C^∞-REGULARITY FOR THE POROUS MEDIUM EQUATION

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ABSTRACT

The equation

$$u_t = (u^m)_{xx},\ x \in \mathrm{R},\ t > 0$$
$$u(\cdot, 0) = u_0$$

with $m > 1$ models the expansion of a gas or liquid with initial density u_0 in a one dimensional porous medium. Denote by $t \to s_\pm(t)$ the vertical boundaries of the support of u. Caffarelli and Friedman have shown that $s_\pm \in C^1(t_\pm, \infty)$ where $t_\pm := \sup\{t : s_\pm(t) = s_\pm(0)\}$ is the waiting time. Using their result we prove that

$$s_\pm \in C^\infty(t_\pm, \infty).$$

Moreover, we show that the pressure $v := u^{m-1}$ is infinitely differentiable up to the free boundaries $s_\pm$ after the waiting time. Our proof is based on a priori estimates in weighted norms which reflect the regularizing effect near the free boundaries.

AMS (MOS) Subject Classifications: 35K55, 35R35

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C^∞-REGULARITY FOR THE POROUS MEDIUM EQUATION

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1. Introduction. We consider the porous medium equation

$$\begin{aligned} &u_t - (u^m)_{xx} = 0,\ x \in \mathrm{R},\ t > 0, \\ &u(\cdot, 0) = u_0 \end{aligned} \tag{1}$$

for $m > 1$ and continuous positive initial data u_0 with connected compact support.

It is well known [3,9,10] that problem (1) has a unique weak solution and that the support of $u(\cdot, t)$ remains bounded for all t, i.e.

$$\operatorname{supp} u(\cdot, t) = [r(t), s(t)].$$

The curves r, s are Lipschitz continuous [7], but in general not C^1. As was first observed by Aronson [1] r' (and similarly s') can have a jump for t equal to

$$t_r := \sup\{t : r(t) = r(0)\}.$$

Caffarelli and Friedman [4] proved that a classical solution of problem (1) exists up to the free boundaries for $t > \max(t_r, t_s)$. By considering the equation for $v := u^{m-1}$ (cf. (2.1) below) they showed that

(i) v_t, v_x, v_{xx} are continuous on the set $\Omega_r := \{(x,t) : r(t) \leq x < s(t),\ t > t_r\}$
(ii) $r \in C^1(t_r, \infty)$
(iii) $r'(t) = -\frac{m}{m-1} v_x(r(t), t),\ t > t_r$.

The corresponding statement holds for the right free boundary s. In particular, the functions in (i) are continuous on the closed support of u if

$$v_0'(r(0))\ v_0'(s(0)) \neq 0 \tag{2}$$

where $v_0 := v(\cdot, 0)$. With the aid of an interesting idea of Gurtin, McCamy and Socolovsky [5] it has been recently shown [6] that $r \in C^\infty(0, T]$ if v_0 is sufficiently smooth, (2) holds and T is sufficiently small. However, this method does not yield regularity of v.

In this paper we obtain the following optimal regularity result.

Theorem. $v \in C^\infty(\Omega_r),\ r \in C^\infty(t_r, \infty)$.

Our approach is different from the method in [6]; it is based on the smoothing effect of the porous medium equation in a neighborhood of the free boundaries. We prove in section 2 the following a priori estimate.

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Proposition 1. Let u be a solution of (1) for which $v \in C^\infty(\Omega_r)$ and assume that

$$\begin{aligned} &s(0) - r(0) < \kappa^{-1} \\ &\kappa < v_0'(r(0)),\ |v_0'| < \kappa^{-1} \\ &|v_0'(r(0)+y) - v_0'(r(0))| < \lambda(y),\ y \leq \kappa, \end{aligned} \tag{3}$$

where κ is a positive constant and λ is a smooth function with $\lambda(0) = 0,\ \lambda' \geq 0$. Then, for any $k \in \mathrm{N}$, there exist positive constants δ, T, A such that

$$|r|_{k,[T/2,T]} \ + \ |v|_{k,\Omega(\delta,T)} \ \leq \ A \tag{4}$$

where $\Omega(\delta,T) := \{(x,t) : r(t) \leq x \leq r(t) + \delta, T/2 \leq t \leq T\}$ and $|\ |_{k,\Omega}$ denotes the norm on $W^k_\infty(\Omega)$. The constants δ, T, A depend on κ, λ, k; in addition, T, A depend on $|v_0|_{2k+4,[r(0)+\delta/2,r(0)+\kappa]}$.

In section 3 we show existence of smooth solutions for smooth data.

Proposition 2. If $v_0 \in C^\infty(\operatorname{supp} v_0)$ and (2) holds, then $v \in C^\infty(\operatorname{supp} v)$ and $r \in C^\infty(0,\infty)$.

The Theorem follows from Propositions 1,2 by an approximation argument. Assume that $\bar{u}$ is a solution of problem (1). By the result of Caffarelli and Friedman, (i)–(iii) are valid for $\bar{v}$ and $\bar{r}$. Let $t_{\bar{r}} < t_1 < t_2$. For any $\tau \in [t_1, t_2]$, $v_0 := \bar{v}(\cdot,\tau)$ satisfies the assumptions (3) of Proposition 1 with a constant κ and a modulus of continuity λ which depend on $\bar{v}, t_1, t_2$ but not on τ. For each (fixed) τ we approximate v_0 by a sequence of smooth functions $v_{0,j} \in C^\infty(\operatorname{supp} v_0)$ for which (3) remains uniformly valid and which converge to v_0 in $\mathrm{L}_\infty(\operatorname{supp} v_0)$. In addition we require that (2) holds for $v_{0,j}$ and

$$\begin{aligned} &\operatorname{supp} v_{0,j} = \operatorname{supp} v_0 \\ &v_{0,j}(x) > 0,\ r(0) < x < s(0), \\ &\sup_j |v_{0,j}|_{2k+4,[r(0)+\delta/2,r(0)+\kappa]} \ < \ \infty. \end{aligned} \tag{5}$$

Let $(v_j)^{1/(m-1)}$ denote the solutions of (1) with initial data $u_0 = (v_{0,j})^{1/(m-1)}$. By Proposition 2, $v_j \in C^\infty(\operatorname{supp} v_j)$. Moreover, the conclusion (4) of Proposition 1 is valid for v_j and the corresponding left free boundary r_j, uniformly in j. Passing to the limit $j \to \infty$ it follows that

$$\begin{aligned} &r \in W^k_\infty[\tau + T/2, \tau + T] \\ &v \in W^k_\infty(\{(x,t) : r(t) \leq x \leq r(t) + \delta,\ \tau + T/2 \leq t \leq \tau + T\}). \end{aligned}$$

Since $k \in \mathrm{N}$, $\tau \in [t_1, t_2]$ were arbitrary and in the interior of $\operatorname{supp} v$ the regularity is known, the Theorem follows.

2. A priori estimates.

Troughout this section we assume that u is a solution of (1.1) for which v satisfies the assumptions of Proposition 1. Substituting $u = v^{1/(m-1)}$ in (1.1) we obtain

$$\begin{aligned} &v_t - mvv_{xx} - nv_x^2 = 0 \\ &v(\cdot,0) = v_0 \end{aligned} \tag{1}$$

where $n := 1/(m-1)$. The change of variables

$$y = x - r(t),\ v(x,t) = w(y,t)$$

transforms the left free boundary to the vertical axis $\{y = 0\}$. Since by (iii)

$$y_t = -r'(t) = nw_y(0,t)$$

the problem for w is

$$\begin{aligned} &w_t - mww_{yy} - nw_y^2 + nw_y(0,\cdot)w_y = 0 \\ &w(\cdot,0) = w_0 := v_0(\cdot + r(0)). \end{aligned} \tag{2}$$

For the proof of Proposition 1 it is sufficient to show that

$$|\partial_y^j w|_{0,[0,\delta]\times[T/2,T]} \ \le\ A',\ j \le 2k. \tag{3}$$

We need several auxiliary Lemmas.

Lemma 1. $\int_0^\delta f(y)^2 \text{ dy} \ \le\ c_1 \int_0^\delta y^2(\delta^{-2} f(y)^2 + f'(y)^2) \text{ dy}.$

Proof. By scaling we may assume that $\delta = 1$. Then,

$$\begin{aligned} \int_0^1 f^2 &= f(1)^2 \ -\ 2\int_0^1 yff' \\ &\le f(1)^2 \ +\ 1/2 \int f^2 \ +\ 2\int y^2(f')^2, \end{aligned}$$

where the first term on the right hand side can be estimated by the standard Sobolev inequality.

Lemma 2. $\sup_{0\le y\le\delta} |yf(y)^2| \ \le\ c_2 \int_0^\delta y^2(\delta^{-2} f(y)^2 + f'(y)^2) \text{ dy}.$

Proof. Again, by scaling, let $\delta = 1$. Then,

$$\begin{aligned} zf(z)^2 &= f(1)^2 \ -\ \int_z^1 f(y)^2 + 2yf(y)f'(y) \text{ dy} \\ &\le f(1)^2 \ +\ 2\int f^2 \ +\ \int y^2(f')^2, \end{aligned}$$

and the Lemma follows from Lemma 1 and the standard Sobolev inequality.

Lemma 3. Let $Q(\delta,T) := [0,\delta]\times[0,T]$, $\partial Q := [0,\delta]\times\{0\}\cup\{\delta\}\times[0,T]$ and assume that $p := \min_{\partial Q} w_y > 0$. Then

$$\min_{\partial Q} w_y \le \min_Q w_y \le \max_Q w_y \le \max_{\partial Q} w_y.$$

Proof. Set $\eta(t) := (p-\epsilon)\exp(-\epsilon t)$ with $0 < \epsilon < p$. We differentiate (2) with respect to y and subtract $\eta' + \epsilon\eta = 0$. This yields

$$[w_{yt} - \eta_t] + [-mww_{yyy}] + [((-m-2n)w_y + nw_y(0,\cdot))w_{yy}] + [-\epsilon\eta] = 0.$$

Assume that $w_y(\tilde{y},\tilde{t}) = \eta(\tilde{t})$ where

$$\tilde{t} := \sup\{t : w_y(\cdot,t) > \eta(t)\}.$$

If $(\tilde{y},\tilde{t}) \in Q \backslash \partial Q$ all terms in square brackets are nonpositive. Since $\eta \neq 0$ this is not possible, i.e. we must have $\eta < w_y$ on Q. Letting $\epsilon \to 0$ proves the first inequality of the Lemma and the last inequality is proved similarly.

Lemma 4. If $2\delta < \kappa$, $\lambda(2\delta) < \kappa/4$, then there exist constants T and c_3 which depend on $\kappa, \delta, k, |v_0|_{2k+4,[\delta/2,\kappa]}$ such that

$$\begin{aligned} &\max_{Q(\delta,T)} w_y - \min_{Q(\delta,T)} w_y \leq 4\lambda(\delta) \\ &\kappa/2 \leq w_y(y,t) \leq 2\kappa^{-1},\ (y,t) \in Q(\delta,T), \\ &|\partial_t^\nu \partial_y^\mu w(\delta,t)| \leq c_3,\ 2\nu+\mu \leq 2k+3,\ t \leq T. \end{aligned} \tag{4}$$

Proof. The maximum principle is valid for problem (1.1), i.e. $u_0^- \leq u_0^+$ implies that $u^- \leq u^+$ and $r^- \geq r^+$. By (1.3) and our assumption on δ,

$$v_0'(y) > 3\kappa/4,\ y - r(0) \leq 2\delta.$$

Using this and (1.3),

$$\begin{aligned} v_0^- &:= \max\{0,(y-r(0))(r(0)+2\delta-y)/2\} \leq v_0 \leq \\ &\max\{0,(y-r(0))(r(0)+4\kappa^{-1}-y)\} =: v_0^+. \end{aligned}$$

For the solutions of (1.1) with initial data $u_0^\pm = (v_0^\pm)^{1/(m-1)}$ the assertions (i)–(iii) are valid with $t_r = 0$. Therefore, by the above comparison principle,

$$\begin{aligned} c &< v(y,t) < c^{-1} \\ -c^{-1}t &< r(t) - r(0) < -ct \end{aligned}$$

if $\delta/2 \leq y \leq 3\delta/2,\ t \leq 1$. The constant c depends on δ, k. We choose $T' \leq 1$ so that

$$|r(t) - r(0)| < \delta/4,\ t \leq T',$$

which also yields

$$c < w(y,t) < c^{-1} \text{ if } 3\delta/4 \leq y \leq 5\delta/4,\ t \leq T'.$$

On the rectangle $[3\delta/4, 5\delta/4] \times [0,T']$ the problem (2) is nondegenerate and the last inequality in (4) follows from parabolic regularity theory if $T \leq T'$ [8]. We set $T := \min\{T', \lambda(\delta)/c_3\}$. Then

$$|w_y(\delta,t) - w_y(\delta,t')| \leq \frac{\lambda(\delta)}{c_3}|w_{yt}(\delta,t'')| \leq \lambda(\delta)$$

which yields the first two inequalities for $(y,t) \in \partial Q$ and therefore, in view of Lemma 3, also for $(y,t) \in Q$.

Proof of Proposition 1. Let $0 = T_{-1} < T_0 < \ldots < T_{2k+1} = T/2$. We prove by induction on l that for sufficiently small δ,

$$\max_{T_l \le t \le T} \int_0^\delta y\partial_y^{l+1} w(y,t)^2 \text{ dy } + \int_{T_l}^T \int_0^\delta y^2 \partial_y^{l+2} w(y,t)^2 \text{ dydt } \le A''(l), \quad 0 \le l \le 2k+1. \quad (5)$$

The constants A'' depend on $\kappa, \delta, \lambda, k, T_\nu, |v_0|_{2k+4,[\delta/2,\kappa]}$. By Lemma 1,

$$\begin{aligned} |\partial_y^j w(\cdot,t)^2|_{0,[0,\delta]} &\le c_\delta \int_0^\delta \partial_y^j w(\cdot,t)^2 + \partial_y^{j+1} w(\cdot,t)^2 \\ &\le c_\delta c_1 \delta^{-2}(A''(j-1) + 2A''(j) + A''(j+1)) \end{aligned}$$

which shows that (5) implies (3).

Since w and w_y are bounded, inequality (5) is obviously valid for $l = -1$. We assume that (5) holds for $l < j$ and set $W_l(y,t) := \partial_y^{j+1} w(y, t + T_{j-1})$. Differentiating (2) $(j+1)$ times with respect to y and replacing t by $t + T_{j-1}$ we obtain

$$(W_j)_t - mW_{-1}W_{j+2} - ((2n + (j+1)m)W_0 - nW_0(0,\cdot))W_{j+1} - \sum_{\substack{1 \le \nu \le \mu \le j \\ \nu+\mu = j+1}} c_{\nu\mu} W_\nu W_\mu = 0 \quad (6)$$

where $c_{\nu\mu}$ are constants which depend on j. We multiply (6) by $t^2 y W_j$ and integrate over the interval $[0, \delta]$,

$$\begin{aligned} \frac{1}{2}\Big(\int_0^\delta t^2 y W_j^2 \text{ dy}\Big)_t + m\int_0^\delta t^2 y W_{-1} W_{j+1}^2 \text{ dy} = \\ \int t y W_j^2 \\ + m t^2 \delta W_{-1}(\delta,t) W_{j+1}(\delta,t) W_j(\delta,t) \\ - m \int t^2 (yW_{-1})_y W_{j+1} W_j \\ + \int t^2 y[(2n + (j+1)m)W_0 - nW_0(0,\cdot)]W_{j+1}W_j \\ + \sum c_{\nu\mu} \int t^2 y W_\nu W_\mu W_j. \end{aligned} \quad (7)$$

The third term on the right hand side of (7) equals

$$-mt^2(W_{-1}(\delta,t) + \delta W_0(\delta,t))W_j(\delta,t)^2/2 + m\int t^2(W_0 + yW_1/2)W_j^2.$$

Proceeding similarly with the fourth term on the right hand side and using (1.3) and (4) we deduce from (7) that

$$\begin{aligned} \frac{1}{2}\Big(\int_0^\delta t^2 y W_j^2\Big)_t + \frac{m\kappa}{2}\int_0^\delta t^2 y^2 W_{j+1}^2 \le \\ c_4 c_3^2 + \int t y W_j^2 \\ - \int t^2[-mW_0 + (n + (j+1)m/2)W_0 - \frac{n}{2}W_0(0,\cdot)]W_j^2 \\ + c_5 \max_{\substack{1 \le \nu \le \mu \le j \\ \nu+\mu = j+1}} \Big|\int t^2 y W_\nu W_\mu W_j\Big| \end{aligned} \quad (8)$$

where the constant c_4 depends on κ and the constant c_5 depends on j. We estimate each of the integrals appearing on the right hand side of (8) seperately. By the definition of W_l and the induction hypothesis

$$\int_0^{\delta} tyW_j(y,t)^2 \text{ dy } \le \\ \epsilon \int t^2 W_j(y,t)^2 \text{dy} \; + \; \epsilon^{-1} \int y^2 \partial_y^{j+1} w(y, t+T_{j-1})^2 \text{ dy}. \tag{9}$$

By (4), $|W_0(y,t) - W_0(0,t)| \le 4\lambda(\delta)$ and $\kappa/2 < W_0(0,t) < 2\kappa^{-1}$. Therefore the term in square brackets in the second integral on the right hand side of (8) can be estimated by

$$\begin{aligned}[\ldots] &\ge \begin{cases} -c_6', & \text{if } j = 0 \\ n\kappa/4 - c_6'\lambda(\delta), & \text{if } j > 0 \end{cases} \\ &\ge n\kappa/4 - c_6\lambda(\delta) - \max(0, 1-j)c_6 \end{aligned} \tag{10}$$

where c_6 depends on j, κ. Finally we estimate $|\int t^2 y W_\nu W_\mu W_j|$. Set $\tilde{W}_0(y,t) := W_0(y,t) - W_0(0,t)$. Integrating by parts and using (4) it follows that

$$\begin{aligned} &|\int t^2 y W_1 W_j^2| \le \\ &t^2 \delta \tilde{W}_0(\delta,t) W_j(\delta,t)^2| + |\int t^2 \tilde{W}_0 W_j^2| + 2|\int t^2 y \tilde{W}_0 W_j W_{j+1}| \le \\ &4\lambda(\delta)c_3^2 + 8\lambda(\delta)\int t^2 W_j^2 + 8\lambda(\delta)\int t^2 y^2 W_{j+1}^2 \end{aligned} \tag{11}$$

if $\delta, t \le 1$. We have

$$|\int t^2 y W_\nu W_\mu W_j| \; \le \; \epsilon \int t^2 W_j^2 + \epsilon^{-1} B_{\nu\mu}(t) \tag{12}$$

where $B_{\nu\mu}(t) := \int t^2 y^2 W_\nu^2 W_\mu^2$. If $\nu \le \mu < j$ it follows from Lemma 2 that

$$\begin{aligned} B_{\nu\mu}(t) \; &\le \; t^2 \big(\max_{0 \le y \le \delta} |yW_\nu(y,t)^2| \big) \times \big(\int_0^{\delta} yW_\mu(y,t)^2 \text{ dy} \big) \\ &\le \; c_2 \delta^{-2} \big(\int y^2 (W_\nu^2 + W_{\nu+1}^2) \big) \times \big(\int y W_\mu^2 \big). \end{aligned}$$

Therefore, using the induction hypothesis,

$$\int_0^{T - T_{j-1}} B_{\nu\mu}(t) \text{ dt} \le c_2 \delta^{-2} (A''(\nu - 1) + A''(\nu)) \times A''(\mu) \le c_7 A''(j-1)^2. \tag{13}$$

Combining the estimates (9–12) it follows from (8) that

$$\begin{aligned} \frac{1}{2}\Big(\int t^2 y W_j^2\Big)_t \ + \ \frac{m\kappa}{2}\int t^2 y^2 W_{j+1}^2 \ \le \\ c_4 c_3^2 + \epsilon \int t^2 W_j^2 + \epsilon^{-1} b(t) \\ - \big(n\kappa/4 - c_6\lambda(\delta) - \max(0, 1-j)c_6\big)\int t^2 W_j^2 \\ + c_5\Big(4\lambda(\delta)c_3^2 + 8\lambda(\delta)\int t^2 W_j^2 + 8\lambda(\delta)\int t^2 y^2 W_{j+1}^2\Big) \\ + c_5\Big(\epsilon\int t^2 W_j^2 + \epsilon^{-1}\max_{\substack{1\le\nu\le\mu<j\\ \nu+\mu=j+1}} B_{\nu\mu}(t)\Big) \end{aligned} \tag{14}$$

where $b(t) = \int y^2 \partial_y^{j+1} w(y, t+T_{j-1})^2$ dy. We choose δ, ϵ so that

$$8c_5\lambda(\delta) \le \frac{m\kappa}{4}$$
$$\epsilon + c_6\lambda(\delta) + 8c_5\lambda(\delta) + c_5\epsilon \le n\kappa/4.$$

Then we obtain from (14) that

$$\begin{aligned} \frac{1}{2}\Big(\int t^2 y W_j^2\Big)_t \ + \ \frac{m\kappa}{4}\int t^2 y^2 W_{j+1}^2 \ \le \\ c_4 c_3^2 \ + \ \epsilon^{-1} b(t) \\ + c_6\max(0, 1-j)\int t^2 W_j^2 \\ + c_5\epsilon^{-1}\max B_{\nu\mu}(t). \end{aligned}$$

Since, induction hypothesis,

$$\int_0^{T-T_{j-1}} b(t)\ \mathrm{dt} \ \le \ A''(j-1)$$

it follows from (4) and (13) that for any $t \in [0, T-T_{j-1}]$,

$$\begin{aligned} \frac{1}{2}\int t^2 y W_j(y,t)^2\ \mathrm{dy} \ + \ \frac{m\kappa}{4}\int_0^t\int \tau y^2 W_{j+1}(y,\tau)^2 \mathrm{dyd}\tau \ \le \\ c_4 c_3^2 t + \epsilon^{-1}A''(j-1) + 4t^3\kappa^{-2} + c_5\epsilon^{-1}c_7 A''(j-1)^2 t. \end{aligned}$$

This completes the induction step.

3. Existence of smooth solutions. In this section we outline the proof of Proposition 2 which justifies the approximation argument in the introduction. Similarly as in section 2 we transform the equation (2.1) to a fixed domain. Let $\xi \in C^\infty[0,1]$ satisfy $\xi' \le 0,\ 0 \le \xi \le 1,\ \xi(y) = 1$ for $0 \le y \le \kappa,\ \xi(y) = 0$ for $2\kappa \le y \le 1$ and set $\eta(y) := \xi(1-y)$. Assuming without loss that $r(0) = 0,\ s(0) = 1$ the change of variables

$$\begin{aligned} y &= x - \xi(y)r(t) - \eta(y)(s(t)-1) \\ v(x,t) &= w(y,t) \end{aligned} \tag{1}$$

transforms the free boundaries to the vertical lines $\{y = 0\}$ and $\{y = 1\}$. One easily verifies that the transformed equation for w is

$$\begin{aligned} &w_t - (m/\chi^2)ww_{yy} - (n/\chi^2)w_y^2 + (n/\chi)\xi w_y(0,\cdot)w_y + (n/\chi)\eta w_y(1,\cdot)w_y \\ &\quad + (m\chi_y/\chi^3)ww_y = 0,\ 0 \le y \le 1,\ t \ge 0, \\ &w(\cdot,0) = w_0 := v_0 \end{aligned} \tag{2}$$

where

$$\chi(y,t) = 1 - n\xi'(y)\int_0^t w_y(0,\tau)\ \mathrm{d}\tau - n\eta'(y)\int_0^t w_y(1,\tau)\ \mathrm{d}\tau.$$

In a neighborhood of the left boundary $\{y = 0\}$ we have $\chi(y) = 1$ and equation (2) coincides with equation (2.2). Therefore an analogous a priori estimate is valid.

Lemma 5. Assume that $w \in C^\infty([0,1] \times [0,T])$ and that $w_0'(0)\, w_0'(1) \ne 0$. Then for any $l \in \mathrm{N}$

$$\Big(\max_{0\le t\le T}\int_0^1 y(1-y)\partial_y^l w(y,t)^2\ \mathrm{d}y\Big) + \Big(\int_0^T\int_0^1 y^2(1-y)^2\partial_y^{l+1}w(y,t)^2\ \mathrm{d}y\mathrm{d}t\Big) \le c \tag{3}$$

where c depends on l, T, v_0.

The **proof** of this Lemma is completely analogous to the proof of Proposition 1. Instead of multiplying equation (2.6) by $t^2 y W_j$, we multiply the corresponding equation obtained by differentiating (2) by $y(1-y)\partial_y^{j+1}w(y,t)$. Because of the weight $y(1-y)$ no boundary terms appear when the appropriate terms are integrated by parts. The estimates are somewhat more complicated because of additional terms involving χ. But, these complications are merely of technical nature.

Given the above a priori estimate it is straightforward to prove a corresponding local existence result via finite difference or finite element approximation. This completes the (outline of the) proof of Proposition 2.

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