CONVERGENCE OF CARDINAL SERIES

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Abstract. The result of this paper is a generalization of our characterization of the limits of multivariate cardinal splines. Let M_n denote the n-fold convolution of a compactly supported function $M \in L_2(\mathbf{R}^d)$ and denote by

$$S_n := \{ \sum_{j \in \mathbf{Z}^d} c(j) M_n(\cdot - j) : c \in l_2(\mathbf{Z}^d)\}$$

the span of the translates of M_n. We prove that there exists a set Ω with $vol_d(\Omega) = (2\pi)^d$ such that for any $f \in L_2(\mathbf{R}^d)$,

$$dist(f, S_n) \to 0 \quad \text{as} \quad n \to \infty,$$

if and only if the support of the Fourier transform of f is contained in $\bar{\Omega}$.

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Convergence of Cardinal Series

Carl de Boor[1], Klaus Höllig[1,2] and Sherman Riemenschneider[3]

1. Introduction. We extract the essential features of our earlier arguments [1–4] concerning the limits of box-splines as their degree tends to infinity. Somewhat surprisingly, the resulting discussion, although covering a more general situation, is very much shorter.

We start with a compactly supported (nonzero) L_2-function M on $\mathbf{R}^d$ for which the Fourier transform

$$\hat{M}(\xi) := \int M(x)\exp(-ix\xi)dx$$

satisfies

$$|\hat{M}(\xi)| = O(|\xi|^{-1}),\ |\xi| \to \infty. \tag{1}$$

With

$$M_n := M * \cdots * M$$

denoting the n-fold convolution of M, we consider approximation in L_2 from the *span*

$$S_n := \{\sum_{j\in\mathbf{Z}^d} c(j)M_n(\cdot - j)\ :\ c \in l_2(\mathbf{Z}^d)\}$$

of the integer translates of M_n. We wish to characterize the class

$$S_\infty := \{f \in L_2(\mathbf{R}^d)\ :\ \lim_{n\to\infty} dist(f, S_n) = 0\}.$$

For this we introduce the set

$$\Omega := \{\xi \in \mathbf{R}^d : |\hat{M}(\xi + 2\pi j)| < |\hat{M}(\xi)|,\ j \in \mathbf{Z}^d\backslash 0\}$$

and establish the following

Proposition. Ω is a fundamental domain, i.e.

$$\Omega \cap (\Omega + 2\pi j) = \emptyset,\ j \neq 0$$
$$\cup_j (\bar{\Omega} + 2\pi j) = \mathbf{R}^d.$$

The class S_∞ consists of functions of exponential type characterized by the set Ω.

Theorem. $f \in S_\infty$ iff $supp\, \hat{f} \subset \bar{\Omega}$.

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2. Proof of the Proposition. The assumption (1) implies that for any positive C,

$$\#\{j \,:\, |\hat{M}(\xi + 2\pi j)| \geq C\} < \infty. \tag{2}$$

Let

$$D := \{\xi \in \mathbf{R}^d \,:\, \hat{M}(\xi) \neq 0\}.$$

On D, the quotient

$$a_j(\xi) := \hat{M}(\xi + 2\pi j)/\hat{M}(\xi)$$

is well defined. In particular,

$$\Omega = \{\xi \in \mathbf{R}^d \,:\, |a_j(\xi)| < 1 \text{ for } j \in \mathbf{Z}^d\backslash 0\}.$$

Lemma. For all $\xi \in \mathbf{R}^d$ there is $j \in \mathbf{Z}^d$ such that $\xi + 2\pi j \in \bar{\Omega}$.

Proof. Since $\hat{M}$ is an entire function, it is sufficient to prove this for $\xi \in D$. The set

$$J(\xi) := \{j \in \mathbf{Z}^d \,:\, |\hat{M}(\xi + 2\pi j)| = \sup_k |\hat{M}(\xi + 2\pi k)|\}$$

is finite and nonempty, by (2). Hence we are done unless $\#J(\xi') > 1$ for all ξ' in some neighborhood of ξ. In this case at least one of the real analytic functions

$$f_j - f_{j'}$$

with

$$f_k := |\hat{M}(\cdot + 2\pi k)|^2$$

vanishes on some open set, hence must vanish identically. But this implies that

$$|\hat{M}| = |\hat{M}(\cdot + 2\pi r)|$$

for $r := j - j' \neq 0$, contradicting (1).

To finish the proof of the Proposition, assume that ξ and $\xi + 2\pi j$ are both in Ω. Then, the assumption $j \neq 0$ leads to the contradiction

$$1 > 1/|\hat{M}(\xi + 2\pi j)/\hat{M}(\xi)| = |\hat{M}(\xi + 2\pi j - 2\pi j)/\hat{M}(\xi + 2\pi j)| > 1.$$

3. Proof of the Theorem. We introduce the trigonometric polynomial

$$P_n(\xi) := \sum_j M_n(j) \exp(ij\xi) = \sum_j \hat{M}_n(\xi + 2\pi j) = \hat{M}_n(\xi) \sum_j (a_j(\xi))^n$$

with the last equality holding, at least, on D. For any $j \neq 0$ and $\xi \in \Omega$,

$$|a_j(\xi)| \leq 1 - \epsilon(j, \xi) \tag{3.1}$$

for some positive $\epsilon(j, \xi)$, while, by (1) and (2),

$$|a_j(\xi)| \leq 1/(1 + C|j|) \tag{3.2}$$

for some positive C uniformly for all but finitely many j. Consequently, for $\xi \in \Omega$,

$$P_n(\xi)/\hat{M}_n(\xi) = \sum_j (a_j(\xi))^n \to 1, \ n \to \infty, \tag{4}$$

and the convergence is uniform on compact subsets Ω_1 of Ω. This shows, in particular, that, for large enough n, P_n does not vanish on such Ω_1.

(i) Assume that $f \in L_2$ and $supp \hat{f} \subset \bar{\Omega}$ and denote by χ the characteristic function of such a set Ω_1. Since Ω is a fundamental domain, we can expand $\hat{f}\chi/P_n$ in a Fourier series,

$$(\hat{f}\chi/P_n)(\xi) = \sum_j c_n(j) \exp(ij\xi), \ \ \xi \in \Omega,$$

with coefficients $c_n \in L_2$. This implies that

$$s_n := \sum_j c_n(j) M_n(\cdot - j) \ \in \ L_2.$$

Since $\hat{f}$ vanishes outside $\bar{\Omega}$,

$$|\hat{f} - \hat{s}_n|^2_{L_2(\mathbf{R}^d)} = |\hat{f} - \hat{s}_n|^2_{L_2(\Omega)} + \sum_{j \neq 0} |\hat{s}_n(\cdot + 2\pi j)|^2_{L^2(\Omega)}.$$

The first term is estimated by

$$|\hat{f} - \hat{s}_n|_{L_2(\Omega)} \leq |\hat{f} - \chi\hat{f}|_{L_2(\Omega)} + |\chi\hat{f} - \chi\hat{f}\hat{M}_n/P_n|_{L_2(\Omega)}.$$

The first norm on the right hand side is small if Ω_1 is chosen close to Ω. For fixed Ω_1, the second norm is small by (4) if n is sufficiently large.

For the terms in the sum it follows from (2) and (3.$*$) that

$$\begin{aligned} |\hat{M}_n(\cdot + 2\pi j)(\hat{f}\chi/P_n)|_{L_2(\Omega)} &= |(a_j)^n \hat{M}_n(\hat{f}\chi/P_n)|_{L_2(\Omega)} \\ &\leq (|a_j|_{L_\infty(\Omega_1)})^n |\hat{M}_n/P_n|_{L_\infty(\Omega_1)} |\hat{f}|_{L_2(\Omega_1)} \ \to \ 0, \ n \to \infty. \end{aligned}$$

(ii) Assume that $s_n = \sum_j c_n(j) M_n(\cdot - j)$ converges to f in L_2. From

$$\hat{s}_n(\xi + 2\pi j) = (a_j(\xi))^n \hat{s}_n(\xi), \ \ \xi \in D,$$

we see that for $j \neq 0$

$$|\hat{s}_n|_{L_2(\Omega_1 + 2\pi j)} \leq (|a_j|_{L_\infty(\Omega_1)})^n \ |\hat{s}_n|_{L_2} \to 0.$$

It follows from (3.$*$) that, as an element of L_2, $\hat{f}$ vanishes off $\bar{\Omega}$.

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