B-splines Without Divided Differences

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Abstract

This note develops the basic B-spline theory without using divided differences. Instead, the starting point is the definition of B-splines via recurrence relations. This approach yields very simple derivations of basic properties of spline functions and algorithms.

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1. Basic Properties

Let $\mathbf{t} := \cdots,\ t_{-1},\ t_0,\ t_1, \cdots$ be a nondecreasing, biinfinite sequence of real numbers with $\lim_{i\to\pm\infty} t_i = \pm\infty$. The **B-splines** corresponding to the "knot sequence" $\mathbf{t}$ are defined by the recurrence relation

$$B_{i,k} := \omega_{i,k} B_{i,k-1} \ + \ (1-\omega_{i+1,k}) B_{i+1,k-1} \tag{1a}$$

with

$$\begin{aligned} B_{i,1}(t) &:= \begin{cases} 1, & \text{if } t_i \le t < t_{i+1} \\ 0, & \text{otherwise} \end{cases} \\ \omega_{i,k}(t) &:= \begin{cases} \frac{t-t_i}{t_{i+k-1}-t_i}, & \text{if } t_i < t_{i+k-1} \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{1b}$$

This gives $B_{i,k}$ in the form

$$B_{i,k} = \sum_{j=i}^{i+k-1} b_{j,k} B_{j,1}, \tag{2}$$

with each $b_{j,k}$ a polynomial of degree $< k$ since it is the sum of products of $k-1$ linear polynomials. ¿From this we read off that $B_{i,k}$ is a piecewise polynomial of degree $< k$ which vanishes outside the interval $[t_i, t_{i+k}]$ and has possible breakpoints $t_i, \cdots, t_{i+k}$. In particular, $B_{i,k}$ is just the zero function in case $t_i = t_{i+k}$. Also, by induction, $B_{i,k}$ is positive on the open interval (t_i, t_{i+k}), since both $\omega_{i,k}$ and $1-\omega_{i+1,k}$ are positive there. But it is a bit of a miracle that the recurrence relations produce smooth functions. We discuss this question in Theorem 2 below.

A **spline of order k with knot sequence $\mathbf{t}$** is, by definition, a linear combination of the B-splines $B_{i,k}$ associated with that knot sequence. Let

$$S_{k,\mathbf{t}} := \{ \sum_{i=-\infty}^{\infty} a_i B_{i,k} \ : \ a_i \in \mathbf{R} \} \tag{3}$$

denote the collection of all such splines. We now explore this space.

We deduce from the recurrence relation that

$$\sum a_i B_{i,k} = \sum \big(a_i \omega_{i,k} + a_{i-1}(1-\omega_{i,k})\big) B_{i,k-1}. \tag{4}$$

On the other hand, arguing as in [2], for the special sequence

$$a_i := \psi_{i,k}(\tau) := (t_{i+1}-\tau)\cdots(t_{i+k-1}-\tau)$$

(with $\tau \in \mathbf{R}$), we find for $B_{i,k-1} \neq 0$, i.e., for $t_i < t_{i+k-1}$ that

$$a_i\omega_{i,k} + a_{i-1}(1-\omega_{i,k}) = \psi_{i,k-1}(\tau)\Big((t_{i+k-1}-\tau)\omega_{i,k} + (t_i - \tau)(1-\omega_{i,k})\Big)$$
$$= \psi_{i,k-1}(\tau)(\cdot - \tau)$$

since $f(t_{i+k-1})\omega_{i.k} + f(t_i)(1-\omega_{i,k})$ is the straight line which agrees with f at t_{i+k-1} and t_i. This shows that

$$\sum \psi_{i,k}(\tau)B_{i,k} = (\cdot - \tau)\sum \psi_{i,k-1}(\tau)B_{i,k-1},$$

hence, by induction, that

$$\sum \psi_{i,k}(\tau)B_{i,k} = (\cdot - \tau)^{k-1}\sum \psi_{i,1}(\tau)B_{i,1}.$$

This proves the following identity due to Marsden.

Theorem 1. For any $\tau \in \mathbf{R}$,

$$(\cdot - \tau)^{k-1} = \sum_i \psi_{i,k}(\tau)B_{i,k}, \tag{5}$$

with $\psi_{i,k}(\tau) := (t_{i+1}-\tau)\cdots(t_{i+k-1}-\tau)$.

Since τ in (5) is arbitrary, it follows that $S_{k,\mathbf{t}}$ contains all polynomials of degree $< k$. More than that, we can even give an explicit expression for the required coefficients, as follows.

By differentiating (5) with respect to τ, we obtain the identities

$$\frac{(\cdot - \tau)^\nu}{\nu!} = \sum_i \frac{(-D)^{k-1-\nu}\psi_{i,k}(\tau)}{(k-1)!}\, B_{i,k}, \quad \nu < k, \tag{6}$$

with Df the derivative of the function f. On using this identity in the Taylor formula

$$p = \sum_{\nu=0}^{k-1} D^\nu p(\tau)\frac{(\cdot - \tau)^\nu}{\nu!}$$

for a polynomial p of degree $< k$, we conclude that any such polynomial can be written in the form

$$p = \sum_i \lambda_{i,k}p\ B_{i,k}, \tag{7}$$

with $\lambda_{i,k}$ given by the rule

$$\lambda_{i,k}f := \sum_{\nu=0}^{k-1} \frac{(-D)^{k-1-\nu}\psi_{i,k}(\tau)}{(k-1)!}\, D^\nu f(\tau). \tag{8}$$

Here are two special cases of particular interest. For $p = 1$, we get

$$1 = \sum_i B_{i,k} \tag{9}$$

since $D^{k-1}\psi_{i,k} = (-1)^{k-1}(k-1)!$, and this shows that the $B_{i,k}$ form a **partition of unity**. Further, since $D^{k-2}\psi_{i,k}$ is a linear polynomial which vanishes at $t_i^* := (t_{i+1} + \cdots + t_{i+k-1})/(k-1)$,

$$\ell = \sum_i \ell(t_i^*) B_{i,k} \text{ for every linear polynomial } \ell. \tag{10}$$

The identity (5) also gives us various **piecewise** polynomials contained in $S_{k,\mathbf{t}}$: If $\tau = t_j$, then $\psi_{i,k}(\tau)$ vanishes for $i = j-k+1, \cdots, j-1$. Since $B_{i,k}(t) = 0$ for $i < j-k+1$ and $t < t_j$, it follows that

$$(\cdot - t_j)_+^{k-1} = \sum_{i=j}^{\infty} \psi_{i,k}(t_j) B_{i,k} \tag{11}$$

with $\alpha_+ := \max\{\alpha, 0\}$ the positive part of the number α. The same observation applied to (6) shows that

$$(\cdot - t_j)_+^{k-\mu} \in S_{k,\mathbf{t}} \text{ for } 1 \le \mu \le \#t_j := \#\{t_i : t_i = t_j\}. \tag{12}$$

Theorem 2. If $t_i < t_{i+k}$, then the B-splines $B_{i,k}$ are linearly independent and the space $S_{k,\mathbf{t}}$ coincides with the space $\tilde{S}$ of all piecewise polynomials of degree $< k$ with breakpoints t_i which are $k - 1 - \#t_i$ times continuously differentiable at t_i.

Proof. It is sufficient to prove that, for any finite interval $I := [a, b]$, the restriction $\tilde{S}_{|I}$ of the space $\tilde{S}$ to the interval I coincides with the restriction of $S_{k,\mathbf{t}}$ to that interval. The latter space is spanned by all the B-splines having some support in I, i.e., all $B_{i,k}$ with $(t_i, t_{i+k}) \cap I \ne \emptyset$. The space $\tilde{S}_{|I}$ has a basis consisting of the functions

$$(\cdot - a)^{k-\nu},\ \nu = 1, \cdots, k;\ (\cdot - t_i)_+^{k-\nu},\ \nu = 1, \cdots, \#t_i, \text{ for } a < t_i < b. \tag{13}$$

This follows from the observation that a piecewise polynomial function f with a breakpoint at t_i which is $k - 1 - \#t_i$ times continuously differentiable there can be written uniquely as

$$f = p + \sum_{\nu=1}^{\#t_i} a_\nu (\cdot - t_i)_+^{k-\nu},$$

with p a suitable polynomial of degree $< k$ and suitable coefficients a_ν. Since each of the functions in (13) lies in $S_{k,\mathbf{t}}$, by (6) and (12), we conclude that

$$\tilde{S}_{|I} \subset (S_{k,\mathbf{t}})_{|I}. \tag{14}$$

On the other hand, the dimension of $\tilde{S}_{|I}$, i.e., the number of functions in (13), equals the number of B-splines with some support in I (since it equals $k + \sum_{a<t_i<b} \#t_i$), hence is an

upper bound on the dimension of $(S_{k,\mathbf{t}})_{|I}$. This implies that equality must hold in (14), and that the set of B-splines having some support in I must be linearly independent over I.

Corollary 1. All B-splines having some support on a given interval are linearly independent over that interval.

Corollary 2. If $\hat{\mathbf{t}}$ is a refinement of the knot sequence $\mathbf{t}$, then $S_{k,\mathbf{t}} \subset S_{k,\hat{\mathbf{t}}}$.

Corollary 3. If $t_i < t_{i+k-1}$, then the derivative of a spline in $S_{k,\mathbf{t}}$ is a spline of degree $< k-1$ with respect to the same knot sequence, i.e., $DS_{k,\mathbf{t}} = S_{k-1,\mathbf{t}}$.

The identity (7) can be extended to all spline functions. For this, we agree, consistent with (1b), that all derivatives in (8) are to be taken as limits from the right in case τ coincides with a knot.

Theorem 3. If τ in definition (8) of $\lambda_{i,k}$ is chosen in the interval $[t_i, t_{i+k})$, then

$$\lambda_{i,k}\Big(\sum_j a_j B_{j,k}\Big) = a_i. \tag{15}$$

It is remarkable that τ can be chosen arbitrarily in the interval $[t_i, t_{i+k})$. The reason behind this is that $\lambda_{i,k} f$ does not depend on τ at all if f is a polynomial of degree less than k.

Proof. Assume that $\tau \in [t_l, t_{l+1}) \subset [t_i, t_{i+k})$ and let p_j be the polynomial which agrees with $B_{j,k}$ on (t_l, t_{l+1}). Then

$$\lambda_{i,k} B_{j,k} = \lambda_{i,k} p_j.$$

On the other hand,

$$p_j = \sum_{i=l+1-k}^{l} \lambda_{i,k} p_j \; p_i,$$

since this holds by (7) on $[t_l, t_{l+1})$, while, by Corollary 1 or directly from (7), $p_{l+1-k}, \cdots, p_l$ are linearly independent. Therefore necessarily $\lambda_{i,k} p_j$ equals 1 if $i = j$ and 0 otherwise.

2. Algorithms

In this section, the basic algorithms for computing with the B-spline representation are derived. Unless otherwise stated, all algorithms refer to the spline function

$$s = \sum_i a_i B_{i,k}. \tag{16}$$

In the following, the subscript k is omitted whenever possible, e.g. $B_i := B_{i,k}$, $\psi_i := \psi_{i,k}$, etc.

The spline s can be evaluated using the recurrence relation. By (4),

$$\sum_i a_i B_{i,k} = \sum_i \Big(\omega_{i,k} a_i + (1 - \omega_{i,k}) a_{i-1}\Big) B_{i,k-1} =: \sum_i a_i^1 B_{i,k-1}.$$

Iterating this identity one finally arrives at

$$s = \sum_i a_i^{k-1} B_{i,1}$$

and, by the definition of $B_{i,1}$, the right hand side equals a_j^{k-1} on $[t_j, t_{j+1})$. This yields

Algorithm 1 . On the interval $[t_j, t_{j+1})$, $s = a_j^{k-1}$, with the polynomials a_i^r computed as follows:

$$\begin{aligned} a_i^0 &:= a_i \\ a_i^{r+1} &:= \omega_{i,k-r} a_i^r + (1 - \omega_{i,k-r}) a_{i-1}^r, \; j - k + r + 1 < i \le j. \end{aligned}$$

For the relevant range of indices, $\omega_{i,k-r}(t) \in [0,1]$ so that $s(t)$ is computed by repeatedly forming convex combinations of the B-spline coefficients.

It follows from Theorem 2 that the derivative of s is a spline of degree $< k - 1$ with respect to the same knot sequence, i.e.

$$Ds =: \sum_i a_i' B_{i,k-1}. \tag{17}$$

By Theorem 3,

$$a_i' = \lambda_{i,k-1}(Ds) \tag{18}$$

if τ is chosen in the interval (t_i, t_{i+k-1}). To relate a' to a, we express $\lambda_{i,k-1} D$ as a linear combination of the functionals $\lambda_{i,k}$, making use of the fact that $\lambda_{i,k}$ **depends linearly on** $\psi_{i,k}$ and that

$$(t_{i+k-1} - t_i)\psi_{i,k-1} = \psi_{i,k} - \psi_{i-1,k}. \tag{19}$$

¿From the definition (8),

$$(\lambda_{i,k} - \lambda_{i-1,k}) f(\tau) = \sum_{\nu=0}^{k-1} \frac{(-D)^{k-1-\nu} (\psi_{i,k} - \psi_{i-1,k})(\tau)}{(k-1)!} D^\nu f(\tau)$$

and

$$\lambda_{i,k-1} D f(\tau) = \sum_{\nu=0}^{k-2} \frac{(-D)^{k-2-\nu} \psi_{i,k-1}(\tau)}{(k-2)!} D^{\nu+1} f(\tau) = \sum_{\mu=0}^{k-1} \frac{(-D)^{k-1-\mu} \psi_{i,k-1}(\tau)}{(k-2)!} D^\mu f(\tau),$$

the last equality by setting $\mu := \nu+1$ and using the fact that $D^{k-1}\psi_{i,k-1} = 0$. Comparison of these two lines shows with the aid of (19) that

$$\lambda_{i,k-1} D = \frac{k-1}{t_{i+k-1} - t_i} \Big(\lambda_{i,k} - \lambda_{i-1,k} \Big).$$

Assuming that $B_{i,k-1} \neq 0$, i.e., that $t_i < t_{i+k-1}$, we can choose $\tau \in (t_i, t_{i+k-1}) = (t_{i-1}, t_{i+k-1}) \cap (t_i, t_{i+k})$. By Theorem 3, this yields

Algorithm 2. Compute the coefficients for $\sum a_i' B_{i,k-1} := D \sum a_i B_{i,k}$ by

$$a_i' = \frac{a_i - a_{i-1}}{(t_{i+k-1} - t_i)/(k-1)}, \text{ if } t_i < t_{i+k-1}.$$

By Corollary 3, $S_{\mathbf{t}} \subset S_{\hat{\mathbf{t}}}$ for any refinement $\hat{\mathbf{t}}$ of the knot sequence $\mathbf{t}$, and therefore any spline $s \in S_{\mathbf{t}}$ can be written as a linear combination $\sum \hat{a}_i \hat{B}_i$ of the B-splines $\hat{B}_i$ which correspond to the refined knot sequence. The computation of the new coefficients $\hat{a}_i$ from the a_i constitutes the **knot insertion** or **subdivision** algorithm used in CAGD [1,4]. For this, we need to express $\hat{a}_i$ in terms of the a_i. By Theorem 3, this is equivalent to comparing the corresponding $\hat{\lambda}_i$ with λ_i. Since λ_i depends linearly on ψ_i, this requires nothing more than to express

$$\hat{\psi}_i = (\hat{t}_{i+1} - \cdot) \cdots (\hat{t}_{i+k-1} - \cdot)$$

as a linear combination of the ψ_i.

This is particularly easy when $\hat{\mathbf{t}}$ is obtained from $\mathbf{t}$ by adding just one knot, say the point $\hat{t} \in [t_j, t_{j+1})$ so that

$$\hat{t}_i = \begin{cases} t_i, & \text{if } i \leq j; \\ \hat{t}, & \text{if } i = j+1; \\ t_{i-1}, & \text{if } i > j+1. \end{cases}$$

Then

$$\hat{\psi}_i = \begin{cases} \psi_i, & i < j-k+2; \\ \psi_{i-1}, & i > j, \end{cases}$$

hence there is some actual computing necessary only for $j-k+2 \le i \le j$. For this case, observe that $\hat{t}_{i+1} \le \hat{t} = \hat{t}_{j+1} \le \hat{t}_{i+k-1}$, hence

$$\begin{aligned}\alpha\psi_i + \beta\psi_{i-1} &= (t_{i+1} - \cdot)\cdots(t_{i+k-2} - \cdot)\big[\alpha(t_{i+k-1} - \cdot) + \beta(t_i - \cdot)\big] \\ &= \hat{\psi}_i\end{aligned}$$

provided $\alpha(t_{i+k-1} - \cdot) + \beta(t_i - \cdot) = (\hat{t} - \cdot)$, i.e.,

$$\alpha = \omega_i(\hat{t}) \text{ and } \beta = (1 - \omega_i(\hat{t})).$$

Since $\hat{t}_{i+1} \le \hat{t} < \hat{t}_{j+2} \le \hat{t}_{i+k}$, we can choose τ in the definition (8) in the interval $(\hat{t}_i, \hat{t}_{i+k}) = (t_{i-1}, t_{i+k-1}) \cap (t_i, t_{i+k})$. This proves

Algorithm 3. If the knot sequence $\hat{\mathbf{t}}$ is obtained from the knot sequence $\mathbf{t}$ by addition of the point $\hat{t} \in [t_j, t_{j+1})$, then the coefficients $\hat{a}_i$ for the spline s with respect to the refined knot sequence are given by

$$\hat{a}_i = \begin{cases} a_i, & \text{if } i < j-k+2; \\ \omega_i(\hat{t})\ a_i \ + \ (1 - \omega_i(\hat{t}))a_{i-1}, & \text{if } j-k+2 \le i \le j; \\ a_{i-1}, & \text{if } i > j. \end{cases} \tag{20}$$

Observe that $\omega(\hat{t}) \in [0,1]$ for the relevant range of indices and thus the coefficients $\hat{a}$ are convex combinations of the coefficients a.

If $r := \#\hat{t} \le k-1$, then, after just $(k-1-r)$-fold insertion of $\hat{t}$, we obtain a knot sequence $\tilde{\mathbf{t}}$ in which the number $\hat{t}$ occurs exactly $k-1$ times. This means that there is exactly one B-spline for that knot sequence which is not zero at $\hat{t}$. Hence it must equal 1 at $\hat{t}$ and its coefficient must provide the value of s at $\hat{t}$. This makes it less surprising that the calculations in Algorithms 1 and 3 are identical.

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