ON THE DISTRIBUTION OF THE SINGULAR VALUES OF TOEPLITZ MATRICES

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ABSTRACT

In 1920, G. Szegõ proved a basic result concerning the distribution of the eigenvalues $\{\lambda_j^{(n)}\}$ of the Toeplitz sections $T_n[f]$ where $f(\Theta) \in L_\infty(-\pi,\pi)$ is a real-valued function. Simple examples show that this result cannot hold in the case where $f(\Theta)$ is not real valued. In this note, we give an extension of this theorem for the singular values of $T_n[f]$ when $f(\Theta) = f_0(\Theta)\, R_0(\Theta)$ with $f_0(\Theta)$ real-valued and $R_0(\Theta)$ continuous, periodic (with period 2π) and $|R_0(\Theta)| = 1$. In addition, we apply the basic theorem of Szegõ to resolve a question of C. Moler.

1. INTRODUCTION

The results in this note were motivated by a question raised by Cleve Moler at the Second SIAM Conference on Linear Algebra, Raleigh, NC, 1985. Consider the matrix

$$A = (a_{ij}) \tag{1.1}$$

with

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$$a_{ij} = \frac{1}{j - i + 1/2} \quad , \quad i,j = 1,2,\ldots,N \tag{1.2}$$

with N a number ~ 30. Using Matlab, Moler computed the singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N \geq 0$ of this matrix. The remarkable result is most of these singular values (say, the first 20 when $N = 30$) were equal to $\pi - \epsilon$ with ϵ very small. In the case $N = 20$ the singular values (to four decimal places) are

$\sigma(1) = \sigma(j) = 3.1416 \quad j = 1,2,\ldots 14$.

$\sigma(15) = 3.1415$

$\sigma(16) = 3.1407$

$\sigma(17) = 3.1323$

$\sigma(18) = 3.0631$

$\sigma(19) = 2.6463$

$\sigma(20) = 1.1705$

In Section 2, we give a qualitative explanation of this phenomena. This discussion is based on a theorem of Szegõ [6] concerning the asymptotic distribution of the eigenvalues of the Toeplitz matrices $T_n[f]$ where $f(\Theta)$ is a real-valued bounded measurable function which is periodic with period 2π. Simple examples show that a similar theorem for the case where $f(\Theta)$ is not real valued is impossible. In Section 3, we prove an interlacing theorem for singular values. While this theorem is stated in more general terms than one finds in the literature (see [2], page 286) the proof is essentially the proof of the interlacing theorem for Hermitian matrices. We include the proof for the sake of completeness. In Section 4, we apply this theorem to obtain extensions of the Szegõ theorem to the singular values of $T_n[f]$ when f is not a real-valued function.

Let $f(\Theta) \in L_\infty(-\pi,\pi)$ and have the Fourier expansion

$$f(\Theta) \sim \sum_{-\infty}^{\infty} c_k e^{ik\Theta} \tag{1.3}$$

Let $T_n[f]$ denote the $(n + 1) \times (n + 1)$ matrix

$$T_n[f] = (t_{ij}) \ , \quad i,j = 0,1,\ldots,n \tag{1.4a}$$

with

$$t_{ij} = c_{j-1} \tag{1.4b}$$

Observe that when $f(\Theta)$ is a real-valued function

$$c_k = \bar{c}_{-k} \tag{1.5}$$

and $T_n[f]$ is a hermitian matrix.

A basic result, which is easily verified, is the following formula for the computation of inner products. Let

$$x = (x_0, x_1, \ldots, x_n)^T \ , \quad y = (y_0, y_1 \cdots y_n)^T \ . \tag{1.6a}$$

Set

$$\hat{x}(\Theta) = \sum_0^n x_k e^{-ik\Theta} \ , \quad \hat{y}(\Theta) = \sum_0^n y_k e^{-ik\Theta} \ . \tag{1.6b}$$

Let $\hat{y}^*(\Theta)$ denote the complex conjugate of $\hat{y}(\Theta)$, that is

$$\hat{y}^*(\Theta) = \sum_0^n \bar{y}_k e^{ik\Theta} \ . \tag{1.6c}$$

Then

$$y^* T_n[f] x = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{y}^*(\Theta) f(\Theta) \hat{x}(\Theta) d\Theta \ . \tag{1.7a}$$

Of course

$$y^* x = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{y}^*(\Theta) \hat{x}(\Theta) d\Theta \ . \tag{1.7b}$$

When $f(\Theta)$ is real valued, this formula yields the basic estimate; let $\lambda_{n+1}^{(n)} \leq \lambda_n^{(n)} \leq \ldots \leq \lambda_2^{(n)} \leq \lambda_1^{(n)}$ be the eigenvalues of $T_n[f]$, then

$$m \leq \lambda_j^{(n)} \leq M \tag{1.8a}$$

where

$$m = \inf f(\Theta) \quad , \quad M = \sup f(\Theta) \quad . \tag{1.8b}$$

Another basic result is the following distribution theorem.

Theorem I (Szegö). Let $f(\Theta) \in L_\infty[-\pi,\pi]$ be real valued. Let m, and M be as in (1.8b). Let $F(\lambda) \in C[m,M]$. Then

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{j=1}^{n+1} F(\lambda_j^{(n)}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(f(\Theta))d\Theta \quad . \tag{1.9}$$

Moreover, for any fixed $j \geq 1$,

$$\lambda_j^{(n)} \to M \quad , \quad \lambda_{n+2-j}^{n} \to m \quad as \quad n \to \infty \quad . \tag{1.10}$$

Proof: See [3], Chapter 5, pp. 64-65. □

Remarks. Theorems on the rate of convergence in (1.10) are given in [4], [5].

In Section 4, we prove an extension of this theorem.

Theorem II. Let $f(\Theta) \in L_\infty[-\pi,\pi]$. Let

$$\sigma_1^{(n)} \geq \sigma_2^{(n)} \geq \cdots \sigma_{n+1}^{(n)} \geq 0 \quad ,$$

be the singular values of $T_n[f]$. Suppose $f(\Theta)$ can be written as

$$f(\Theta) = f_0(\Theta)R_0(\Theta) \tag{1.11a}$$

where $f(\Theta)$ is a real-valued function and $R_0(\Theta)$ is a continuous periodic function with period 2π which also satisfies

$$|R_0(\Theta)| = 1 \tag{1.11b}$$

Let

$$M = sup|f(\Theta)| \tag{1.12}$$

and let $F(\lambda) \in C[0,M]$. Then

$$\lim_{n \to N} \frac{1}{n+1} \sum_{1}^{n+1} F(\sigma_j^{(n)}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(|f(\Theta)|)d\Theta \ . \tag{1.13}$$

2. MOLER'S PROBLEM

Let A be the $N \times N$ matrix given by (1.1), (1.2). Let B be the $(2N) \times (2N)$ hermitian matrix given by

$$B = \frac{1}{i}\begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix} \ , \quad i = \sqrt{-1} \ . \tag{2.1}$$

Since B is a hermitian matrix, its singular values are merely the absolute values of its eigenvalues. At the same time, the singular values of B are the singular values of A -- each with multiplicity 2.

Let P be the permutation on $\{1,2,\ldots,2N\}$ given by

$$P(j) = 2j - 1 \ , \quad j = 1,2,\ldots,N \ , \tag{2.2a}$$

$$P(N + j) = 2j \ , \quad j = 1,2,\ldots,N \ . \tag{2.2b}$$

Let $\mathbb{P}$ be the associated permutation matrix. Let

$$\mathbb{P}^T B \mathbb{P} = D \tag{2.3}$$

Then a direct, but detailed, calculation shows that

$$D = T_{2N-1}[g] \tag{2.4a}$$

where

$$g(\Theta) = \sum_{k=-\infty}^{\infty} \frac{1}{(2k-1)i} e^{(2k-1)i\Theta} \tag{2.4b}$$

and, in fact, $g(\Theta)$ is the "square wave" given by

$$g(\Theta) = \begin{cases} -\pi, & -\pi < \Theta < 0 \\ \pi, & 0 < \Theta < \pi \end{cases} \tag{2.4c}$$

Remarks. To obtain (2.4a), (2.4b), it is easiest to make the change of variables

$$x_j = y_{2j-1} \ , \quad j = 1,2,\ldots,N$$

$$x_{N+j} = y_{2j} \ , \quad j = 1,2,\ldots,N \ .$$

To obtain (2.4c), one can calculate or check any elementary text, e.g., see problem 3, page 64 of [1]. Then for any $\epsilon > 0$ we see that only $o(N)$ of the eigenvalues of $T_{2N-1}[g]$ satisfy

$$\left| |\lambda_j^{(2N-1)}| - \pi \right| > \epsilon \ .$$

To see this, we merely need apply Theorem I with $F(\lambda) = |\lambda|$. Then

$$|\lambda_j^{(2N-1)}| \leq \pi \ . \tag{2.5a}$$

and

$$\underset{N\to\infty}{Lim} \frac{1}{2N} \sum_{j=1}^{2N} |\lambda_j^{(2N-1)}| = \pi \ , \tag{2.5b}$$

Thus, "most" of the singular values of A are "close" to π. Another remark which is relevant to the limit relations (1.10): the estimates of [5] show that, for fixed j and every integer $r \geq 1$, there is a constant $C_{r,j}$ such that

$$|\lambda_j^{(2N-1)} + \pi| \leq \frac{C_{r,j}}{N^{2r}} \quad , \tag{2.6a}$$

$$|\lambda_{2N+1-j}^{(2N-1)} - \pi| \leq \frac{C_{r,j}}{N^{2r}} \quad . \tag{2.6b}$$

3. AN INTERLACING THEOREM

Let $B = B^*$ be an $n \times n$ hermitian matrix. Let B_k be the $(n-1) \times (n-1)$ hermitian matrix obtained from B by deletion of the kth row and column. Let $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$ be the eigenvalues of B and let $b_1 \leq b_2 \leq \cdots \leq b_{n-1}$ be the eigenvalues of B_k. Then, as is well known,

$$\beta_1 \leq b_1 \leq \beta_2 \leq b_2 \leq \cdots \; b_{n-1} \leq \beta_n \quad .$$

For our current purpose, we prefer to restate this theorem as follows.

Theorem 3.1. Let $S \leq \mathbb{C}_n$ be a $(n - r)$ dimensional subspace of C_n, the complex n dimensional vector space. Let P be the orthogonal projection onto S. Let

$$B' = PBP \quad . \tag{3.1}$$

Then B' is an hermitian matrix and, viewed as an operator from S to S has eigenvalues $b_1 \leq b_2 \leq \cdots \leq b_{n-r}$, and

$$\beta_k \leq b_k \leq \beta_{k+n} \quad , \quad k = 1,2,\ldots,n-r \quad . \tag{3.2}$$

Proof: The proof follows exactly as the proof of the well-known theorem cited above. We merely observe that S is characterized by r linearly independent vectors y_1, y_2 ... y_r which are orthogonal to S. Then the proof follows the argument given in [7; section 47, page 103]. □

Corollary 1. Let A be a $m \times n$ complex matrix. Let $\overline{m} = \min(m,n)$ and let

$$\sigma_1 \geq \sigma_2 \cdots \geq \sigma_{\overline{m}} \geq 0$$

be the singular values of A. Let P be the projection above and let

$$A' = AP \ .$$

Then A' is an operator from S to $\mathbb{C}_m$ and has singular values $a_1 \geq a_2 \geq \cdots \geq a_l \geq 0$ where $l = \min(m, n - r)$. Finally

$$\sigma_k \geq a_k \geq \sigma_{k+r} \ , \quad k = 1,2,\ldots,\overline{m} - r \ . \tag{3.3}$$

Proof: The values $(\sigma_k)^2$ are the eigenvalues of A^*A while the values $(a_k)^2$ are the eigenvalues of $(A')^*(A') = P^*A^*AP$. The corollary now follows from the theorem. □

Corollary 2: Let A and P be as in Corollary 1. Let $T \supseteq \mathbb{C}_m$ be an $m - \rho$ dimensional subspace of $\mathbb{C}_m$. Let Q be the orthogonal projection of $\mathbb{C}_m$ onto T. Let

$$B = QAP \ .$$

Then B is an operator from S to T and has singular values $b_1 \geq b_2 \geq > \cdots \geq b_\mu \geq 0$ where $\mu = \min(n - r,\ m - \rho)$. Let

$$r_0 = \max(r, \rho) \ .$$

Then

$$\sigma_k \geq b_k \geq \sigma_{k+r+\rho} \ , \quad k = 1,2,\ldots,\overline{m} - 2r_0 \ . \tag{3.4}$$

Proof: The singular values of A^* are the singular values of A. In particular, the values $(a_k)^2$ are also the eigenvalues of $(A')(A')^*$ while the values $(b_k)^2$ are the eigenvalues of $(QA')\,(QA')^*$. That is, the values $(b_k)^2$ are the eigenvalues of $Q[A'(A')^*]Q^*$. Hence, applying Corollary 1,

$$a_k \geq b_k \geq a_{k+\rho} \ . \tag{3.5}$$

Then, using (3.3) we have

$$\sigma_k \geq a_k \geq b_k \geq a_{k+\rho} \geq \sigma_{k+\rho+r} \quad ,$$

which proves the corollary. □

4. THE DISTRIBUTION THEOREM

The asympotic distribution of the singular values of Toeplitz matrices can be expressed in the terminology of the theory of "equal distribution" (see [3, chapter 5]).

Definition: for each $n \geq 1$, we consider sets of $(n+1)$ real numbers $a(n) = \{a_k(n),\ k = 1,2,\ldots,(n+1)\}$ with $a_k(n) \geq a_{k+1}(n)$. Let $b(n) = \{b_k(n)\}$ be another set of the same kind. Assume that for all k and n

$$|a_k(n)| \leq K \quad , \quad |b_k(n)| \leq K \tag{4.1}$$

where K is a constant independent of k and n. We say that $\{a(n)\}$ $\{b(n)\}$, $n \to \infty$ are "equally distributed" in the interval $[-K,K]$ if the following holds: Let $F(t)$ be an arbitrary continuous function defined on the interval $[-K,K]$; then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum \left[f(a_k(n)] - F(b_k(n))\right] = 0 \ . \tag{4.2}$$

In our case, we may assume that $a_k(n) \geq 0$. In this case, it can show that the limit relation (4.2) holds for all continuous functions $F(t)$ if it holds for all $F(t) \in C^1[0,K]$ which also satisfy $F^1(t) \geq 0$ (see [3]).

Lemma 4.1: Let $\{a(n)\}$, $\{b(n)\}$ be two sets of real numbers which satisfy the following interlacing and positivity conditions

$$K \geq a_k(n) \geq a_k(n-1) \geq a_{k+1}(n) \geq 0 \ , \quad 1 \leq k \leq n \ , \tag{4.3a}$$

$$K \geq b_k(n) \geq b_k(n-1) \geq b_{k+1}(n) \geq 0 \ , \quad 1 \leq k \leq n \ ; \tag{4.3b}$$

and for some fixed $r_0 > 0$

$$b_k(n) \geq a_k(n - r_0) \geq b_{k+r_0}(n) \ , \quad k = 1,2,\ldots,(n + 1 - r_0) \ , \tag{4.3c}$$

Then $\{a(n)\}$ and $\{b(n)\}$ are equally distributed.

Proof.

Let $F \in C^1[-K,K]$ with $F^1(t) \geq 0$. Then it is an easy matter to show that

$$\operatorname*{Lim\,inf}_{n\to\infty} \frac{1}{n+1} \sum_{k=1}^{n} [F(b_k(n)) - F(a_k(n))] \geq 0 \tag{4.4a}$$

$$\operatorname*{Lim\,sup}_{n\to\infty} \frac{1}{n+1} \sum [F(b_k(n)) - F(a_k(n))] \leq 0 \quad \square \tag{4.4b}$$

Let $f(\Theta) \in L_\infty[-\pi,\pi]$ and have the Fourier expansion

$$f(\Theta) \sim \sum_{k=-\infty}^{\infty} c_k e^{ik\Theta} \ . \tag{4.5}$$

Let $T_{m,n}[f]$ be the $(m + 1) \times (n + 1)$ matrix

$$T_{m,n}[f] = (t_{ij}) \ , \quad i = 0,1,\ldots,m, \quad j = 0,1,2,\ldots,n \ , \tag{4.6a}$$

where

$$t_{ij} = c_{j-i} \ . \tag{4.6b}$$

If $(m - n) \leq r_1$, a fixed integer, then the results of Sec. 3 and Lemma 4.1 imply that the singular values of $T_{m,n}[f]$ are equally distributed as the singular values of $T_{n,n}[f] = T_n[f]$. Indeed, we can even allow

$$\frac{|m - n|}{\bar{m}} \to 0$$

where $\bar{m} = \min(m,n)$, that being the case, we limit ourselves to the singular values of the square matrices

$T_n[f]$.

Let $p(\Theta)$, $q(\Theta)$ be two fixed trigonometric polynomials with non-negative indices of the same order. That is

$$p(\Theta) = \sum_{k=0}^{r-1} p_k e^{ik\Theta} \ , \tag{4.7a}$$

$$q(\Theta) = \sum_{j=0}^{r-1} q_j e^{ik\Theta} \ . \tag{4.7b}$$

Let $P \in ₵_{n+1}$, $Q \in ₵_{m+1}$ be subspaces described by the conditions.

$$x \in P \Leftrightarrow \hat{x}(\Theta) = p(\Theta)\ S_{n+1-r}(\Theta) \tag{4.8a}$$

$$y \in Q \Leftrightarrow \hat{y}(\Theta) = q(\Theta)\ t_{n+1-r}(\Theta) \tag{4.8b}$$

where $S_{n+1-r}(\Theta)$ and $t_{n+1-r}(\Theta)$ are of the form

$$\sum_{j=0}^{n-r} \xi_j e^{ij\Theta} \ . \tag{4.8c}$$

As in Sec. 3, let P, Q denote the orthogonal projection onto P and Q respectively. Let

$$B_n[f,p,q] = \mathrm{Q} T_n[f] \mathrm{P} \ . \tag{4.9}$$

Remark: We have not required that $p_{r-1} \neq 0$, $q_{r-1} \neq 0$. Nevertheless, P and $Q \subset ₵_{n+1}$ and are both of dimension $(n + 1 - r)$.

We now turn to the following question. What is the relationship between the singular values of $B_n[f,p,q]$ and the singular values of $T_{n-r}[\bar{q}fp]$? We begin by recalling

Lemma 4.2: Let A be an $n \times n$ complex matrix with singular values $\sigma_1 \geq \sigma_2 > \cdots \sigma_n \geq 0$. Then

$$\sigma_k = \underset{dim S = k}{Max} \ \underset{\substack{x \in S \\ x \neq 0}}{Min} \frac{\|Ax\|_2}{\|x\|_2} \ . \tag{4.10}$$

Proof: See [2, chapter 8]. □

Corollary: Let

$$M = sup|f(\Theta)| ,$$

and let $\sigma_j^{(n)}$, $j = 1,2,\ldots n + 1$ be the singular values of $T_n[f]$. Then

$$0 \le \sigma_j^{(n)} \le M . \tag{4.10a}$$

Proof: This estimate follows from (4.10) and the basic formulae (1.7a), (1.7b) together with the fact that

$$\|T_n[f]x\|_2 = \frac{sup|y^* T_n[f]x|}{\|y\|_2} . \quad \square$$

Let $k \le n + 1 - r$. There is a one-to-one correspondence between the k dimensional subspaces S' of $\mathbb{C}_{n+1}$ and the k dimensional subspaces S' of P. For every vector $x \in S$, the vector $x' \in S'$ is determined by the relationship

$$\hat{x}'(\Theta) = p(\Theta)x(\Theta) . \tag{4.11}$$

For each such $x \in S$, we have

$$\|x\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{x}(\Theta)|^2 d\Theta , \tag{4.12a}$$

$$\|x'\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |p(\Theta)|^2 |\hat{x}(\Theta)|^2 d\Theta . \tag{4.12b}$$

We define

$$\|x\|_p^2 = \|x'\|_2^2 . \tag{4.13}$$

Similarly, each $y \in C_{n+1-r}$ is in a one-to-one correspondence with a $y' \in Q$ determined by

$$\hat{y}'(\Theta) = q(\Theta)\hat{y}(\Theta) \ . \tag{4.14}$$

As above, we have

$$\|y\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{y}(\Theta)|^2 d\Theta \ . \tag{4.15a}$$

We define

$$\|y\|_q^2 = \|y'\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |q(\Theta)|^2 \, |\hat{y}(\Theta)|^2 d\Theta \ . \tag{4.15b}$$

For every such $x \in S$, $y \in ¢_{n-r}$ we set

$$[y,x] = y^* T_{n-r} \, [\bar{q}fp]x = \frac{1}{2\pi} \int_{2\pi} \hat{y}^*(\Theta)\bar{q}(\Theta)f(\Theta)p(\Theta)\hat{x}(\Theta)d\Theta \ , \tag{4.16a}$$

We observe that $[y,x]$ can also be interpreted as

$$[y,x] = (y')^* B_n[f,p,q](x') \ . \tag{4.16b}$$

Therefore,

$$\frac{\|B_n[f,p,q](x')\|_2}{\|x'\|_2} = \sup_{y \neq 0} \frac{[y,x]}{\|y\|_q \, \|x\|_p} \tag{4.17a}$$

while

$$\frac{\|T_{n-r}[\bar{q}fp]x\|_2}{\|x\|_2} = \sup_{y \neq 0} \frac{[y,x]}{\|x\|_2 \|y\|_2} \tag{4.17b}$$

Lemma 4.3: Let $f(\Theta) \in L_\infty[-\pi,\pi]$ be of the form

$$f(\Theta) = f_0(\Theta)R_0(\Theta) \ , \tag{4.18a}$$

where $f_0(\Theta)$ is real valued and $R_0(\Theta)$ is a continuous periodic function with period 2π which satisfies

$$|R_0(\Theta)| = 1 \quad . \tag{4.18b}$$

Let ϵ, $0 < \epsilon < 1$ be given. There are polynomials $p(\Theta)$, $q(\Theta)$ of the form (4.9a), (4.9b) which satisfy

$$1 - \epsilon \leq |p(\Theta)| \leq 1 + \epsilon \quad , \quad |q(\Theta)| = 1 \quad . \tag{4.19}$$

Let $\{\alpha_k(n - r); \ k = 1,2,\ldots,n + 1 - r\}$ be the singular values of $T_{n-r}\ [\bar{q}fp]$ while $\{\beta_k(n - r),$ $k = 1,2,\ldots,n + 1 - r\}$ are the singular values of $B_n[f,p,q]$. Then

$$\frac{\beta_k}{1 + \epsilon} \leq \alpha_k \leq \frac{\beta_k}{1 - \epsilon} \quad . \tag{4.20}$$

Finally, let $\{\gamma_k(n - r), \ k = 1,2,\ldots,(n + 1 - r)]$ be the singular values of $T_{n-r}\ [f_0]$. Then

$$|\alpha_k - \gamma_k| \leq \epsilon M \tag{4.21a}$$

where

$$sup|f(\Theta)| = M \tag{4.21b}$$

Proof.

Applying Fejer's Theorem [8, pp. 89, 90] we find a trigonometric polynomial.

$$g(\Theta) = \sum_{j=-r_1}^{r_1} g_k e^{ik\Theta} \tag{4.22}$$

such that

$$|g(\Theta) - R_0^{-1}(\Theta)| < \epsilon \quad .$$

Or, since (4.20b) holds

$$|R_0(\Theta)g(\Theta) - 1| < \epsilon \quad . \tag{4.23}$$

Let

$$P(\Theta) = e^{ir_1\Theta} g(\Theta) \ , \quad q(\Theta) = e^{ir_1\Theta} \ , \quad r = 2r_1 \ . \tag{4.24}$$

Then (4.19) holds. Applying Lemma 4.2 (4.17a), and (4.17b), we have (4.20). Finally

$$f_0 - \bar{q} f_0 R_0 p = f_0[1 - gR_\infty] \ .$$

Hence

$$|f_0 - \bar{q} f_0 R_0 p| \leq M\epsilon \ .$$

Thus, (4.21) follows from standard perturbation arguments, see [2]. □

Proof of Theorem II:

Let $\{\sigma_k(n),\ k = 1,2,\ldots,n+1\}$ be the singular values of $T_n[f]$. By Corollary 2 of Theorem 3.1 and Lemma 4.1, the set $\{\sigma_k(n)\}$ and $\{\beta_k(n)\}$ are equally distributed. By (4.20) and (4.21) we see that

$$|\gamma_k(n) - \beta_k(n)| \leq 2M\epsilon \ . \tag{4.25}$$

Let $\epsilon > 0$ be given. Choose the appropriate $p(\Theta)$, $q(\Theta)$. Let $F(t) \in C^1[0,M]$ and $|F^1(t)| \leq \delta$. Then

$$[F(\sigma_k(n)) - F(\gamma_k(n))] = [F(\sigma_k(n)) - F(\beta_k(n))] + [F(\beta_k(n)) - F(\gamma_k(n))] \ .$$

Hence

$$\frac{1}{n+1} |\sum [F(\sigma_k(n)) - F(\gamma_n(n))]| \leq 2M\delta\epsilon + \tau_n$$

where

$$\frac{1}{n+1} |\sum [F(\sigma_k(n)) - F(\beta_k(n))]| = \tau_n \to 0 \ as \ n \to \infty$$

therefore

$$0 \leq Lim \begin{matrix} sup \\ inf \end{matrix} \frac{1}{n+1} |\sum[F(\sigma_n(n)) - F(\gamma_n(n))]| \leq 2M\delta\epsilon \ .$$

Hence, $\{\sigma_k(n)\}$ and $\{\gamma_n(n)\}$ are equally distributed. The Theorem now follows from Theorem I. □

5. REMARKS

Lemma 4.3 has some striking consequences. Let $\{\sigma_k^n, k = 1,2,\ldots,n+1\}$ be the singular values of $T_n[f]$. Suppose $|f(\Theta)| = f_0(\Theta)$ and

$$0 < m \leq |f(\Theta)| \leq M \ . \tag{5.1}$$

Applying Corollary 2 of Theorem 3.1, we see that

$$\sigma_k^n \geq \beta_k(n) \geq \sigma_{k+2r}^n \ , \quad k = n + 1 - 2r \ . \tag{5.2}$$

From (4.25) we see that

$$\sigma_k^m + 2M\epsilon \geq \gamma_k(n) \geq \sigma_{k+2r}^n - 2M\epsilon \ , \quad k \leq n + 1 - 2r \ . \tag{5.3}$$

However, (1.10) implies

$$m \leq \gamma_n(n) \leq M \ .$$

Hence, since $\sigma_k^n \leq M$ [see (4.10a)], we have

$$m - 2M\epsilon \leq \sigma_{n+1-2r}^n \leq \sigma_1^n \leq M \ . \tag{5.4}$$

That is, all but a finite number, at most $2r$, of the singular values of $T_n[f]$ are within $2M\epsilon$ of the range of $|f(\Theta)|$.

Example: Let $g(\Theta)$ be a real valued continuous function with period 2π, $(g(-\pi) = g(\pi))$. Let

$$f(\Theta) = e^{ig(\Theta)} \ . \tag{5.5}$$

Let $\epsilon > 0$ be given. Then for all $n \geq n_0$, all but a finite number of the singular value $\sigma_k^n(f)$ of $T_n[f]$ satisfy

$$|\sigma_k^n - 1| < \epsilon \quad . \tag{5.6}$$

One can easily verify that

$$f(\Theta) = \frac{\pi}{i} e^{-i\frac{\Theta}{2}} \cdot sgn\Theta \ , \quad -\pi \leq \Theta \leq \pi$$

is the function used by Moler. That is

$$f(\Theta) \sim \sum \frac{e^{ik\Theta}}{k + ½} \ .$$

However, because $e^{-\frac{i\Theta}{2}}$ is not continuous, we are unable to apply Theorem II or the remarks above. Hence, the trickery'' used in Section 2. It seems reasonable to conjecture that one can weaken the hypothesis of Theorem II. We do not see how to do this at this time.

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