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Abstract

Multivariate B-splines are defined as volume densities of convex polyhedra. Two special cases, simplex splines and box splines, give rise to natural generalizations of univariate spline functions. While simplex splines yield smooth piecewise polynomial spaces on fairly general triangular meshes, box splines correspond to regular triangulations and share many of the computationally attractive features of tensor products. In this paper, the basic properties of these new classes of spline functions are discussed as well as their application to surface approximation.

AMS class. 41A15

Key words: splines, multivariate, approximation, surfaces, algorithms.

MULTIVARIATE SPLINES

Klaus Höllig[1]

In this paper the construction of multivariate splines on triangular meshes via multivariate B-splines is described. B-splines in several variables can be defined geometrically, as volume densities of convex polyhedra. From this general definition smoothness properties and recurrence relations are derived. The B-splines corresponding to simplices and parallelepipeds give rise to natural generalizations of univariate splines. For both cases it is shown how linear combinations of B-splines have to be selected to yield a smooth spline space which admits a local representation of polynomials. This yields the standard approximation properties for piecewise polynomials familiar from univariate theory.

For simplex splines, the underlying mesh can be chosen almost arbitrarily while maximal smoothness is preserved. While this is a definite advantage over tensor products, new ideas are still needed to overcome computational difficulties resulting from the fairly complicated structure of the mesh. Box splines are defined on regular (triangular) meshes. Therefore, many of the advantages of tensor products and Bezier representations are maintained. In particular, efficient algorithms based on subdivision techniques have been developed and this has led to application box spline methods in computer aided design.

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There are several equivalent ways of defining the univariate B-spline $B(\cdot|t_0,\ldots,t_n)$. Perhaps the least common approach would be to use a variant of the Hermite-Genocchi formula

$$\int_{\mathbf{R}} B(x|t_0,\ldots,t_n)\phi(x)\ dx = n! \int_{\sigma(n)} \phi(\sum_{\nu=0}^{n} \lambda(\nu)t_\nu)\ d\lambda(1)\ldots d\lambda(n) \tag{1}$$

or the geometric interpretation of the B-spline due to Curry and Schoenberg,

$$B(x|t_0,\ldots,t_n) = \mathrm{vol}_{n-1}(T \cap (\{x\} \times \mathbf{R}^{n-1}))/\mathrm{vol}_n(T). \tag{2}$$

Here, $\sigma(n) := \{(\lambda(1),\ldots,\lambda(n)) : \lambda(\nu) \geq 0,\ \sum_{\nu=0}^{n} \lambda(\nu) = 1\}$ is the n-simplex with vertices $e_0 = (0,\ldots,0)$, $e_1 = (1,0,\ldots,0)$, $\ldots$, $e_n = (0,\ldots,0,1)$ and T is an n-simplex for which the first component of each vertex coincides with one of the knots t_ν. Both of the above identities admit a natural generalization to several variables.

Definition 1 [BH82]. For $n \geq m$ denote by $P : \mathbf{R}^n \to \mathbf{R}^m$ the canonical projection and let $Q \subset \mathbf{R}^n$ be a convex polyhedron with affine dimension $m + k$. The multivariate B-spline B is the linear functional defined by

$$< B, \phi >:= \int_Q \phi \circ P,\ \phi \in C_0(\mathbf{R}^m), \tag{3}$$

where the integral is taken with respect to $m + k$ dimensional measure.
If $\mathrm{vol}_m(PQ) > 0$, B can be identified with the bounded function

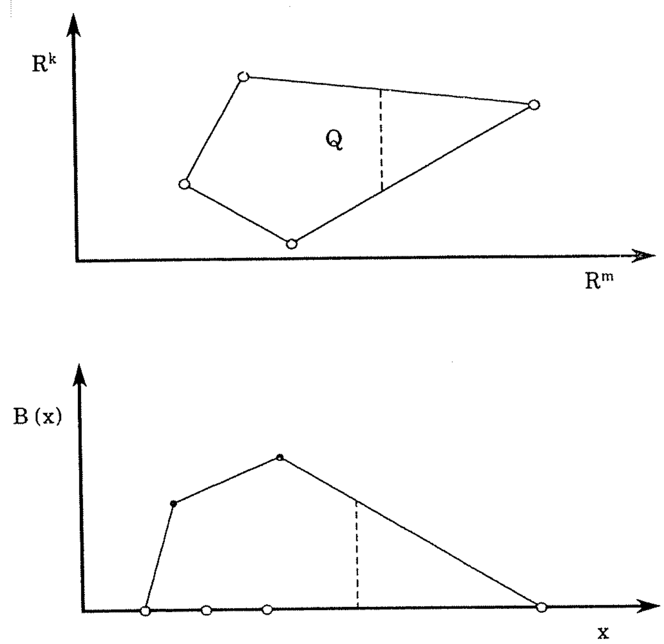
$$B(x) := \mathrm{vol}_k(Q \cap P^{-1}x), \tag{4}$$

i.e. $B(x)$ is the k dimensional volume of the cross section of Q which is projected onto x (cf. Figure 1).

The equivalence of (3) and (4) follows from Fubini's Theorem since, if $\mathrm{vol}_m(PQ) > 0$, the right hand side of (3) can be written as

$$\int_{PQ} dx\ (\int_{Q \cap P^{-1}x} \phi(x)dy) = \int_{PQ} \mathrm{vol}_k(Q \cap P^{-1}x)dx.$$

Strictly speaking, the pointwise definition (3) is valid only for almost every x (in the sense of Lebesgue measure). In the univariate case this difficulty is less apparent since a consistent definition of B at discontinuities is possible, e.g. all B-splines are assumed to be continuous from the right. In several variables there does not seem to exist a simple convention which is compatible with the recurrence relations of Theorem 1 below. However, if B is continuous, which is the case of practical interest, the problem does not arise.



< Figure 1 >

The geometric definition (4) is essentially due to de Boor [B76] who considered the special case when Q is a simplex. The usefulness of the analytical definition (3) for analyzing simplicial B-splines was discovered by Micchelli [M80] which finally led to the general definition given in [BH82].

It is obvious that B is nonnegative (as a functional on $C_0(\mathbf{R}^m)$) with *support* equal to PQ. If $\mathrm{vol}_m(PQ) > 0$, it follows from Theorem 1 below that B is a polynomial of degree $\le k$ on any subset of $\mathbf{R}^m$ which is not intersected by the projection of any $(m-1)$ dimensional face of Q. Theorem 1 also implies that B is $(r-1)$ times continuously differentiable where r is the smallest integer for which an $(m+k-1-r)$ dimensional face of T is projected by P into an $(m-1)$ dimensional set.

Denote by D_ξ the derivative in the direction ξ, i.e. $(D_\xi\phi)(x) := \sum_\nu \xi(\nu)\partial_\nu\phi(x)$ where ∂_ν is the derivative with respect to the ν-th variable. Moreover, denote by Q_i the $(m+k-1)$ dimensional faces making up the boundary of Q, by η_i the corresponding outward normals and by B_i the B-splines corresponding to the polyhedra Q_i (cf. Figure 2).

Theorem 1 [BH82]. Assume that $\mathrm{vol}_n(Q) > 0$, i.e. that $k = n - m$.

(i) For any $z \in \mathbf{R}^n$,

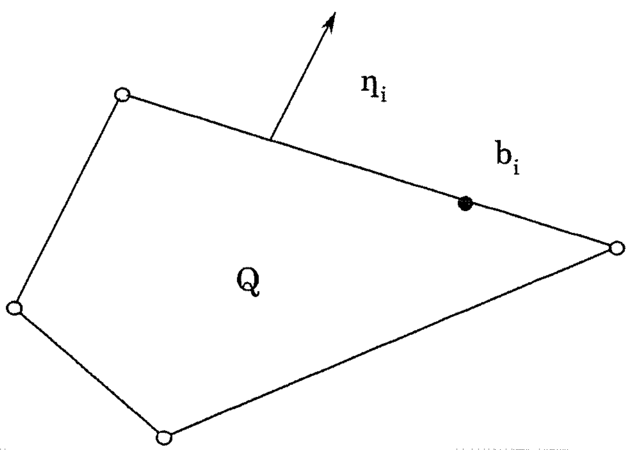
$$D_{Pz}B = -\sum_i (z \cdot \eta_i) B_i.$$

(ii) For all points $x = Pz$ where B and B_i are continuous,

$$kB(x) = \sum_i ((b_i - z) \cdot \eta_i) B_i(x)$$

where b_i is any point in the hyperplane containing Q_i.

The assumption that the polyhedron Q is nondegenerate is not essential. If $k < n - m$, the affine hull of Q can be identified with $\mathbf{R}^{m+k}$ and the Theorem applies.



< Figure 2 >

A repeated application of the Theorem yields that, for $\xi \in \mathbf{R}^m$, $(D_\xi)^r B$ is a linear combination of B-splines corresponding to $(m + k - r)$ dimensional faces Q_i^r. For $r > k$ the supports PQ_i^r of of these B-splines (interpreted as linear functionals in the sense of definition (3)) are contained in hyperplanes. Therefore, B is a polynomial of degree $\leq k$ on any region which is not intersected by any of the sets PQ_i^{k+1} (since all $(k+1)$-th order derivatives of B vanish on such a region).

If $\mathrm{vol}_m(PQ_i^r) > 0$, the B-spline corresponding to Q_i^r can be identified with a bounded function. Therefore, if $\mathrm{vol}_m(PQ_i^r) > 0$ for all i, the derivatives of order r of B are bounded which implies that B is $(r-1)$ times continuously differentiable.

Proof of Theorem 1. The proof of (i) is immediate:

$$\begin{aligned} < D_{Pz}B, \phi > &= - < B, D_{Pz}\phi > = -\int_Q (D_{Pz}\phi)(Py)dy \\ &= -\int_Q (D_z(\phi \circ P))(y)dy = -\sum_i \int_{Q_i} (z \cdot \eta_i)\phi(Py)dy \\ &= -\sum_i (z \cdot \eta_i) < B_i, \phi > . \end{aligned}$$

This uses the fact that, by definition, the derivative $D_\xi B$ is the linear functional given by $\phi \to - < B, D_\xi \phi >$ and that, by the chain rule,

$$D_z(\phi \circ P) = (D_{Pz}\phi) \circ P.$$

Define $(D\phi)(x) := (D_x\phi)(x)$. The recurrence relation (ii) is a consequence of the identity

$$DB = kB - \sum_i (b_i \cdot \eta_i) B_i. \tag{5}$$

With $x = Pz$ it follows from (i) and (5) that

$$\begin{aligned} 0 &= ((D - D_{Pz})B)(x) \\ &= kB(x) - \sum_i (b_i \cdot \eta_i) B_i(x) + \sum_i (z \cdot \eta_i) B_i(x) \end{aligned}$$

if B and B_i are continuous at x.

It remains to prove (5). By definition of D and the chain rule,

$$(D\phi)(Py) = (D_{Py}\phi)(Py) = (D_y(\phi \circ P))(y) = (D(\phi \circ P))(y). \tag{6}$$

Denote by χ_ν the ν-th coordinate function, i.e. $\chi_\nu(x) = x(\nu)$. Then, integrating by parts and using definition (3),

$$\begin{aligned} - < DB, \phi > &= - < \sum_{\nu=1}^{m} \chi_\nu \partial_\nu B, \phi > = < B, \sum_\nu \partial_\nu (\chi_\nu \phi) > \\ &= \sum_\nu \int_Q (\partial_\nu(\chi_\nu \phi)) \circ P = m \int_Q \phi \circ P + \sum_\nu \int_Q (\chi_\nu \partial_\nu \phi) \circ P \\ &= m < B, \phi > + \int_Q (D\phi) \circ P, \end{aligned}$$

and similarly,

$$\sum_{\nu=1}^{n} \int_Q \partial_\nu(\chi_\nu(\phi \circ P)) = n < B, \phi > + \int_Q D(\phi \circ P).$$

By (6), the last integral in the first identity equals the last integral in the second. Therefore,

$$< DB, \phi > = (n - m) < B, \phi > - \sum_{\nu=1}^{n} \int_Q \partial_\nu(\chi_\nu(\phi \circ P)).$$

This proves (5) since, with $\eta(y)$ denoting the boundary normal of Q at y,

$$\sum_{\nu=1}^{n} \int_Q \partial_\nu(\chi_\nu(\phi \circ P)) = \int_{\partial Q} (\eta(y) \cdot y) \phi(Py) dy$$

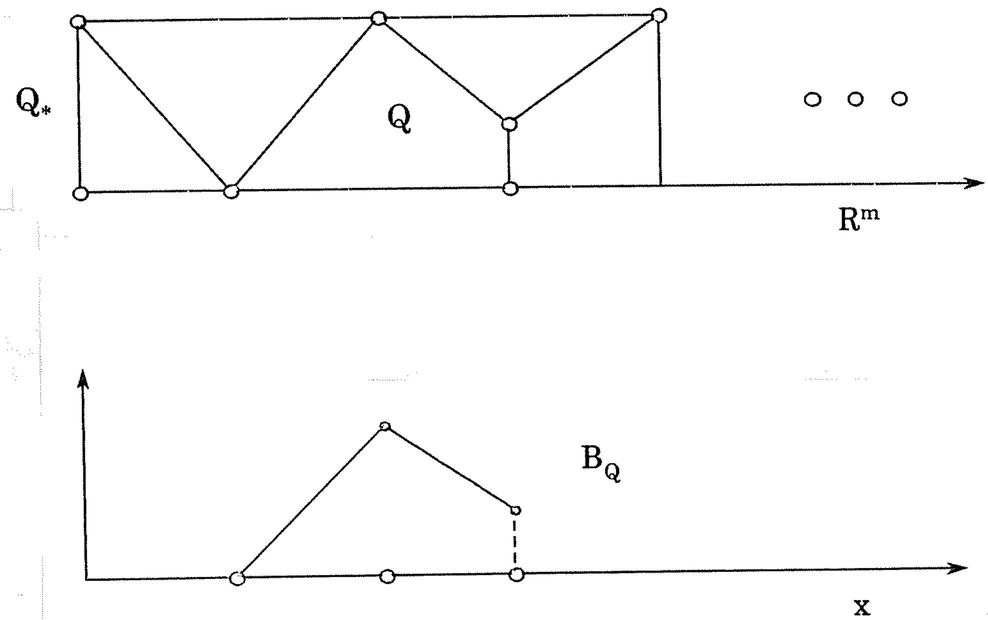
and, for $y \in Q_i$, $\eta(y) = \eta_i$ and $\eta_i \cdot y$ is constant.

Multivariate splines are, by definition, linear combinations of B-splines. However, it is not obvious, how the B-splines should be selected to yield good approximation properties of the resulting spline space. De Boor [B76] suggested the following geometric construction.

Definition 2. Let $Q_* \subset \mathbf{R}^k$ be a convex polyhedron and assume that the collection of convex polyhedra $\{Q : Q \in \Delta\}$ forms a partition of $\mathbf{R}^m \times Q_*$ and that $\mathrm{vol}_m(PQ) > 0$ for all $Q \in \Delta$. The spline functions corresponding to the partition Δ are defined by

$$S(\Delta) := \{\sum_{Q\in\Delta} a_Q B_Q : a_Q \in \mathbf{R}\} \tag{7}$$

where B_Q denotes the B-spline corresponding to the polyhedron Q.



< Figure 3 >

It is clear from (4) that the B-splines B_Q, $Q \in \Delta$, form a partition of unity, i.e. that

$$\sum_{Q} B_Q(x) = \mathrm{vol}_k(Q_*) \tag{8}$$

for all x where the B-splines are continuous. This implies that the spline spaces $S(\Delta)$ are dense in continuous functions as the partition Δ is refined.

Proposition 1. Set $h := \max\{\mathrm{diameter}(Q) : Q \in \Delta\}$ and choose $x_Q \in PQ$. Then, for any continous function f,

$$\|f - \sum_{Q} f(x_Q)(B_Q/vol_k(Q_*))\|_\infty \leq \omega(h)$$

where $\| \|_\infty$ denotes the L_∞ norm on $\mathbf{R}^m$ and ω is the modulus of continuity of f.

Proof. By (8) and since the B-splines are nonnegative, we have for almost every x

$$\begin{aligned} |f(x) - \sum f(x_Q)B_Q(x)| &= |\sum [f(x) - f(x_Q)](B_Q(x)/vol_k(Q_\cdot))| \\ &\leq \max_{B_Q(x)\neq 0} |f(x) - f(x_Q)|, \end{aligned}$$

and for all Q for which $B_Q(x)$ is nonzero $|x - x_Q| \leq h$.

In this generality, little more can be said about the approximation properties of the spline spaces $S(\Delta)$. However, a particularly rich theory results if Q is either a simplex or a parallelepiped. This is due to the fact that in both cases the faces which make up the boundary of Q are of the same type as Q itself.

Simplex Splines

Historically, the case when Q is a simplex has been considered first. Simplicial B-splines were defined by de Boor in [B76] generalizing the geometric interpretation of univariate B-splines due to Curry and Schoenberg. Micchelli [M80] discovered the recurrence relations. Then, the author [H82] and independently Dahmen and Micchelli [DM82] described an appropriate choice for the space $S(\Delta)$ which yields the approximation properties familiar from the univariate theory. Subsequently many interesting results have been obtained and the reader is referred to the survey article [DM84$_1$].

Denote by U a collection of vectors $\{u : u \in U\}$ which need not be distinct, by $\#U$ the number of vectors in U counting multiplicities and by $[U]$ the convex hull of the vectors in U.

Definition 1S. Let $[U]$ be a simplex in $\mathbf{R}^n$ ($\#U = n + 1$) and denote by $V := \{v = Pu : u \in U\}$ the projections of the vertices of $[U]$. The normalized simplicial B-spline M_V is defined by

$$M_V := B_{[U]}/\text{vol}[U]. \tag{9}$$

To justify this definition, one has to show that the right hand side of (9) does only depend on the projections of the vertices $v \in V$. This follows from Definition (3) by a change of variables. Let G be an affine mapping of the simplex $\sigma(n)$ onto $[U]$. Then,

$$\text{vol}[U]^{-1} \int_{[U]} \phi(Py)dy = \int_{\sigma(n)} \phi(\sum_{\nu=0}^{n} \lambda(\nu)PGe_\nu)\ d\lambda(1)\ldots\lambda(n), \tag{10}$$

where $Ge_0, \ldots, Ge_n$ are the vertices of $[U]$.

Theorem 1S [M80]. Let V be a collection of $n+1$ points in $\mathbf{R}^m$ which span a proper convex set.

(i) If $\xi = \sum_{v \in V} \lambda(v) v$ with $\sum_{v \in V} \lambda(v) = 0$, then

$$D_\xi M_V = n \sum_V \lambda(v) M_{V \backslash v}$$

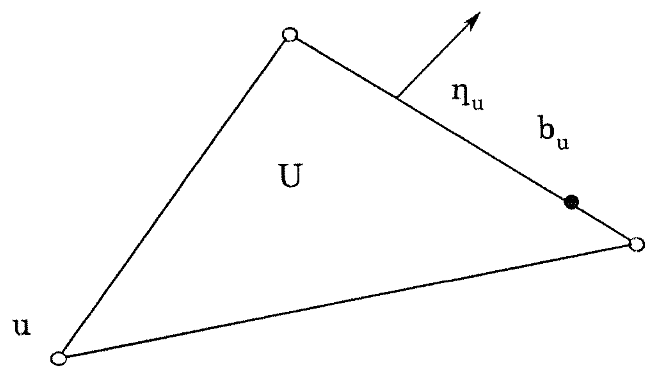
where $V \backslash v$ is obtained from V by decreasing the multiplicity of v by one (i.e. by deleting v if this vector occurs only once in V).

(ii) If $x = \sum_{v \in V} \lambda(v) v$ with $\sum_{v \in V} \lambda(v) = 1$ and $M_{V \backslash v}$, $v \in V$, are continuous at x, then

$$M_V(x) = \frac{n}{n-m} \sum_V \lambda(v) M_{V \backslash v}.$$

To derive this Theorem from Theorem 1 let $[U]$ be a simplex in $\mathbf{R}^m$ with $\{v := Pu : u \in U\} = V$. Set $B := \mathrm{vol}[U]\ M_V$ and $B_v := \mathrm{vol}[U \backslash u]\ M_{V \backslash v}$ and denote the normal of the face $[U \backslash u]$ by η_u. Then, for any $b_u \in [U \backslash u]$,

$$(b_u - u') \cdot \eta_u = \begin{cases} n\ \mathrm{vol}[U]/\mathrm{vol}[U \backslash u], & \text{if } u = u'; \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$



< Figure 4 >

To prove (i), fix $u' \in U$ and set

$$z := \sum_{u \in U} \lambda(v) u = \sum_{u \neq u'} \lambda(v)(u - u')$$

using that the sum of the weights $\lambda(v)$ is zero. By (11), for $u \neq u'$,

$$-z \cdot \eta_u = -\lambda(v)(u - u') \cdot \eta_u = n\ \lambda(v) \frac{\mathrm{vol}[U]}{\mathrm{vol}[U \backslash u]},$$

and similarly,

$$-z \cdot \eta_{u'} = -n \sum_{u \neq u'} \lambda(v) \mathrm{vol}[U]/\mathrm{vol}[U \backslash u'] = n\ \lambda(v') \frac{\mathrm{vol}[U]}{\mathrm{vol}[U \backslash u']},$$

and (i) follows from the normalization of the simplicial B-splines.

To prove (ii) we define z as before and note that

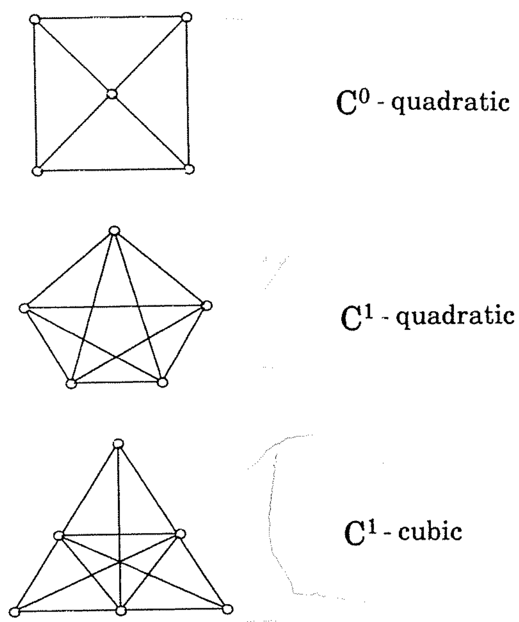
$$b_{u'} - z = \sum_{u \in U} \lambda(v)(b_{u'} - u).$$

Again, by (11),

$$(b_{u'} - z) \cdot \eta_{u'} = \lambda(u')(b_{u'} - u') \cdot \eta_{u'} = n\ \lambda(u') \frac{\mathrm{vol}[U]}{\mathrm{vol}(U \backslash u')}.$$

In view of the remarks following Theorem 1, the simplicial B-spline is a piecewise polynomial of degree $\leq k = n - m$ which is $(r - 1)$ times continously differentiable where r is the smallest integer for which $(m + k - r)$ points from the "knot set" V ly in a hyperplane. Thus, if the knots are in "general" position, M_V is $(k - 1)$ times continuously differentiable.

Figure 5 below gives a few examples of knot sets and corresponding meshes for simplicial B-splines in two variables. While in some cases the structure of the mesh (i.e. the hyperplanes where derivatives of M_V are discontinuous) is fairly complicated, this is no disadvantage in itself since the explicit form of M_V on each of the subregions is not needed in computations.



< Figure 5 >

Example 1. Let $[W]$ be a proper simplex in $\mathbf{R}^m$ with vertices $\{w : w \in W\}$ and denote by $\varrho_w(x)$ the barycentric coordinates of x with respect to W, i.e.

$$x = \sum_{w \in W} \varrho_w(x) w$$
$$1 = \sum_{w \in W} \varrho_w(x).$$

If the knot set V consists of the vertices of $[W]$ with multiplicities $\alpha(w)$, $w \in W$, then, up to a normalizing factor, the simplicial B-spline coincides on $[W]$ with a multivariate (Bernstein) polynomial, i.e.

$$M_V(x) = \frac{n!}{m!} \prod_{w \in W} \varrho_w(x)^{\alpha(w)}/\alpha(w)!. \tag{12}$$

This is most easily seen by checking that the right side of (12) satisfies the recurrence relation (ii) of Theorem 1S.

In principle, simplex splines can be defined by (7) with $Q_* := \sigma(k)$ and $B_Q := \mathrm{vol}[U] M_V$. However, without further restrictions, the simplicial B-splines M_V, $[U] \in \Delta$, need not be linearly independent. Nevertheless, their linear span does contain all polynomials of degree $\leq k$ which is the minimal requirement for good local approximation properties.

Theorem 2S [DM82, H82]. For $\xi \in \mathbf{R}^m$ define the mapping

$$(x, y) \to G_\xi(x, y) := (x, (1 + \xi \cdot x) y) \;:\; \mathbf{R}^m \times \sigma(k) \to \mathbf{R}^m \times \mathbf{R}^k.$$

If all B-splines M_V are continous at x, then

$$(1 + \xi \cdot x)^k = \sum_{[U] \in \Delta} c_V(\xi) M_V(x) \tag{13}$$

where

$$c_V(\xi) := (k!/n!)\ sign(U)\ \det\|G_\xi U\|$$

with $\det\|G_\xi U\|$ denoting the determinant of the $(n+1) \times (n+1)$ matrix with columns

$$\begin{bmatrix} G_\xi u \\ 1 \end{bmatrix}, \ u \in U,$$

and $sign(U) \in \{-1, +1\}$ chosen so that $c_V(0)$ is positive.

Identity (13) is the multivariate analogue of Marsden's identity for univariate splines. As in the univariate case, this identity is the basis for the construction of dual linear functionals and local approximation schemes [DM82, H82]. In two variables the identity is due to Goodman and Lee [GL81] who also obtained a more explicit formula for the B-spline coefficients.

Proof. For fixed x both sides of (13) are polynomials in ξ and we may therefore assume that $||\xi||$ is small. Small perturbations of the vertices do not change the combinatorial structure of a triangulation. Moreover, G_ξ maps the hyperplanes which form the boundary of $\mathbf{R}^m \times \sigma$ onto hyperplanes. Therefore, for fixed x and small ξ, the simplices

$$[G_\xi U] := [\{G_\xi u : u \in U\}]$$

form a partition of $\Omega := G_\xi(\mathbf{R}^m \times \sigma(k))$ in a neighborhood of x. This implies that (cf. Figure 6)

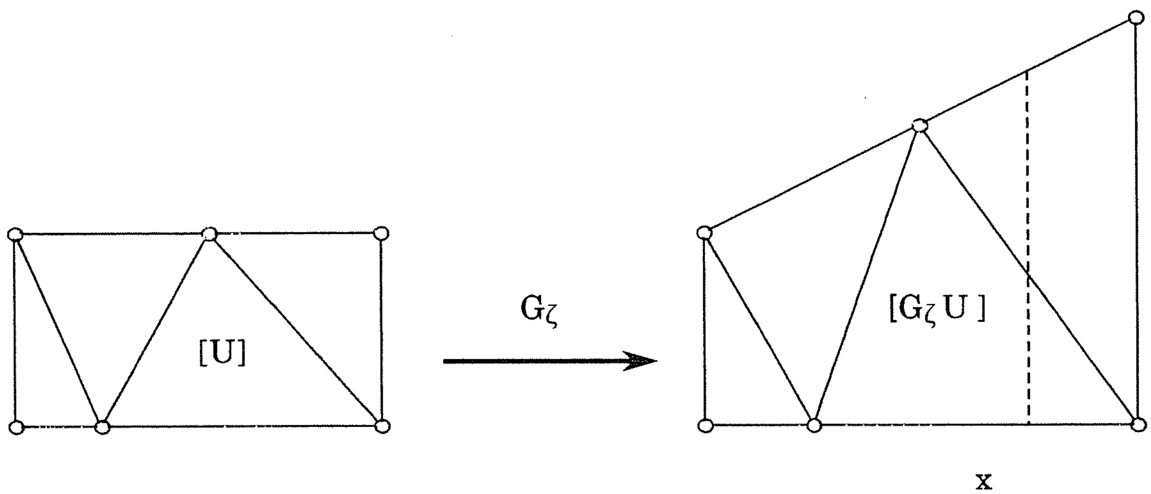
$$(x, (1 + \xi \cdot x)\sigma(k)) = (x, \mathbf{R}^k) \cap \bigcup_{x \in P[U]} [G_\xi U].$$

Computing the volume on both sides of this identity it follows from (4) and (9) that

$$\frac{1}{k!}(1 + \xi \cdot x)^k = \sum_{x \in P[U]} \mathrm{vol}_k([G_\xi U] \cap P^{-1}x)$$

$$\sum \mathrm{vol}_n [G_\xi U] M_V(x)$$

which yields the Theorem.



< Figure 6 >

A drawback of definition (7) is that the spline space $S_M(\Delta)$ is defined via a triangulation in n dimensions while the B-splines depend only on the knots in $\mathbf{R}^m$. In [H82] a method for constructing (simplicial) spline spaces from a triangulation of $\mathbf{R}^m$ was described. This construction is a generalization of the process of "pulling apart" knots, i.e. of obtaining smooth splines as a perturbation of piecewise polynomials without smoothness constraints.

Denote by

$$[W_i] := [w_{i(0)}, \ldots, w_{i(m)}], \; i \in I,$$

the simplices of a triangulation Δ_m of $\mathbf{R}^m$ with vertices $W := \{\ldots, w_{-1}, w_0, w_1, \ldots\}$. Moreover, assume that the vertices are consistently ordered, i.e. if

$$i(\nu) = i'(\nu'), \; i(\mu) = i'(\mu'), \text{ with } \nu < \mu \text{ and } i, i' \in I,$$

then

$$\nu' < \mu'.$$

Denote by Γ all "index" sets of the form

$$\gamma = ((\alpha(0), \beta(0)), \ldots, (\alpha(n), \beta(n)))$$

with

$$\begin{aligned} &\alpha(\nu) \in \{i(0), \ldots, i(m)\} \text{ for some } i \in I \\ &\beta(\nu) \in \{0, \ldots, k\} \end{aligned} \tag{14}$$

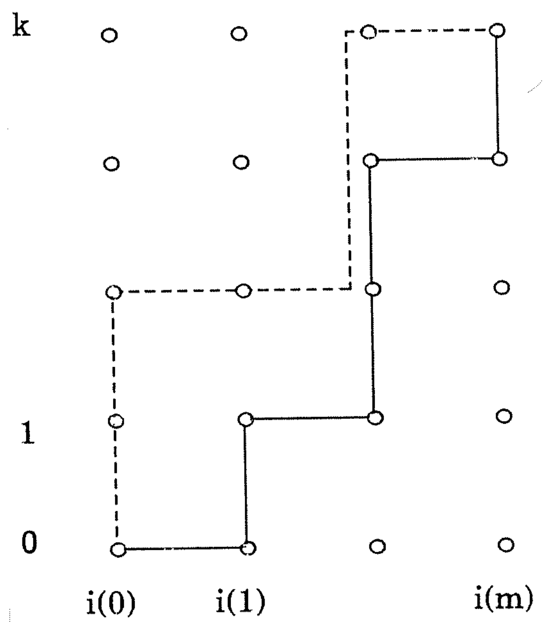
and

$$\alpha(\nu) \leq \alpha(\nu+1),\ \beta(\nu) \leq \beta(\nu+1)$$

where one of the inequalities is strict.

As is indicated in Figure 7 below, the index sets γ corresponding to a simplex $[W_i]$ can be identified with the ordered sequences of length $n+1 = m+k+1$ from the set

$$\{w_{i(0)}, \ldots, w_{i(m)}\} \times \{0, \ldots, k\}.$$



< Figure 7 >

Definition 2S [H82]. Let F be a mapping from $\{\ldots, -1, 0, 1 \ldots\} \times \{0, \ldots, k\}$ to $\mathbf{R}^m$ and denote by $F(\gamma)$ the collection of vectors $\{F(\alpha(0), \beta(0)), \ldots, F(\alpha(n), \beta(n))\}$. Assume that the union of the sets $[F(\gamma)]$ covers $\mathbf{R}^m$, that the range of F has no limit point and that each $x \in \mathbf{R}^m$ in contained in at most finitely many of the sets $[F(\gamma)]$. Then, the spline space $S(F, \Gamma)$ is defined as the linear span of the B-splines $M_{F(\gamma)}$, $\gamma \in \Gamma$.

Note, that the mapping F can be chosen almost arbitrarily, i.e. in analogy with the univariate situation there is almost no restriction on the placement of the "knots" $F(\gamma)$. However, there is no canonical choice for F which yields maximal smoothness or a well conditioned B-spline basis. This must still be viewed as one of the major drawbacks of simplicial spline spaces. However, on the other hand, for "almost all" choices of F, the space $S(F,\Gamma)$ consists of piecewise polynomials of smoothness $k-1$ and degree k which is, in general fairly difficult to achieve with other constructions.

Example 2. Let W be a partition of $\mathbf{R}$, i.e. the "simplices" $[W_i]$ are the intervals $[w_i, w_{i+1}]$. Define F by

$$F(\alpha,\beta) := t_{\alpha(k+1)-\beta}$$

where $\{\ldots, t_{-1}, t_0, t_1, \ldots\}$ is an increasing sequence of knots. Since $m=1$ the index sets γ are of the simple form

$$\gamma = ((i,0),(i,1),\ldots,(i,j),(i+1,j),\ldots,(i+1,k))$$

where $0 \le j \le k$ and i is any integer. Thus $F(\gamma)$ consists of the $k+2$ consecutive knots

$$t_{i(k+1)-j},\ t_{i(k+1)-j+1},\ \ldots, t_{(i+1)(k+1)-j}$$

and therefore $S(F,\Gamma)$ is the standard space of univariate splines. However, Definition 2S is more general since the sequence of knots does not have to be monotone increasing.

Example 3. For the particular choice

$$F_*(\alpha,\beta) := w_\alpha,\ \gamma = (\alpha,\beta) \in \Gamma, \tag{15}$$

$S(F_*,\Gamma)$ consists of all piecewise polynomials of degree $\le k$ with respect to the triangulation Δ_m. This can be seen as follows. For F_* defined by (15), the B-splines which correspond to different index sets i via (14) have disjoint support. Therefore, restricted to a simplex $[w_{i(0)}, \ldots, w_{i(m)}]$ of Δ_m, the spline space $S(F,\Gamma)$ reduces to the linear span of $M_{F(\gamma)}$ where

$$F(\gamma) = (w_{\alpha(0)}, \ldots, w_{\alpha(n)}) \text{ with } \alpha(\nu) \in \{i(0), \ldots, i(m)\},$$

i.e. the linear span of B-splines with multiple knots. From Figure 7 it is clear that all combinations of multiplicities occur and by Example 1 the corresponding B-splines coincide with the Bernstein polynomials.

A small perturbation of the mapping F_* can be interpreted as "pulling apart" multiple knots, i.e. as deforming the space of (nonsmooth) piecewise polynomials into a space of smooth splines. However, Definition 2S allows arbitrary perturbations as long as the combinatorial relationship between the knot sets is preserved.

Theorem 3 [H82]. With

$$U := \{(F(\alpha(0),\beta(0)),e_{\beta(0)}),\ldots,(F(\alpha(n),\beta(n)),e_{\beta(n)})\}$$

and $[U] \in \Delta$ replaced by $\gamma \in \Gamma$, Theorem 2S remains valid for the spline space $S(F,\Gamma)$.

The proof of this result is based on the fact that the Fourier transform of the identity (13) is an entire function of the knots. Therefore, if the identity holds for small perturbations of the knots, it remains valid globally.

Under additional assumptions on F, the linear independence of the B-splines $M_{F(\gamma)}$, $\gamma \in \Gamma$ can be established. Moreover, the standard error estimates are valid for simplex splines. The practical implementation of algorithms for computing with simplex splines still seems to be the major unsolved problem. However, one might think that, similarly as for box splines, new algorithms based on subdivision techniques can be developed.

Box Splines

The other natural choice for Q in Definitions (3,4) is a parallelepiped which leads to the definition of box splines. These splines have been introduced by de Boor and DeVore in [BD83] and their basic properties were studied in [BH82/3]. Box splines can be viewed as generalizations of univariate cardinal splines. A variety of results on interpolation operators [BHR85], combinatorial problems [DM85] and smooth piecewise polynomials on regular meshes [$\text{BH83}_{1,2}$] have been obtained. Moreover, efficient algorithms for manipulating box spline surfaces have been developed [Bö83, CLR84, DM84_2, P83/84] which is the basis for for applying box spline techniques to computer aided design.

Definition 1B [BD82, BH82/3]. Denote by $]U[$ a parallelepiped in $\mathbf{R}^n$ which is spanned by the vectors $\{u : u \in U\}$, i.e.

$$]U[= \{\sum_{u\in U} \lambda(u)u : 0 \leq \lambda(u) \leq 1\}$$

and $\#U = n$. The corresponding normalized B-spline is defined as

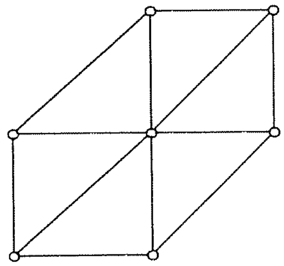
$$N_V := B_{]U[}/\text{vol}]U[\tag{16}$$

where $V := \{v := Pu : u \in U\}$.

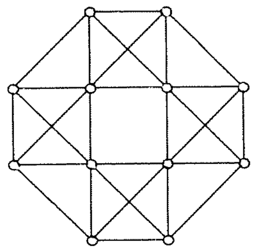
As for the simplicial B-spline, the right hand side of (16) does only depend on the projections of the vectors in U and

$$< N_V, \phi >= \text{vol}]U[^{-1} \int_{]U[} \phi(Py)dy = \int_{[0,1]^n} \phi(\sum_{v\in V} \lambda(v)v)d\lambda. \tag{17}$$

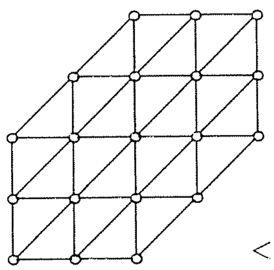
By the remarks following Theorem 1, N_V is a piecewise polynomial of degree $k = n - m$ which is $(r-1)$ times continuously differentiable where r is the smallest integer for which $(m+k-r-1)$ of the vectors in V do not span $\mathbf{R}^m$. In contrast to the simplicial B-spline the mesh for N_V is quite regular. It consists of translates of hyperplanes which are spanned by $(m-1)$ linearly independent vectors in V. Figure 8 below shows a few examples of meshes for bivariate box splines.



C^0 - linear



C^1 - quadratic



C^2 - quartic

< Figure 8 >

Example 4. (i) If $m = 1$ and $v = 1$ for all $v \in V$, N_V is the forward cardinal B-spline $B(\cdot|0,\ldots,k+1)$. To see this let $]U[$ be a parallelepiped with $v = Pu = 1$ for all $u \in U$ and consider the standard triangulation of $]U[$ into $n!$ simplices $[U^\nu]$ with equal volume. For all simplices $[U^\nu]$ the projections of the vertices are the integers $0, 1, \ldots, k+1$. Therefore,

$$B_{]U[} = \sum_\nu B_{[U^\nu]} = (\sum_\nu \mathrm{vol}[U^\nu])\ M_{\{0,1,\ldots,k+1\}} = \mathrm{vol}]U[\ B(\cdot|0,\ldots,k+1).$$

(ii) If V consists of the unit vectors $e_1, \ldots, e_m$ with multiplicities $\alpha(1), \ldots, \alpha(m)$ respectively, then N_V coincides with the tensor product B-spline with equally spaced knots,

$$(x(1),\ldots,x(m)) \to \prod_{\nu=1}^m B(x(\nu)|0,\ldots,\alpha(\nu)).$$

This could be verified directly from (17), but is more easily seen from formula (19) below for the Fourier transform of N_V.

(iii) For $m = 2$ and $V = \{(1,0),(1,1),(0,1)\}$, N_V is the standard linear finite element. Adding the vector $(1,-1)$ to V, one obtains the quadratic element which has been independently derived by Zwart [Z73], Powell and Sabin [PS77]. Further examples can be found in the work of Frederickson [F71].

Theorem 1B [BH82/3]. Let V be a collection of n vectors which span $\mathbf{R}^m$.

(i) If $\xi = \sum_{v \in V} \lambda(v) v$, then

$$D_\xi N_V = \sum_V \lambda(v)\big(N_{V\setminus v} - N_{V\setminus v}(\cdot - v)\big).$$

(ii) If $x = \sum_{v \in V} \lambda(v) v$ and the B-splines $N_{V\setminus v}$, $v \in V$, are continuous at x, then

$$N_V(x) = \frac{1}{n-m} \sum_V \big(\lambda(v) N_{V\setminus v}(x) + (1-\lambda(v)) N_{V\setminus v}(x-v)\big).$$

The recurrence relation (i) has a particularly simple form if $\xi = v$ for some $v \in V$. Then,

$$D_v N_V = \nabla_v N_{V\setminus v},$$

where $(\nabla_v f)(x) := f(x) - f(x-v)$ is the backward difference operator. With $D_W := \prod_{w\in W} D_w$ and $\nabla_W := \prod_{w \in W} D_w$, this yields

$$D_W N_V = \nabla_W N_{V\setminus W}.$$

In particular,

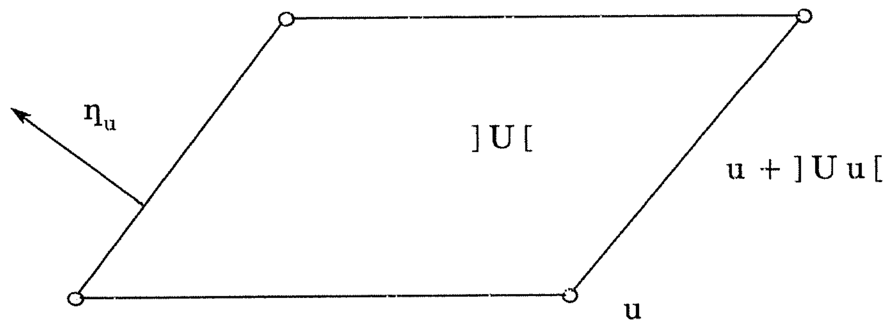
$$D_V N_V = \nabla_V \delta,$$

where δ denotes point evaluation at 0, i.e. $< \delta, \phi > := \phi(0)$. Therefore,

$$\int_{\mathbf{R}^m} N_V D_V \phi = (\Delta_V \phi)(0), \tag{18}$$

which gives an integral representation for the forward difference operator Δ_V in terms of the B-spline N_V.

The derivation of the recurrence relations is almost identical with the proof of Theorem 1S. Let $]U[$ be a parallelepiped in $\mathbf{R}^n$ for which $V = \{v := Pu : u \in U\}$ and apply Theorem 1 with $Q :=]U[$ and $B := \mathrm{vol}]U[\ N_V$. The boundary faces of $]U[$ consist of the parallelepipeds $]U\backslash u[$ and their translates $u+]U\backslash u[$ with normals η_u and $-\eta_u$ respectively. The corresponding B-splines are $B_{]U\backslash u[} = \mathrm{vol}]U\backslash u[\ N_{V\backslash v}$ and $B_{u+]U\backslash u[} = \mathrm{vol}]U\backslash u[\ N_{V\backslash v}(\cdot - v)$ (cf. Figure 9).



< Figure 9 >

To prove (i), set $z = \sum_{u \in U} \lambda(v)u$. Then, since the vectors u, $u \neq u'$, span the boundary face $]U\backslash u'[$,

$$-z \cdot \eta_{u'} = -\sum_U \lambda(v)u \cdot \eta_{u'} = -\lambda(v')u' \cdot \eta_{u'} = \lambda(v')\frac{\mathrm{vol}]U[}{\mathrm{vol}]U\backslash u'[}$$

and the assertion follows from the normalization of the B-spline.

To prove (ii), define z as before and choose the points b_u in the boundary faces $]U\backslash u[$ and $u+]U\backslash u[$ as 0 and u respectively. Then,

$$(0 - z) \cdot \eta_u = \lambda(v)\frac{\mathrm{vol}]U[}{\mathrm{vol}]U\backslash u[},$$

and

$$(u - z) \cdot (-\eta_u) = (1 - \lambda(v))u \cdot (-\eta_u) = (1 - \lambda(v))\frac{\mathrm{vol}]U[}{\mathrm{vol}]U\backslash u[}.$$

Setting $\phi(x) = \exp(-iy \cdot x)$ in (17), one sees that the Fourier transform of N_V is

$$\hat{N}_V(y) = \prod_{v \in V} \frac{1 - \exp(-iy \cdot v)}{iy \cdot v}. \tag{19}$$

From this it follows that

$$N_{V \cup V'} = N_V * N_{V'} \tag{20}$$

where $f * g(x) := \int f(x-y)g(y)dy$ denotes the convolution of f and g. In particular, if V' consists out of a single vector ξ,

$$N_{V \cup \xi}(x) = \int_0^1 N_V(x - \lambda\xi)\, d\lambda. \tag{20'}$$

This identity provides an alternative definition for N_V via repeated "averaging" in direction of the vectors $v \in V$.

Definition 2B [BD83, BH82/3]. Assume that the vectors in V have integer coordinates and that V contains the unit vectors $e_1, \ldots, e_n$. The space of box splines corresponding to V is defined as

$$S(V) := \{ \sum_{j \in \mathbf{Z}^m} a_j N_V(\cdot - j) : a_j \in \mathbf{R} \} \tag{21}$$

where $\mathbf{Z}$ denotes the integers.

Definition 2B is a special case of Definition 1 with $Q_* := [0,1]^{n-m}$ and the partition Δ consisting of translates of the parallelepiped which is spanned by the vectors $(e_1, 0), \ldots, (e_m, 0)$ and $(v_{m+1}, e_{m+1}), \ldots, (v_n, e_n)$ where $V = \{e_1, \ldots, e_m, v_{m+1}, \ldots, v_n\}$. The assumption that V contains the unit vectors is no loss of generality since this can always be achieved by a change of variables. However, for the proof of Theorem 2B below it is essential that all vectors v are chosen from a common lattice which, again by a change of variables, can assumed to be the lattice of vectors with integer coefficients.

Theorem 2B [BH82/3]. Denote by $< W >$ the linear span of the vectors $\{w : w \in W\}$ and define

$$\Lambda := \{W \subset V : < V \backslash W > \neq \mathbf{R}^m\}.$$

Then,

$$\pi \cap S(V) = \bigcap_{W \in \Lambda} \ker D_W, \tag{22}$$

where π denotes the space of polynomials.

Example 5. (i) As was pointed out in Example 4 (ii) for the tensor product B-spline, V consists of the unit vectors. Assume that each unit vector occurs with multiplicity α, then Λ contains the sets

$$W_\nu = \{\underbrace{e_\nu, \ldots, e_\nu}_{\alpha \text{ times}}\}, \ \nu = 1, \ldots, m,$$

and any other set in Λ contains one of these sets as subset. Therefore, by (22), a polynomial p is in $S(V)$ if and only if it is annihilated by

$$D_{W_\nu} = \partial_\nu^\alpha, \ \nu = 1, \ldots, m.$$

(ii) If the vectors in V are in "general" position, then all sets in $W \in \Lambda$ satisfies $\#W > k$. Thus, by (22), all polynomials of degree $\leq k$ are in $S(V)$. This is, e.g., the case for the quadratic B-spline of Example 4 (iii).

Proof of Theorem 2B. Let

$$p := \sum_{j \in \mathbf{Z}^m} a_j N_V(\cdot - j) \in \pi \cap S(V).$$

By the remark following Theorem 1B,

$$D_v p = \sum a_j \big(N_{V\backslash v}(\cdot - j) - N_{V\backslash v}(\cdot - j - v)\big) = \sum (a_j - a_{j-v}) N_{V\backslash v}(\cdot - j),$$

where it was used that v has integer coefficients. Repeating this argument,

$$D_W p = \sum (\nabla_W a)_j N_{V\backslash W}(\cdot - j). \tag{23}$$

For $W \in \Lambda$, the B-splines $N_{W\backslash V}(\cdot - j)$ have support on a set of measure zero which implies that the polynomial $D_W p$ vanishes identically, i.e. lies in the kernel of D_W.

For the converse statement we first prove that

$$L := \bigcap_{W \in \Lambda} \ker D_W \subset \pi. \tag{24}$$

Fix $\xi \in \mathbf{R}^m$. If $V' \notin \Lambda$, then ξ can be written as linear combination of the vectors in $V\backslash V'$,

$$\xi = \sum_{w \in V\backslash V'} a(w) w.$$

Therefore,

$$(D_\xi)^r D_{V'} = (D_\xi)^{r-1} \sum a(w) D_{V' \cup w}.$$

Iterating this identity, replacing $(D_\xi)^{r'} D_{V''}$ by a linear combination of $(D_\xi)^{r'-1} D_{V'' \cup w'}$, $w' \in V\backslash V''$, one arrives at

$$(D_\xi)^r = \Big(\sum_{V' \in \Lambda,\ \#V' \le r} a(V')(D_\xi)^{r - \#V'} D_{V'}\Big) + \Big(\sum_{\substack{V' \subset V,\ V' \notin \Lambda \\ \#V' = r}} a(V') D_{V'}\Big) \tag{25}$$

with certain coefficients $a(V')$.

This proves (24) since, for $r > \#V$, the second sum on the right hand side of (25) is empty and the derivatives in the first sum vanish on functions in L.

To complete the proof of the Theorem we show by induction on r that

$$\pi_r \cap L \subset S(V)$$

where π_r denotes the space of polynomials of degree $\le r$. For the induction step, we prove that

$$p \in \pi_r \cap L \qquad \text{implies} \qquad q := p - \sum_j p(j) N_V(\cdot - j) \in \pi_{r-1} \cap L.$$

By (23),

$$D_W(p-q)=\sum_j(\nabla_W p)(j)N_V(\cdot-j).$$

If $W\in\Lambda$, then $D_W p=0$ and by (18) also $(\nabla_W p)(j)=(\Delta_W p)(j')=0$. which shows that $p-q\in L$.

By (25) and since $q\in L$ (i.e. $D_{V'}q=0$ for $V'\in\Lambda$),

$$(D_\xi)^r q=\sum_{\substack{V'\subset V,\ V'\notin\Lambda\\ \#V'=r}}a(V')\big(D_{V'}p-\sum_j(\nabla_{V'}p)(j)N_{V\setminus V'}(\cdot-j)\big).$$

Since p is a polynomial of degree $\le r$, $D_{V'}p=\nabla_{V'}p$, and since $\sum_j N_{V\setminus V'}(\cdot-j)=1$ it follows that $(D_\xi)^r q=0$.

From Theorem 2B one can derive error estimates for approximation by box splines. Moreover, the result in useful for studying approximation order for piecewise polynomials on regular triangulations. For this and further results, the reader is referred to the work by de Boor and the author [BH82/3, $BH83_{1,2}$] and Dahmen and Micchelli [$DM84_1$, $DM85_1$].

Surface Approximation

As pointed out in section 3, box splines are natural generalizations of tensor product splines. The underlying triangular meshes yield more flexibility in the choice of degree and smoothness while some of the attractive computational features of tensor products are maintained. In this section a simple approximation scheme is described and shape preserving properties of box spline expansions are discussed.

Denote by N_V^h the (bivariate) B-spline corresponding to a grid of meshsize h and, slightly changing the notation of the previous section, assume that N_V^h is centered at 0, i.e

$$N_V^h(x):=N_V(x/h-\xi_V)\tag{26}$$

where $\xi_V:=\sum_{v\in V}v/2$ is the center of the B-spline N_V defined in (16). Moreover, denote by N_*^h the piecewise linear B-spline corresponding to the directions $V_*:=\{(1,0),(0,1),(1,1)\}$.

In the following it is always assumed that V contains V_*. This excludes tensor product splines and certain degenerate cases where the translates of the B-splines N_V are not linearly independent and therefore is no significant loss of generality.

Define the approximation scheme

$$f\to S_V^h f:=\sum_{j\in\mathbf{Z}^2}f(jh)N_V^h(\cdot-jh).\tag{27}$$

which is a generalization of Schoenberg's univariate variation diminishing spline approximant. In particular, if $V = V_*$, then $S_*^h f$ is the piecewise linear interpolant to f with respect to the triangulation of $\mathbf{R}^2$ which is generated by the three directions $(1,0)$, $(0,1)$ and $(1,1)$.

Proposition 2. The method (27) is second order accurate, i.e.

$$f(x) - (S_V^h f)(x) = O(h^2) \tag{28}$$

for any smooth function f.

Proof. The piecewise linear interpolant $S_*^h f$ is second order accurate. Therefore, arguing by induction, it is sufficient to show that the estimate (28) remains valid if a vector w is added to the set of vectors V. From (20′) and (26) one sees that

$$N_{V\cup w}^h(x) = h^{-1}(N_w^h * N_V^h)(x) := \int_{-1/2}^{1/2} N_V^h(x - \lambda h w)\,d\lambda \tag{29}$$

which implies

$$S_{V\cup w}^h f = h^{-1} N_w^h * S_V^h f. \tag{30}$$

Write the left hand side of (28) in the form

$$(f - h^{-1}N_w^h * f)(x) + (h^{-1}N_w^h * (f - S_V^h f))(x).$$

The second term is of order h^2 since convolution by $h^{-1}N_w^h$ does not increase the maximum norm. The first term equals

$$\int_{-1/2}^{1/2} (f(x) - f(x - \lambda h w))\,d\lambda.$$

Adding $0 = \int_{-1/2}^{1/2}(D_w f)(x)\lambda h\,d\lambda$ to this expression, it follows that this term is also of order $O(h^2)$.

Obviously, S_V^h is a positive operator, i.e., if f is nonnegative, then so is $S_V^h f$. Moreover, S_V^h preserves monotonicity and convexity which is made more precise below.

Proposition 3 [DM85$_2$, G85].

(i) If, for some $\xi \in \mathbf{R}^2$, $D_\xi S_*^h f$ is nonnegative, then so is $D_\xi S_V^h f$.

(ii) If $S_*^h f$ is convex, then so is $S_V^h f$.

The piecewise linear spline $S_*^h f$ is called the "control polygon" of the box spline $S_V^h f$. It interpolates the B-spline coefficients at the points $j \in \mathbf{Z}^2$. The Proposition states that the box spline has roughly the same "shape" as its control polygon which is a desirable feature for design purposes.

The **Proof** of Proposition 3 is quite simple: It follows from the identity (30) and the observation that convolution with a positive kernel preserves monotonicity and convexity.

E.g., for the proof of (ii) assume by induction that $S_V^h f$ is convex. Then, for $x = \sum_\nu \varrho(\nu) x_\nu$ with $\sum_\nu \varrho(\nu) = 1$ and $\varrho(\nu) \geq 0$,

$$(S_{V\cup w}^h f)(x) = \int_{-1/2}^{1/2} (S_V^h f)((\sum_\nu \varrho(\nu) x_\nu) - \lambda h w) d\lambda$$
$$\leq \sum_\nu \int (S_V^h f)(x_\nu - \lambda h w) d\lambda = \sum_\nu (S_{V\cup w}^h f)(x_\nu).$$

In principle, box splines can be evaluated via the recurrence relation of Theorem 1B (ii). However, for approximate evaluation as is required, e.g. for rendering techniques, algorithms based on subdivision techniques are considerably faster. For box splines such algorithms have been developed independently by Böhm [Bö83], Cohen, Lyche and Riesenfeld [CLR84], Dahmen and Micchelli [$\mathrm{DM84}_2$] and Prautsch [P83/84]. The idea can be described as follows.

A box spline $\sum_j a_V^h(j) N_V^h(\cdot - jh)$ can be rewritten as a linear combination of the B-splines $N_V^{h/2}(\cdot - jh/2)$ corresponding to a refined grid, i.e.

$$\sum_j a_V^h(j) N_V^h(x - jh) = \sum_j a_V^{h/2}(j + \xi_V) N_V^{h/2}(x - (j + \xi_V)h/2), \tag{31}$$

where $\xi_V := \sum_{v\in V} v/2$. The shift by ξ_V is necessary only if $\xi_V \notin \mathbf{Z}^2$ since then the mesh for $N_V^{h/2}$ is not a refinement of the mesh corresponding to N_V^h. The subdivision process can be repeated and, as has been shown in [D85], the sequence of control polygons converges to the box spline at a quadratic rate. The coefficients $a_V^{h/2}$ in (31) can be computed via the following

Algorithm.

(i) Define

$$a_{V_*}^{h/2}(i) := \sum_j a_V^h(j) N_{V_*}^h(ih/2 - jh), \; i \in \mathbf{Z}^2.$$

(ii) Set $V' := V_*$.

(iii) if $V' = V$ stop

else choose $w \in V \backslash V'$ and define

$a_{V'\cup w}^{h/2}(j + \xi_{V'} + w/2) := (a_V^h(j + \xi_V) + a_V^h(j + \xi_V + w))/2$ for $j \in \mathbf{Z}^2$.

(iv) Set $V' := V' \cup w$ and go to step (iii).

Example 6. As was first observed by Böhm [Bö83], the algorithm takes on a particularly simple form if

$$V = \underbrace{V_* \cup \ldots \cup V_*}_{r \text{ times}},$$

i.e. if V contains the vectors $(1,0)$, $(0,1)$ and $(1,1)$ with equal multiplicity r. In this case three applications of step (iii) of the Algorithm can be combined which results in

$$a^{h/2}_{V'\cup V_*}(j) := \frac{1}{8}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} a^h_{V'}(j),\ j \in \mathbf{Z}^2$$

where the weights in the square matrix are applied centered at the (double) index j.

Derivation of the Algorithm. For $V = V_*$, the new coefficients are obtained by linear interpolation since the control polygon interpolates the B-spline coefficients. This explains step (i) of the algorithm. Now, one has to show that (31) remains valid if a vector w is added to the set V and the coefficients $a^{h/2}_{V\cup w}$ are computed via step (iii). Convolve both sides of (31) with $h^{-1}N^h_w$. Then, by (29), on the left hand side N^h_V is replaced by $N^h_{V\cup w}$. For the right hand side one obtains

$$\begin{aligned} h^{-1}(N^h_w * N^{h/2}_V)(x) &= \int_{-1/2}^{1/2} N^{h/2}_V(x - \lambda h w)d\lambda \\ &= (1/2)\int_{-1}^{1} N^{h/2}_V(x - \lambda(h/2)w)d\lambda \\ &= (1/2)(\int_0^1 \dots + \int_{-1}^0 \dots) \\ &= (1/2)(N^{h/2}_{V\cup w}(x - hw/4) + N^{h/2}_{V\cup w}(x + hw/4)). \end{aligned}$$

Therefore, using that $y := x - (j + \xi_V)(h/2) - (h/2)(w/2) = x - (j + \xi_{V\cup w})(h/2)$, the right hand side of (31) equals

$$\begin{aligned} &\sum_j a^{h/2}_V(j + \xi_V)\ (1/2)\ (N^{h/2}_{V\cup w}(x - (j + \xi_{V\cup w})h/2) + (N^{h/2}_{V\cup w}(x - (j + \xi_{V\cup w})h/2 + hw/2)) \\ &\quad = \sum_j\ (1/2)\ (a^{h/2}_V(j) + a^{h/2}_V(j + w))\ N^{h/2}_{V\cup w}(x - (j + \xi_{V\cup w})h/2) \end{aligned}$$

which establishes the formula for the coefficients.

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