and q (cf. A1 of the appendix). The first condition states that the points p_2, d, q_1 are collinear. Using this and the fact that γ is a free parameter, the second condition becomes

$$[\beta^2(q_1 - q_2)] + [p_1 - p_2] + [\delta(q_1 - p_2)] = 0$$

for some constant δ [which equals $1 - \gamma/(1+\beta)$]. In other words, the three vectors in square brackets form a triangle which implies in particular that the points p_1, p_2, d, q_1, q_2 are coplanar. This geometric interpretation yields an elegant description of the (G^2) condition.

Geometric $\mathbf{C^2}$-continuity [F82$_2$, Bö85]. Denote by $d_i = a_{i,0}$, $a_{i,1}$, $a_{i,2}$, $a_{i,3} = d_{i+1}$ the Bézier coefficients of the i-th segment of a piecewise cubic curve and assume that $a_{i,\nu+1} - a_{i,\nu} \neq 0$. Then, the curve is twice continuously differentiable at d_i if and only if the condition

$$(a \wedge b \wedge (c1 \vee c2)) \vee d$$

is satisfied ["$\wedge$" and "$\vee$" stand for "and" and "or" respectively] where

a: the points $a_{i-1,2}$, d_i, $a_{i,1}$ are collinear;

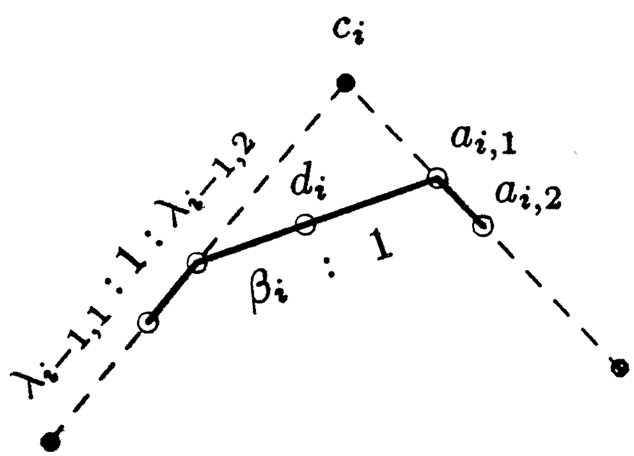
b: the polygon connecting $a_{i-1,1}$, $a_{i-1,2}$, $a_{i,1}$, $a_{i,2}$, $a_{i-1,1}$ bounds a [planar] quadrilateral;

c1: the lines through $a_{i-1,1}, a_{i-1,2}$ and $a_{i,1}, a_{i,2}$ intersect at a [control] point c_i and the ratios $\beta_i, \lambda_{j,\nu}$ defined in Figure 4 satisfy

$$\beta_i^2 = \lambda_{i,1}/\lambda_{i-1,2}; \tag{4}$$

c2: $a_{i-1,2} - a_{i-1,1} = \beta_i^2(a_{i,2} - a_{i,1})$;

d: the points $a_{i-1,1}$, $a_{i-1,2}$, d_i, $a_{i,1}$, $a_{i,2}$ are collinear.



⟨ Figure 4 ⟩

The "generic" (G^2) condition is $a \wedge b \wedge c1$. In this case, equation (4) yields a simple algorithm for computing the Bézier coefficients from the control points.

Algorithm 2 [Bö85].

step 1: choose the control points c_i;

step 2: choose the shape parameters $\lambda_{i,\nu}$;

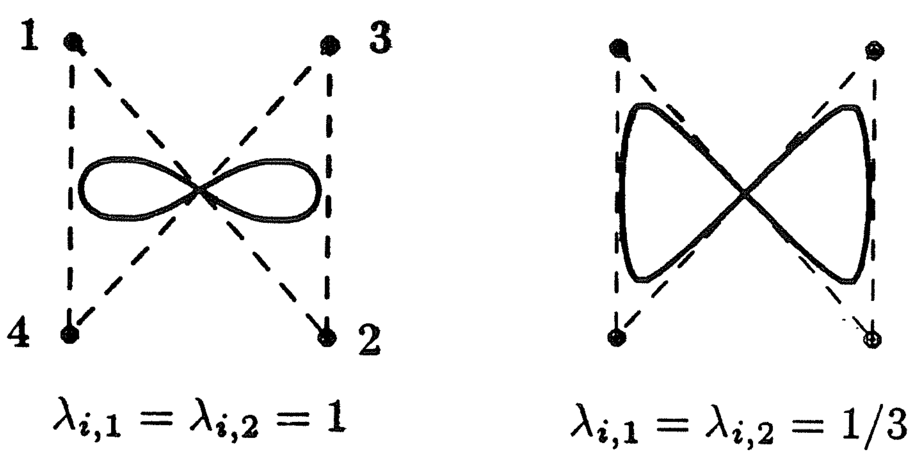
step 3: set $\lambda_i := 1 + \lambda_{i,1} + \lambda_{i,2}$ and compute

$$a_{i,1} := ((1 + \lambda_{i,2})c_i + \lambda_{i,1}c_{i+1})/\lambda_i$$
$$a_{i,2} := ((\lambda_{i,2}c_i + (1 + \lambda_{i,1})c_{i+1})/\lambda_i;$$

step 4: set $\beta_i^2 := \lambda_{i,1}/\lambda_{i-1,2}$ and compute

$$d_i = a_{i-1,3} = a_{i,0} := (a_{i-1,2} + \beta_i a_{i,1})/(1+\beta_i).$$

The algorithm provides a simple method for evaluating cubic β-splines since, once the Bézier coefficients are known, de Casteljau's algorithm (cf. [Da86]) can be applied. For $\lambda_{i,1} = \lambda_{i,2} = 1$, all i, the algorithm reduces to the algorithm for converting from B-spline to Bézier representation for standard cubic splines. The effect of the shape parameters is similar to the example considered in the introduction. For $\lambda_{i,\nu}$ approaching 0, the curve approaches the control polygon; the same effect is obtained if, e.g., $\lambda_{i,1} \mapsto \infty$ and $\lambda_{i,2} \mapsto 0$. Figure 5 shows an example.



⟨ Figure 5 ⟩

In view of Algorithm 2, the Bézier coefficients depend linearly on the control points,

$$a_{i,\nu} = \sum_{j=i-1}^{i+2} m_{i,\nu,j}\, c_j. \tag{5}$$

Therefore, the i-th curve segment can be parametrized by

$$t \mapsto \sum_j c_j M_{i,j}(t), \qquad t \in [0,1], \tag{6}$$

where

$$M_{i,j} := \sum_{\nu=0}^{3} m_{i,\nu,j} B_\nu^3$$

are the cubic polynomials which make up the B-spline which corresponds to the control point c_i. The matrix elements $m_{i,\nu,j}$, $\nu = 0,\ldots,3$, are the Bézier coefficients of $M_{i,j}$. Comparing (5) with Algorithm 2, one obtains the explicit formulas

$$\begin{aligned}
m_{i-1,0,i} &= m_{i-2,3,i} = m_{i-1,1,i}\beta_{i-1}/(1+\beta_{i-1}) \\
m_{i-1,1,i} &= \lambda_{i-1,1}/\lambda_{i-1} \\
m_{i-1,2,i} &= (1+\lambda_{i-1,1})/\lambda_{i-1} \\
m_{i-1,3,i} &= m_{i,0,i} = m_{i-1,2,i}/(1+\beta_i) + m_{i,1,i}\beta_i/(1+\beta_i) \\
m_{i,1,i} &= (1+\lambda_{i,2})/\lambda_i \\
m_{i,2,i} &= \lambda_{i,2}/\lambda_i \\
m_{i,3,i} &= m_{i+1,0,i} = m_{i,2,i}/(1+\beta_{i+1})
\end{aligned} \tag{7}$$

for the nonzero coefficients. The above expressions simplify for special choices of the shape parameters; e.g. for uniformly chosen shape parameters, $\lambda_{i,\nu} = \lambda_\nu$ and $\beta_i = \beta$ for all i. In particular, if $\lambda_1 = \lambda_2 = \beta = 1$, one obtains the Bézier coefficients of the standard cubic B-spline with equally spaced knots,

$$\begin{aligned}
m_{i-1,\cdot,i} &= [1\ 2\ 4\ 4]\ /\ 6 \\
m_{i,\cdot,i} &= [4\ 4\ 2\ 1]\ /\ 6.
\end{aligned}$$

Finally, we describe the "converse" of Algorithm 2, the construction of the control points [and thus all Bézier coefficients] from given data d_i and shape parameters. This amounts to solving the linear system obtained from (5) by considering only the equations for $d_i = a_{i-1,3} = a_{i,0}$. The resulting system can be solved separately for each of the three components $c_j(1), c_j(2), c_j(3)$ of the control points. The corresponding matrix L is tridiagonal with entries

$$\begin{aligned}
L_{i,i-1} &= m_{i,0,i-1} = \frac{\lambda_{i-1,2}}{\lambda_{i-1}(1+\beta_i)} \\
L_{i,i} &= m_{i,0,i} = \frac{1+\lambda_{i-1,1}}{\lambda_{i-1}(1+\beta_i)} + \frac{(1+\lambda_{i,2})\beta_i}{\lambda_i(1+\beta_i)} \\
L_{i,i+1} &= m_{i,0,i+1} = \frac{\lambda_{i,1}\beta_i}{\lambda_i(1+\beta_i)}
\end{aligned}$$

where the first and last row of L have to be modified according to the particular boundary conditions. Note that the entries in each row sum to one. This is consistent with the fact that the B-splines form a partition of unity,

$$\sum_j M_{i,j}(t) = 1, \qquad t \in [0,1].$$

Algorithm 3. For given data $d_i = (d_i(1), d_i(2), d_i(3))$, $i = 1,\ldots,n$, and shape parameters $\lambda_{i,\nu}$ compute the control points c_i of an interpolating β-spline curve by solving

$$L^* c(\mu) = d(\mu), \mu = 1, 2, 3.$$

(a) For a closed curve, i.e for periodic boundary conditions,

$$L^*_{i,j} = L_{i,j \bmod n}.$$

(b) If slopes at both endpoints are specified, i.e., if $a_{1,1}$ and $a_{n-1,2}$ are prescribed, then d_1 and d_n are replaced by $a_{1,1}$ and $a_{n-1,2}$ respectively and

$$\begin{aligned}
L^*_{1,1} &= m_{1,1,1} = \frac{1+\lambda_{1,2}}{\lambda_1} \\
L^*_{1,2} &= m_{1,1,2} = \frac{\lambda_{1,1}}{\lambda_1} \\
L^*_{n,n-1} &= m_{n-1,2,n-1} = \frac{\lambda_{n-1,2}}{\lambda_{n-1}} \\
L^*_{n,n} &= m_{n-1,2,n} = \frac{1+\lambda_{n-1,1}}{\lambda_{n-1}} \\
L^*_{i,j} &= L_{i,j} \quad \text{for} \quad (i,j) \neq (1,1),(1,2),(n,n-1),(n,n).
\end{aligned}$$

Note that, if $\lambda_{i,\nu} < \lambda_i/2$, the matrix L^* is diagonally dominant and thus invertible.

3. β-spline curves of arbitrary degree

This section may be skipped by the reader primarily interested in applications; but this reader is referred to [Bö86] where an interesting discussion of the quartic case is given. The section describes briefly the "general case" which is a special case of the results by Dyn and Micchelli [DM85].

The conditions for geometric continuity generalize easily. If $t \mapsto p(t)$ and $t \mapsto q(t)$ with $t \in [0,1]$ are regular polynomial parametrizations of degree k such that the corresponding arcs join at a point $d = p(1) = q(0)$, then the conditions for C^ℓ geometric continuity are [BD85]

$$\left(\frac{d}{dt}\right)^\nu p(t)\,_{|t=1} = \left(\frac{d}{dt}\right)^\nu q(\varphi(t))\,_{|t=0}, \qquad \nu = 1,\ldots,\ell, \tag{G^ℓ}$$

where φ is some strictly increasing function mapping $[0,T]$ onto $[0,1]$. Conditions (G^ℓ) involve the shape parameters $\beta_1 := \varphi'(0) > 0$ and $\beta_\nu := \varphi^{(\nu)}(0)$ [the superscript $^{(\nu)}$ denotes the ν-th derivative] which can be chosen arbitrarily.

Denote by $t \mapsto p_i(t) : [0,1] \mapsto \mathbb{R}^3$ a regular parametrization of the i-th segment of the piecewise polynomial curve of degree k and assume that $p_{i-1}(1) = p_i(0)$. Moreover, denote by $P_i^{(\ell)}$ the matrix of derivatives up to order ℓ of p_i, i.e. the ν-th row of $P^{(\ell)}$ contains the

3-vector $p_i^{(\nu)}$. Then conditions (G^ℓ) for adjacent curve segments can be written in matrix form,

$$P_{i-1}^{(\ell)}(1) = \Lambda_i^{(\ell)} P_i^{(\ell)}(0), \tag{8}$$

where the $\ell \times \ell$ matrix $\Lambda_i^{(\ell)}$ is computed via the chain rule and depends on the shape parameters $\beta_{i,\nu}$. For example,

$$\Lambda_i^{(3)} = \begin{bmatrix} \beta_{i,1} & 0 & 0 \\ \beta_{i,2} & \beta_{i,1}^2 & 0 \\ \beta_{i,3} & 3\beta_{i,2}\beta_{i,1} & \beta_{i,1}^3 \end{bmatrix}. \tag{9}$$

Generalizing earlier work by Goodman [Go84], Dyn and Micchelli [DM85] showed that many of the standard results on splines extend to the piecewise polynomial curves defined by (8) if one assumes that the matrices $\Lambda_i^{(\ell)}$ are totally positive [K68] (e.g. for (9) total positivity is equivalent to $\beta_{i,\nu} \geq 0$ and $3\beta_{i,2}^2 \geq \beta_{i,1}\beta_{i,3}$). One of the main results is the existence of a positive B-spline basis which forms a partition of unity.

Theorem [DM85]. Let $\ell = k-1$ and assume that the matrices Λ_i^ℓ are totally positive. If (8) holds, then the polynomials p_i can be uniquely written as

$$p_i(t) = \sum_{j=i}^{i+k} c_j M_{i,j}(t) \tag{10}$$

where c_j are the control points of the curve and $M_{i,j}$ are polynomials which depend on the shape parameters β. The polynomials $M_{i,j}$ are positive and form a partition of unity

$$\sum_{j=i}^{i+k} M_{i,j}(t) = 1, \tag{11}$$

i.e. the points on the curve are convex combinations of $k+1$ consecutive control points.

The Bézier coefficients $m_{i,\nu,j}$, $\nu = 0, \ldots, k$, of $M_{i,j}$ and their dependence on the shape parameters can be determined via symbolic manipulation. Choosing only one vector c_j in (10) nonzero, one sees that the polynomials $M_{i,j}$ satisfy conditions (8), i.e.

$$M_{i-1,j}(1) = M_{i,j}(0), \qquad i = j-k, \ldots, j+1,$$

$$M_{i-1,j}^{(\nu)}(1) = \sum_{\mu=1}^{\ell} \left(\Lambda_i^{(\ell)}\right)_{\nu,\mu} M_{i,j}^{(\mu)}(0), \qquad \nu = 1, \ldots, \ell, \quad i = j-k, \ldots, j+1,$$

if one defines $M_{i,j}(t) = 0$ for $i < j-k$ or $i > j$. For fixed j, these equations can be rewritten as an $(k^2+2k) \times (k+1)^2$ homogenous linear system for the Bézier coefficients $m_{i,\nu,j}$, $i = j-k, \ldots j$, $\nu = 0, \ldots, k$. The matrix of this system has full rank but depends on β. Hence there exists a nontrivial solution which can be normalized using equation (11).

With the coefficients $m_{i,\nu,j}$ determined, one has an analogue of Algorithm 2: The Bézier coefficients $a_{i,\nu}$, $\nu = 0,\cdots,k$, of the i-th segment of the β-spline curve can be computed from the control points c_i via

$$a_{i,\nu} = \sum_{j=i}^{i+k} c_j m_{i,\nu,j}. \tag{12}$$

This "algorithm" allows one to evaluate a general β-spline curve almost as easily as in the cubic case: for given control points and shape parameters one computes the Bézier coefficients via (12); then evaluates the Bézier form using de Casteljau's algorithm.

4. Tensor Products

As is the case for [almost] all univariate approximation procedures, the methods described in the preceeding sections can be extended to surfaces via tensor products. For the sake of [notational] simplicity we discuss this for the cubic case.

Relying on the results in section 2, we begin with equation (6), the B-spline representation for cubic β-spline curves. For tensor product cubic β-spline surfaces, we have the corresponding representation,

$$(x_1, x_2) \mapsto p_i(x) := \sum_{J(i)} c_j \; M_{i_1,j_1}(x_1) \; \widetilde{M}_{i_2,j_2}(x_2), \qquad x \in [0,1]^2, \tag{13}$$

for the bi-cubic polynomial p_i which parametrizes the i-th surface patch; $J(i)$ denotes the set of indices $\{(j_1,j_2) : i_1 - 1 \le j_1 \le i_1 + 2,\ i_2 - 1 \le j_2 \le i_2 + 2\}$. The "$\sim$" indicates that $\widetilde{M}$ may be defined by a different set of shape parameters than M, i.e. to each coordinate direction corresponds a different set of shape parameters. The Bézier coefficients $a_{i,\underbrace{\nu_1,\nu_2}}$, $0 \le \nu_1, \nu_2 \le 3$, are computed by substituting the Bézier representation for M and $\widetilde{M}$ into (13). This yields the analogue of (5),

$$a_{i,\nu} = \sum_{J(i)} c_j \; m_{i,\nu_1,j_1} \; \widetilde{m}_{i,\nu_2,j_2}. \tag{14}$$

Proceeding backwards [with respect to section 2], we obtain

Algorithm 4.

step 1: choose the control points c_i and the shape parameters $\lambda_{i_1,\nu}$ and $\widetilde{\lambda}_{i_2,\nu}$;
step 2: compute the Bézier coefficients m and $\widetilde{m}$ from (7);
step 3: compute $a_{i,\nu}$ from (14) in two steps

$$
\begin{aligned}
b_{j_2} &:= \sum_{i_1-1\leq j_1\leq i_1+2} c_{j_1,j_2} m_{i,\nu_1,j_1} \\
a_{i,\nu} &:= \sum_{i_2-1\leq j_2\leq i_2+2} b_{j_2} \widetilde{m}_{i,\nu_2,j_2}.
\end{aligned}
$$

We have described [very] briefly how to construct C^2-continuous cubic β-spline surfaces with respect to rectangular grids or unions of such grids. For a detailed discussion of the properties of such representations we refer to [BBB85].

5. Geometric continuity of piecewise polynomial surfaces

Tensor product methods are limited to surfaces which can be covered by rectangular patches with four patches meeting at each vertex. For example, surfaces homeomorphic to a sphere cannot be modelled by tensor products. In this section, we describe general piecewise polynomial surfaces with entirely local structure which do not require any topological restrictions.

Definition. A piecewise polynomial surface is the union of polynomial patches,

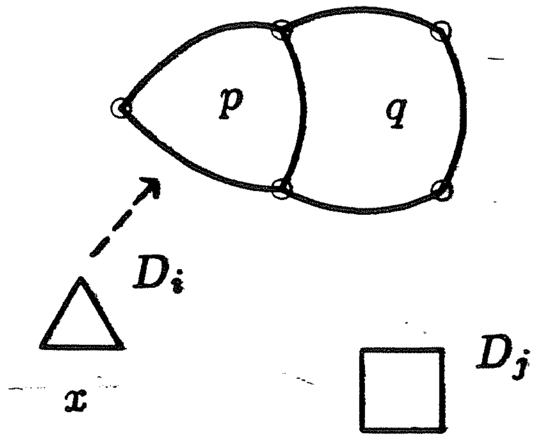
$$
S := \bigcup_i \{p_i(x) : x \in D_i\}, \qquad i = 1, \dots, n,
$$

where D_i is either the unit square or an equilateral triangle with side-length 1 and, correspondingly, the components of $p_i : D_i \mapsto \mathbb{R}^3$ are polynomials of either coordinate or total degree k_i. The Bézier coefficients of p_i are denoted by $a_{i,\nu}$ (cf. the Appendix). Two patches are either disjoint or share a common boundary arc [or vertex]. With ∂_ν denoting the partial derivative with respect to the ν-th variable and "$\times$" denoting the cross product, we assume that

$$
\partial_1 p_i(x) \times \partial_2 p_i(x) \neq 0, \qquad x \in D_i,
$$

and that the natural orientation of D_i induces a consistent orientation on S.

Note that the normalization of the reference domains D_i represents no loss of generality since it can always be achieved by a linear change of variables.



⟨ Figure 6 ⟩

Note that as for curves, the parametric representation of S is not unique. For any family of 1–1, orientation preserving mappings φ_i, the functions $x \mapsto p_i(\varphi_i(x))$ yield an equivalent parametrization of S. Following Barsky and DeRose [BD85, De85], we derive the conditions for geometric continuity of adjacent patches. By definition a surface is of class C^ℓ if for each neighborhood on the surface there exists a parametrization of class C^ℓ. To apply this definition to piecewise polynomial surfaces, consider two adjacent patches which join at a common boundary arc and are parametrized by $p : D \mapsto \mathbb{R}^3$ and $q : E \mapsto \mathbb{R}^3$. Without loss of generality we may assume that $I := D \cap E = \{(0,t) : 0 \le t \le 1\}$ and that

$$p(0,t) = q(0,t), \qquad t \in [0,1]. \tag{15}$$

Let

$$(u_1, u_2) \mapsto \varphi(u) := (\varphi^1(u_1,u_2), \varphi^2(u_1,u_2))$$

be a continuous 1–1 mapping of $D \cup E'$ onto $D \cup E$ with $\varphi(u) = u$ for $u \in D$. Then

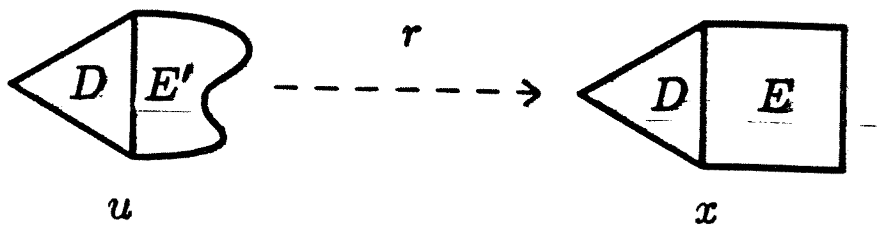
$$u \mapsto r(u) := \begin{cases} p(u), & \text{if } u \in D \\ q(\varphi(u)), & \text{if } u \in E' \end{cases}$$

is an equivalent parametrization of the two patches under consideration. Therefore, the union of the two patches is of class C^ℓ if there exists a mapping φ with the above properties such that φ is of class C^ℓ on E' and r is ℓ times differentiable on I. The latter condition means that

$$\partial_1^\nu\big(p(u) - q(\varphi(u))\big) = 0, \qquad \text{for } u = (0,t),\ t \in [0,1], \text{ and } \nu = 1, \ldots \ell. \tag{GS^ℓ}$$

Note that (15) and the assumption $\varphi(0,t) = (0,t)$ imply that all partial derivatives of order $\le \ell$ which involve differentiation with respect to u_2 are continous for $u = (0,t)$ if (GS^ℓ) holds.

Remark. Smoothness could also be defined in terms of parametrization invariant geometric characteristics such as tangent planes, curvature, etc. However, for C^1- and C^2-continuity this would be equivalent to our approach (cf. [H86]) and for higher order smoothness the "geometric" approach becomes rather complicated.



⟨ Figure 7 ⟩

Writing out the conditions (GS^ℓ) in detail and using the abbreviations $f_{\nu,\mu} := \partial_1^\nu \partial_2^\mu f$, we obtain

$$p_{1,0} = q_{1,0}\varphi_{1,0}^1 + q_{0,1}\varphi_{1,0}^2 \qquad (GS^1)$$

$$\begin{aligned} p_{2,0} =& q_{2,0}\left(\varphi_{1,0}^1\right)^2 + 2q_{1,1}\varphi_{1,0}^1\varphi_{1,0}^2 + q_{0,2}\left(\varphi_{1,0}^2\right)^2 + \\ & q_{1,0}\varphi_{2,0}^1 + q_{0,1}\varphi_{2,0}^2 \end{aligned} \qquad (GS^2)$$

$$\bullet\bullet\bullet$$

where all of the above terms are evaluated at $u = (0,t)$. Note that, in the above conditions, the shape parameters $\varphi_{\mu,0}^\nu(0,t)$ are univariate functions. However, the fact that the components of p and q are polynomials of a given degree imposes restrictions on the possible choices of φ. It is reasonable to make the following

Consistency Assumption. For a given choice of the **shape functions** $\varphi_{\mu,0}^\nu$, either one of the parametrizations p [or q] can be freely chosen, i.e. for any p there exists q [or vice versa] so that the (GS) conditions hold.

Under this assumption, it will be shown that conditions (GS^ℓ) can be rewritten as a system of homogeneous linear constraints on the Bézier coefficients,

$$\Lambda_{i,j}^{(\ell)}(a_i, a_j) = 0, \qquad (16)$$

for adjacent patches p_i and p_j with Bézier coefficients $a_{i,\nu}$ and $a_{j,\nu}$ respectively. The matrices $\Lambda_{i,j}^{(\ell)}$ depend, as for β-spline curves, on a set of **scalar** shape parameters [rather than functions]. Thus, β-spline surfaces can be described as a set of polynomial patches p_i defined in terms of their Bézier coefficients a_i together with constraint relations (16) for each interior edge. Unfortunately, at present, B-splines are not available to represent such surfaces. It seems very complicated to determine control points, i.e. to characterize the degrees of freedom for the general β-spline surfaces. However, Bézier coefficients which satisfy the constraint relations (16) as well as certain design objectives can be determined by local iterative methods.

Algorithm 5. To construct a smooth β-spline surface S which is close to a given piecewise polynomial surface S_* [which is not necessarily smooth; but approximates the desired shape of S], solve the quadratic program

$$\text{minimize } \|a - a_*\|^2 \quad \text{subject to } \Lambda^{(\ell)} a = 0 \tag{17}$$

where a and a_* represent the Bézier coefficients of all patches of S and S_* respectively and $\Lambda^{(\ell)}$ represents all constraint relations. A standard technique for solving (17) is to apply SOR to solve the dual problem which very effectively makes use of the sparsity of the matrix $\Lambda^{(\ell)}$ (cf. [Gr86] where the implementation of a special version of this algorithm for C^1 cubic interpolation is described).

6. Triangular patches

We apply the general discussion of the previous section to triangular patches, i.e. we assume that the p_i in Definition 1 are polynomials of total degree $\leq k$. Consider first condition (GS^1) for continuous differentiability of the surface. Since p and q are polynomials of **total** degree $\leq k$, the functions $f := p_{1,0}(0,\cdot)$, $g := q_{1,0}(0,\cdot)$ and $h := q_{0,1}(0,\cdot)$ are univariate polynomials of degree $\leq k-1$. With $\alpha := \varphi_{1,0}^1$ and $\beta := \varphi_{1,0}^2$, the (GS^1) condition derived in the previous section is

$$f = g\alpha + h\beta. \tag{18}$$

The consistency assumption implies that the right hand side of (18) is a polynomial of degree $\leq k-1$ for any choice of the polynomials g and h. By choosing $g(t) := (0,0,t^2)$, $h(t) := (0,0,0)$, it follows that α must be constant; similarly one concludes that β must be constant. Therefore, condition (GS^1) admits only 2 **scalar** shape parameters. Since $\varphi(u) = (\alpha u_1, \beta u_1 + u_2)$ is invertible and orientation preserving, α must be positive. With this observation we reformulate the (GS^1) condition in terms of the Bézier coefficients (cf. A3 of the appendix). To this end we label the Bézier coefficients adjacent to the common arc by p_ν, q_ν, $\nu = 0,\ldots,k-1$, and d_ν, $\nu = 0,\ldots,k$ as indicated in Figure 8. Using the differentiation formula for the Bézier representation, the Bézier coefficients of the terms in (18) are, up to a normalizing factor,

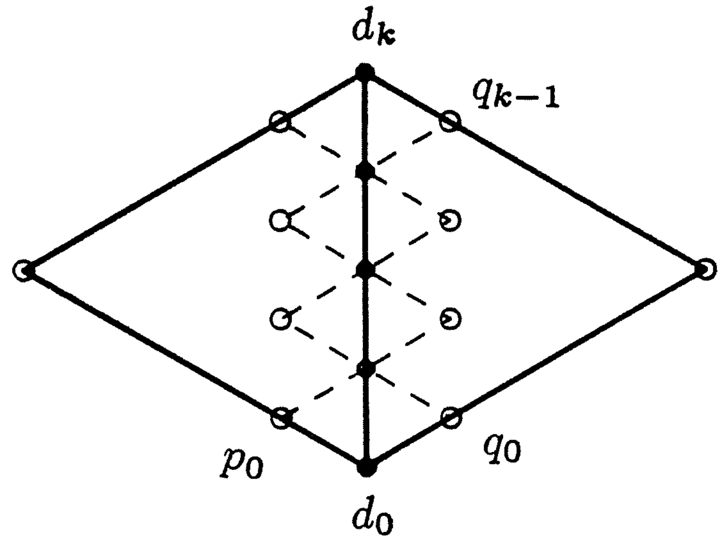
$$\begin{array}{lll} f : & d_{\nu+1} + d_\nu - 2p_\nu, & \nu = 0, \ldots, k-1 \\ g : & 2q_\nu - d_{\nu+1} - d_\nu, & \nu = 0, \ldots, k-1 \\ h : & d_{\nu+1} - d_\nu, & \nu = 0, \ldots, k-1. \end{array} \tag{19}$$

Comparing Bézier coefficients in (18), we obtain the following equivalent formulation of the (GS^1) condition.

Geometric $\mathbf{C^1}$-continuity. The union of the patches is continuously differentiable at the common boundary [corresponding to the common Bézier coefficients d_ν, $\nu = 0, \ldots, k$] if

$$p_\nu = \frac{1+\alpha-\beta}{2}\, d_{\nu+1} + \frac{1+\alpha+\beta}{2}\, d_\nu - \alpha\, q_\nu, \qquad \nu = 0, \ldots, k-1, \tag{20}$$

for some shape parameters $\alpha > 0$ and β. Geometrically, condition (20) means that the points p_ν, $d_{\nu+1}$, d_ν, q_ν are coplanar and that the points p_ν have the same barycentric coordinates with respect to the triangles spanned by $d_{\nu+1}$, d_ν, q_ν for all ν. The condition for strict parametric continuity is the special case corresponding to $\alpha = 1$ and $\beta = 0$.



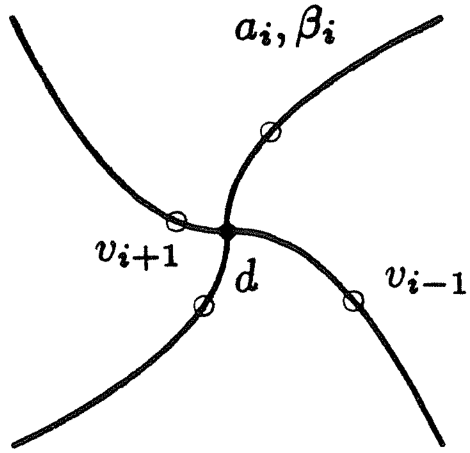
⟨ Figure 8 ⟩

It may seem disappointing that there exist only 2 scalar shape parameters [rather than shape functions]. However, consider again the general (GS^1) condition which may be rephrased as

$$\det \| \, p_{1,0}(0,t) \quad q_{1,0}(0,t) \quad q_{0,1}(0,t) \, \| = 0, \qquad t \in [0,1].$$

This determinant is a polynomial of degree $\leq 3(k-1)$ and therefore the condition is equivalent to $3k-2$ [nonlinear] constraints [all coefficients of the polynomial have to vanish]. This is consistent with (20) which represents $3k$ constraints with 2 free parameters. The above discussion seems to contradict Farin's construction [F82$_1$] of constraint relations involving a linear shape function [2 parameters] and 1 additional scalar free parameter. However, he assumes that the boundary arc is of lower degree, thus imposing 3 additional constraints. Therefore, his conditions are more restrictive, but arise naturally when blending triangular and rectangular patches (cf. the next section).

By (20), to each interior boundary arc there correspond 2 shape parameters. However, at each vertex certain compatibility conditions must hold to ensure consistency of the constraints at a vertex. To derive these conditions, assume that n_d patches meet at an interior vertex d (cf. Figure 9) and denote the shape parameters corresponding to the edges by α_i, β_i. Further, denote the Bézier coefficients adjacent to the vertex by $v_1, \ldots, v_n$ and set $w_\nu := v_\nu - d$.



⟨ Figure 9 ⟩

With this notation, condition (20) implies that

$$v_{i-1} = \frac{1+\alpha_i+\beta_i}{2}\, d \;+\; \frac{1+\alpha_i-\beta_i}{2}\, v_i - \;\alpha_i v_{i+1}$$

where the index i has to be interpreted modulo n. Writing this condition in terms of the vectors w_i yields

$$w_{i-1} + \beta_i' w_i + \alpha_i w_{i+1} = 0, \qquad i = 1, \ldots, n \tag{V^n}$$

with $\beta_i' := (-1-\alpha_i+\beta_i)/2$. The consistency assumption implies the following condition.

Vertex consistency. The shape parameters which correspond to boundary arcs meeting at a vertex have to be chosen so that the system (V^n) has a solution for any choice of two consecutive vectors w_i, w_{i+1}. In accordance with the consistency assumption, this guarantees that the tangent plane [which is spanned by the vectors w_i] can be freely chosen at the vertex.

Since the above condition has to hold for each vertex of a C^1-surface, it may lead to **global** constraints, further restricting the choice of the shape parameters. Below we describe the admissible choices of the shape parameters for two examples.

$\mathbf{n_d = 3}$: For 3 edges joining at a vertex, the system (V^3) becomes

$$\begin{bmatrix} \beta_1' & \alpha_1 & 1 \\ 1 & \beta_2' & \alpha_2 \\ \alpha_3 & 1 & \beta_3' \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Eliminating w_3 in the first equation yields

$$\beta_1' w_1 + \alpha_1 w_2 - \frac{1}{\alpha_2} w_1 - \frac{\beta_2'}{\alpha_2} w_2 = 0.$$

Vertex consistency requires that this equation hold for any w_1 and w_2 which implies that $\beta_1' = 1/\alpha_2$ and $\alpha_1 = \beta_2'/\alpha_2$. Because of the cyclic structure of the matrix, this yields the conditions

$$\begin{aligned} &\alpha_i = 1/\beta_{i-1}', \qquad i = 1,2,3 \\ &\alpha_1\alpha_2\alpha_3 = 1 \end{aligned} \tag{21}$$

which are also sufficient. Therefore, for any choice of $\alpha_i > 0$ with $\alpha_1\alpha_2\alpha_3 = 1$ there exists a consistent choice of the parameters β_i.

$\mathbf{n_d = 4}$: A similar analysis for the case of 4 edges yields the conditions

$$\begin{aligned} &1/\alpha_i - \alpha_{i+2} = \beta_{i-1}'\beta_{i+2}' \\ &\alpha_i\alpha_{i-1}\beta_{i+2}' = \beta_i'. \end{aligned} \tag{22}$$

For parametric C^1-continuity, $\alpha_i = 1$ and $\beta_i' = \beta_i/2 - 1 = -1$, and neither of the conditions (21) or (22) admit a solution in this case. The reason for this paradox is that we have normalized the reference domains in the definition of a piecewise polynomial surface. The above conditions describe the conditions for the required linear reparametrizations to join the patches smoothly.

For triangular patches, the conditions for higher order smoothness do not allow additional shape parameters if the consistency assumption is adopted. For example, if polynomial patches of total degree k are considered, the GS^2 condition is of the form

$$p_{2,0}(0,t) = f(t) + [q_{1,0}\varphi_{2,0}^1 + q_{0,1}\varphi_{2,0}^2](0,t)$$

where the components of $p_{2,0}(0,\cdot)$ and f are polynomials of degree $\le k-2$. Since the components of $q_{1,0}(0,\cdot)$ and $q_{0,1}(0,\cdot)$ are of degree $\le k-1$ and, according to the consistency assumption can be freely chosen, it follows that the shape functions $\varphi^{\nu}_{2,0}$ must vanish. Thus the (GS^2) condition reduces to the condition for C^2-continuity of the parametrization, the only degree of freedom being a linear reparametrization of one of the adjacent patches. The same conclusion holds for higher order smoothness.

7. Rectangular patches

In this section we consider rectangular patches, not necessarily of tensor product type; i.e. the number of patches meeting at a vertex need not equal 4. As in the previous section we apply the results of section 5, beginning with C^1-continuity. We assume that p and q are polynomials of coordinate degree $\le k$ (cf. A2 of the appendix) and rewrite condition (GS^1) as (18) using the same abbreviations as before. There is a slight difference; differentiation with respect to u_1 [i.e. differentiation perpendicular to the common arc] does not reduce the degree of the polynomials restricted to the segment I. Therefore, the polynomials f, g and h in (18) are of degree less than or equal to k, k and $k-1$ respectively. This allows additional freedom in selecting β which, in accordance with the consistency condition, can be a linear function. In other words, 3 scalar shape parameters can be chosen, namely $\alpha > 0$, β_1 and β_2 which denote the values of the linear function β at $t=0$ and $t=1$ respectively.

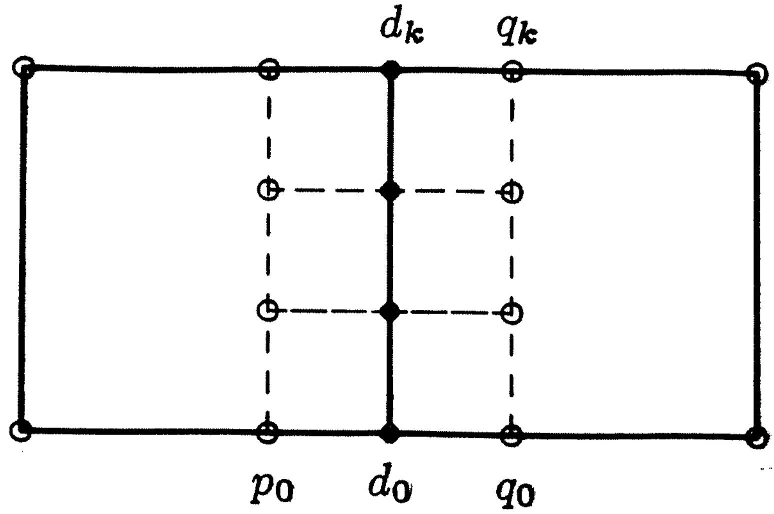
Following Farin [F82$_1$] we reformulate the resulting condition in terms of the Bézier coefficients. To this end we label the Bézier coefficients adjacent to the common edge as in the triangular case by p_ν, d_ν and q_ν (cf. Figure 10) and calculate the Bézier coefficients of the terms appearing in (18). Using the differentiation formula in A2 we obtain

$$\begin{aligned}
f &: \; k(d_\nu - p_\nu), \qquad \nu = 0,\ldots,k\\
g &: \; k(q_\nu - d_\nu), \qquad \nu = 0,\ldots,k\\
h &: \; k(d_{\nu+1} - d_\nu), \qquad \nu = 0,\ldots,k-1.
\end{aligned}$$

From the definition of the Bézier form one verifies that the Bézier representation of the product $h(t)(\beta_0(1-t)+\beta_1 t)$ is given by

$$h\beta \; : \; k\big(\beta_0 \frac{k-\nu}{k}(d_{\nu+1}-d_\nu) + \beta_1 \frac{\nu}{k}(d_\nu - d_{\nu-1})\big), \qquad \nu = 0,\ldots,k.$$

Combining the above expression and simplifying we obtain the Bézier form of the (GS^1) condition.



⟨ Figure 10 ⟩

Geometric $\mathbf{C^1}$ continuity [F82$_1$]. The union of the two patches is continuously differentiable if the Bézier coefficients satisfy

$$\begin{aligned} p_\nu = &\frac{k-\nu}{k}\big((1+\alpha+\beta_0)d_\nu - \beta_0 d_{\nu+1} - \alpha q_\nu\big) \\ &+\frac{\nu}{k}\big((1+\alpha-\beta_1)d_\nu + \beta_1 d_{\nu-1} - \alpha q_\nu\big), \qquad \nu = 0,\ldots,k, \end{aligned} \tag{23}$$

for some shape parameters $\alpha > 0$ and β_1, β_2. This condition can be interpreted as an "average" of two conditions of the form (20) with different β's.

At the vertex d_0, condition (23) becomes

$$p_0 = (1+\alpha+\beta_0)d_0 - \beta_0 d_1 + \alpha q_0.$$

As before, this leads to a compatibility condition at a vertex. Using the notation of Figure 9 [but with d, v_ν now denoting the Bézier coefficients of tensor product polynomial patches; cf. A2], one obtains the conditions

$$w_{i-1} + \beta_i w_i + \alpha_i w_{i+1} = 0, \qquad i = 0,\ldots,n,$$

which are identical with the conditions (V^n) except that β_i' is replaced by β_i. Therefore, the conclusions of the preceeding section apply. Note that in the second case ($n_d = 4$), parametric continuity corresponds to a consistent set of parameters since $\alpha_1 = 1$ and $\beta_i' = \beta_i = 0$.

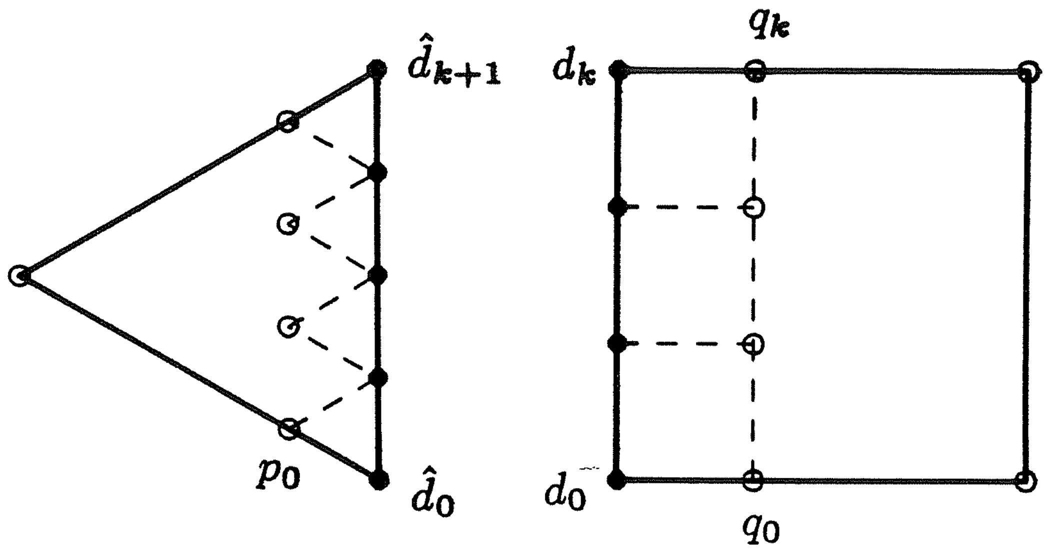
Farin [F82$_1$] also derived the conditions for matching a triangular and rectangular patch. Let p denote the parametrization of the triangular and q the parametrization of the rectangular patch. If q is of degree $\leq k$, it follows from (18) and the consistency assumption that p must be of degree $k+1$. The constraints for the Bézier coefficients are derived as before and we merely state the result and refer to [F82₁] for details.

Geometric C^1-continuity. With the notation as described in Figure 11, the union of the two patches is continuously differentiable if

$$\hat{d}_\nu = \frac{\nu}{k+1} d_{\nu-1} + \frac{k+1-\nu}{k+1} d_\nu, \qquad \nu = 0, \ldots, k+1$$

$$p_\nu = \frac{k-\nu}{k}\big((1+\alpha+\beta_0)d_\nu - \beta_0 d_{\nu+1} - \alpha q_\nu\big) + \frac{\nu}{k}\big((1+\alpha-\beta_1)d_\nu + \beta_1 d_{\nu-1} - \alpha q_\nu\big), \qquad \nu = 0, \ldots, k,$$

where $\alpha > 0$ and β_1, β_2 can be chosen arbitrarily.



⟨ Figure 11 ⟩

Finally, we discuss briefly the conditions for C^2-continuity. We write condition (GS^2) in the form

$$p_{2,0} = F + \big[g\gamma + h\delta\big]$$

where F denotes the terms involving second derivatives of q and $g := q_{1,0}$, $h := q_{0,1}$ are defined as before. For polynomials of coordinate degree, $p_{0,2}$, g are of degree $\leq k$, h is of degree $\leq k-1$ and, with the shape parameters $\varphi^1_{1,0}$, $\varphi^2_{1,0}$ chosen according to the (GS^1) condition, F is of degree $\leq k$. Therefore, by the consistency condition, γ must be constant while δ can be chosen as a linear function. This yields 3 free parameters in addition to the parameters α, β_ν. As before, since δ is a linear function, the (GS^2) condition can be expressed in terms of the Bézier coefficients. However, offhand, the resulting conditions are complicated and new ideas are needed to understand the nature of these constraints.

For a mathematician, used to a theory "for arbitrary k", the results of the last two sections are not quite satisfactory. The author hopes that the special cases described will get the reader interested in some of the open [β-spline] problems.

Appendix: Bézier representation of polynomials

We review the definition of the Bézier form and some properties which were used in these notes (for details cf. [Bo86, F86, Da86]).

A1. Polynomials of one variable: A polynomial p of degree $\leq k$ can be written as

$$p(t) = \sum_{\nu=0}^{k} a_\nu B_\nu^k(t), \qquad t \in [0,1],$$

where

$$B_\nu^k(t) := \binom{k}{\nu}(1-t)^{k-\nu}t^\nu$$

and a_ν are called the Bézier coefficients of p. The corresponding Bézier forms of the derivatives of p are

$$p' = k\sum_{\nu=0}^{k-1}(a_{\nu+1} - a_\nu)B_\nu^{k-1},$$

$$p'' = k(k-1)\sum_{\nu=0}^{k-2}(a_{\nu+2} - 2a_{\nu+1} + a_\nu)B_\nu^{k-2},$$

etc., i.e. differentiation corresponds to differencing of the Bézier coefficients. By the definition of the polynomials B_ν^k, $p(0) = a_0$, $p'(0) = k(a_1 - a_0)$, $p''(0) = k(k-1)(a_2 - 2a_1 + a_0)$, ..., and the analogous statement holds for evaluation of the derivatives at 1.

A2. Bivariate polynomials (coordinate degree): A bivariate polynomial p of degree $\leq k_1, k_2$ in the variables x_1, x_2 respectively can be written as

$$p(x) = \sum_{\nu_1=0}^{k_1}\sum_{\nu_2=0}^{k_2} a_\nu \; B_{\nu_1}^{k_1}(x_1)\; B_{\nu_2}^{k_2}(x_2)$$

where a_ν are the Bézier coefficients. The Bézier representation of the partial derivative of p with respect to x_1 is

$$\partial_1 p(x) = k_1 \sum_{\nu_1=0}^{k_1-1}\sum_{\nu_2=0}^{k_2}\left(a_{\nu_1+1,\nu_2} - a_{\nu_1,\nu_2}\right)\; B_{\nu_1}^{k_1}(x_1)B_{\nu_2}^{k_2}(x_2)$$

and an analogous formula is valid for $\partial_2 p$.

A3. Bivariate polynomials (total degree): Let ξ_ν denote the barycentric coordinates with respect to a triangle D with vertices v_ν, i.e.

$$\begin{aligned} x &= \xi_1(x)v_1 + \xi_2(x)v_2 + \xi_3(x)v_3 \\ 1 &= \xi_1(x) + \xi_2(x) + \xi_3(x). \end{aligned}$$

A polynomial p of total degree $\leq k$ can be written as

$$p(x) = \sum_{\substack{\nu_1,\nu_2,\nu_3 \geq 0 \\ \nu_1+\nu_2+\nu_3=k}} a_\nu \, B_\nu$$

where

$$B_\nu(x) := \frac{(\nu_1+\nu_2+\nu_3)!}{\nu_1!\,\nu_2!\,\nu_3!} \, \xi_1(x)^{\nu_1} \xi_2(x)^{\nu_2} \xi_3(x)^{\nu_3}$$

and a_ν are the Bézier coefficients of p with respect to the triangle D. The derivative of p in the direction $\eta := v_2 - v_1$ is given by

$$D_\eta p = \sum_{\nu_1+\nu_2+\nu_3=k-1} \left(a_{\nu_1,\nu_2+1,\nu_3} - a_{\nu_1+1,\nu_2,\nu_3}\right) B_\nu$$

and analogous formulas hold for derivatives in the directions $v_3 - v_2$ and $v_1 - v_3$.

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