MEDIAL-AXIS-BASED SHAPE SMOOTHING

by

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**Abstract**

Shape smoothing generally requires definitions of boundary context, curvature and prominence. We define shape smoothing as a medial-axis pruning operation. The domain of each medial-axis point is used to select a segment of the shape boundary and to measure its relative prominence. This prominence feature is then used to guide the selection of medial-axis points to be removed. An augmented medial-axis transform (MAT) is defined using a Euclidean distance transform, MAT point interpolation at non-integer coordinates, and a linking procedure which creates a thin, connected skeleton.

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## 1. Introduction

Shape smoothing is an important problem in image understanding because it is the basis for shape simplification, approximation and description. The criteria for shape smoothing must be based on measures which evaluate shape properties in a context relative to other shape features. Most current shape smoothing techniques are based on smoothing object boundaries by computing the "domain" and local curvature of every boundary point.

The main disadvantage with boundary-based methods is that a given shape feature which is "important" within one object (or piece of an object) may be "unimportant" in another object (or a second segment of the same object). That is, boundary evaluation procedures are relatively local, scale dependent, and do not incorporate a boundary segment's context with respect to its description as a complete, often closed, contour. Hence procedures for determining a boundary point's domain, curvature and significance are often very error-prone. For example, in Fig. 1 the two notches in the hourglass shape are qualitatively different (although locally identical) and therefore should be smoothed differently based on the differences between the relative shapes in the narrow "neck" region versus in the large "bulb" region of the hourglass. Similar remarks also often apply in cases in which a given boundary segment is part of two different shapes which are qualitatively very different in overall size.

Hence shape smoothing techniques which are based solely on local boundary smoothing (e.g. Gaussian smoothing [Bra82]) are not appropriate in many environments where scale-invariant, relative shape features provide a better measure of perceptual significance. Even when multiple domain sizes are used, there is no criterion for selecting the appropriate scale for each segment of the boundary.

In this paper we introduce a method for shape smoothing and shape understanding based on properties of the medial-axis transform (MAT) [Blu67]. Rather than define smoothing operations on the shape boundary, we define a procedure which operates on the MAT representation of a shape by combining features of a MAT point and its associated boundary segment to guide the selection of which MAT points are to be removed.

Shape smoothing based on the MAT is motivated by the observation that the MAT (implicitly) contains more global information about the shape than is possible to compute by searching in a local domain around each boundary point. This includes shape radius and a hierarchical representation of the convex segments of the boundary. Using these properties we define new measures of a boundary segment's prominence based on the computation of more meaningful boundary domains and their *scale* with respect to a larger boundary context.

MAT-based methods are advantageous not only because they reflect more global properties of the shape being analyzed, but also because the MAT contains symmetry information, and the skeleton topology is invariant under translation, rotation and scale change. One major disadvantage of the MAT for shape description is that it is very sensitive to noise in the shape boundary. That is, for each sharp angular change in the boundary a "spur" or "branch" is produced in the medial axis. This same argument is also sometimes used to criticize the MAT representation of a rectangle (containing a central "spine" plus pairs of spurs at each end leading to each of the four corners).

In order to reduce the effects of this problem, investigators have studied methods for smoothing the shape boundary before computation of the medial axis (see, for example, [Blu78], [Mon69], [Bad79]). Typically, this means applying some kind of local smoothing operator at each point on the object boundary. But, as stated above, boundary smoothing methods are weak because they use only local context to determine the degree of smoothing to be performed. Since our goal here is to smooth the shape by first pruning and smoothing the MAT, this class of methods is clearly inappropriate.

A few primitive MAT pruning procedures have been previously proposed for shape smoothing. Thresholding the *propagation velocity* of the points on the MAT has been suggested as one means of pruning MAT spurs and smoothing the object boundary [Dud73]. This method, however, may disconnect the MAT and can only locally detect boundary curvature.

Instead of using an absolute threshold of this type, we define the prominence of a segment of the boundary based on the local radius of the shape (i.e., the radius of the MAT disk associated with this segment) and the local smoothness of the segment. Thus a boundary segment will be labeled *noise* (not significant) if the radius of the shape in which it is contained is several orders of magnitude larger than the segment, and will be labeled *significant* if the radius of the shape is of the same order of magnitude. While this relative allowable variation along the shape boundary still requires the specification of a threshold, it is independent of the size of the shape.

Section 2 presents an iterative pruning algorithm which removes "spurs" corresponding to insignificant boundary features with respect to some MAT points. Section 3 describes the generation of the augmented medial axis which is used as the representation for the algorithm in Section 2. Section 4 presents results on several shapes. Section 5 considers the form of the smoothed MAT as a function of the pruning threshold.

## 2. The Smoothing Algorithm

The basic idea of the smoothing algorithm is to look at the boundary of a shape from the point of view of each MAT point. By definition, each point in the MAT is of minimal distance to the shape boundary in at least two points. Assume, for convenience, exactly two such points are associated with each MAT point. This pair of points partitions the (closed) shape into two segments. The segment associated with the minimum angular separation of the vectors from the MAT point to its two boundary points, defines the boundary segment of interest for the given MAT point. This segment is used because it is more "corner-like" than the other segment.

In this way each MAT point determines a boundary domain. Each segment of the boundary is traversed and the maximum distance between the segment and the circular arc defining the maximal disk for the associated MAT point is our measure of *prominence* of this boundary segment. We currently measure prominence by the maximum perpendicular distance between the boundary segment and the line segment connecting the given pair of boundary points. Fig. 2 illustrates this definition.

The *relative prominence* of a given boundary segment is defined as the ratio of prominence to the disk radius associated with the current MAT point. MAT points whose relative prominence is greater than a given threshold are marked as *not to be smoothed* (NSM); otherwise they are marked as *to be smoothed* (SM). Fig. 3 illustrates the results of this labeling process for a noisy rectangle.

After labeling all of the MAT points, we next determine which points are to be removed. As shown in Fig. 3, in the vicinity of the notch this segment of the boundary is "seen" to be prominent and these MAT points are labeled NSM. But as we move along the "ridge" towards the main spine of the MAT, points are labeled SM because from their vantage points the notch is viewed as insignificant.

To distinguish the main spines from the side spurs which are to be removed requires identifying the transitions from SM points to NSM points. This is done iteratively as follows. For each SM point with either an NSM, EL, or SM neighbor such that the radius associated with the current point is less than the radius associated with the neighbor point, re-label the current point EL *(to be eliminated)*. For each NSM point with an EL neighbor such that the radius of the current point is less than the radius of its EL neighbor, re-label the current point EL. This process propagates the EL label along an entire spur, but never along the main spine since spurs contain decreasing radius values, but the main spine does not.

This process iterates until the re-labeling terminates. Each EL point is then removed from the MAT. The smoothed shape can now be easily reconstructed by expanding each of the disks associated with the remaining MAT points. (Alternatively, the remaining MAT points could first be smoothed before reconstruction in order to more completely smooth the shape boundary.) Fig. 4 shows the labeling and smoothing of an hourglass shape.

While this method requires the specification of a threshold, the threshold specifies a scale-invariant measure of the *relative prominence* of shape features, not an absolute size parameter. Furthermore, this pruning process is not very sensitive to the choice of threshold.

Previous techniques for obtaining continuous, noise-free skeletons based on thresholding the "propagation velocity" ([Dud73], [Mon69]) have the disadvantage of not being able to distinguish prominent features from non-prominent features. Consider, for example, applying this method to the shape in Fig. 3. The MAT points along the main, "horizontal" spine will all have infinite velocities. However, the four spurs leading from the main spine to the four corners of the rectangle as well as the spur leading from the center of the main spine to the notch will be comprised of MAT points having some finite velocities. The choice of a threshold that removes the spur leading to the notch (which is a relatively non-prominent feature) will also remove the four spurs leading to the four corners (which are relatively prominent features) of the rectangle.

## 3. Computing the Medial Axis

### 3.1. Euclidean MAT

In order to obtain MATs that are connected and rotation-invariant for any simply connected shape, Euclidean distance must be used. We have implemented a variation on Danielsson's six-pass Euclidean distance transform algorithm [Dan80]. In our case we maintain at each pixel the coordinates of up to two points on the shape boundary which are of minimal Euclidean distance to the given pixel. If there are more than two boundary points of minimal distance to a pixel, then only the two points that subtend the angle closest to 180 degrees are saved. (This is sufficient since our definition of "prominence" in the previous section uses only this "best" pair of boundary points.)

First, we describe one method of generating the MAT based on the (nearly exact) Euclidean distance values computed at each pixel by Danielsson's algorithm. This method will be revised, however, in favor of a better method in Section 3.2. Since we are interested in approximating the continuous MAT, we must mark a pixel as belonging to the MAT if there is a real-valued "ridge" point in the continuous distance function within the unit-square area centered on the given pixel (which has integer coordinates).

A pixel is labeled a MAT point as follows. Consider the distance values associated with a point e and its neighbors:

| | | | |
|---|---|---|---|
| a | b | c | |
| d | e | f | g |
| h | i | j | |

If there exists a pair of opposite neighbors of e, e.g. d and f, and both neighbors have distance values less than e's distance value, then e is marked as a MAT point. In addition, if there exists a pair of opposite neighbors of e such that one neighbor, say d, has distance value less than e's distance, the other neighbor, f, has distance value equal to e, and f's neighbor in the direction opposite e, g, has distance value less than e, then e is also marked as a MAT point. Otherwise, e is not a MAT point.

The first case corresponds to the natural definition of a ridge as a local maximum in at least one direction. The second case is necessary when the true ridge point falls midway between two pixels. Our definition labels both points as MAT points, although this means that MAT skeletons can be two pixels wide.

Most of the MAT points selected by the above procedure will have two boundary points associated with them as a result of the distance labeling procedure described above. For those MAT points with only a single boundary point, we select a second boundary point from one of its neighbors. That is, the MAT point computes the direction to each of the boundary points associated with each of its eight neighbors. The boundary point which has a vector magnitude closest to the magnitude of the first vector is chosen. In cases of ties, the boundary point which forms an angle closest to 180 degrees with the original boundary point is selected as the given MAT point's second boundary point.

The MAT constructed using the above procedure has several problems. Because the final MAT representation contains only points at integer coordinates, if a real-valued MAT point is exactly halfway between two integer coordinate points, it will be represented by both of those integer coordinate points and produce a thick skeleton. This is an awkward representation of the skeleton topology since we need to be able to traverse and prune arcs of the skeleton.

The discrete MAT representation also has another serious problem. Fig. 5 shows a neighborhood of MAT points in which one "real" MAT point does not have integer coordinates. While we can approximate the position of the real MAT point by its nearest integer coordinate point, because the integer coordinate point is slightly closer to the top right boundary of the shape, its two associated MAT vectors both point to the right boundary and hence do not correctly represent the MAT vectors at the real MAT point. (MAT vectors are the two vectors connecting the current MAT point to its two associated boundary points.) That is, other nearby MAT points have one vector pointing to the top left boundary and the other pointing to the top right boundary. This local disruption of the MAT vector field will have detrimental effects on the interpretation of the MAT and the subsequent smoothing process.

In view of the above problems with discrete MATs, we implemented an *interpolation* process that will add MAT points that lie between integer coordinates. This MAT will be more robust both in terms of giving an accurate skeleton topology and correctly guiding the smoothing process.

### 3.2. MAT Interpolation

The interpolation algorithm makes use of the Euclidean distance map as well as the boundary points associated with each of the MAT points generated by the method described in the previous section. At integer coordinate points a criterion similar to, but more stringent than, the ridge point criterion used in [Dye84] is applied to identify those MAT points that coincide with digital points (Fig. 6). The number of MAT points detected this way is very small, especially for noisy shapes, because most of the MAT points occur at non-integer points.

Next, using the vectors associated with each pixel, a ridge point criterion for identifying non-integer coordinate MAT points is applied between all pairs of adjacent pixels. For every pair of 8-adjacent pixels, at most one non-integer MAT point is detected. (This is sufficient since the boundary is also discrete, defined by the line segments between adjacent boundary pixels.) If a new MAT point is found, it is given a pair of MAT vectors by combining the MAT vectors from the two nearest pixels.

After adding these interpolated MAT points, a linking algorithm is used to connect each MAT point to at most two other MAT point neighbors. This completes the construction of our augmented MAT representation.

Finally, the MAT branch pruning process described in Section 2 is applied to do the shape smoothing. The next section presents more details of this interpolation procedure.

#### 3.2.1. The Interpolation Algorithm

Before determining the coordinates of a non-integer MAT point, we first determine whether or not there exists a MAT point between each pair of 8-adjacent pixels. Fig. 6 shows the two cases of distance value cross sections that indicate the presence of a non-integer MAT point.

For each pair of pixels which contains a MAT point, we then compute the MAT point's coordinates and vectors as follows. Each of the two pixels stores either one or two vectors to the boundary. We assume that only the vectors associated with these two pixels are necessary to compute the coordinates of the real MAT point.

The line segment connecting the two pixels is divided into ten equal intervals (so that the location of the interpolated MAT point will be computed to 0.1 pixel precision). Nine interpolation points are considered on this line segment. The (up to) four vectors associated with the two pixels are "propagated" to these nine points as follows. The coordinates of the four boundary points associated with the four vectors are passed to each interpolation point. Each point then computes its own four (real-valued) vectors. From these four candidate vectors the two having the smallest magnitudes are saved.

From the nine interpolation points, select the one that has the least absolute difference between its two vectors' magnitudes. This point is added as a non-integer MAT point. The rationale behind this algorithm is very simple: the selected point most closely satisfies the definition of a MAT point (i.e., a MAT point is one that has at least two equal minimal distance vectors to the boundary).

In examining the four vectors at each interpolation point, if the boundary points of any two vectors are closer together than one pixel, then one of the vectors is discarded. This is to prevent selection of two minimal magnitude vectors that subtend a very small angle when there are better candidate vectors which subtend larger angles.

### 3.3. MAT Point Linking

The next step in computing the augmented MAT is to link each MAT point to up to two neighboring MAT points. This will produce a tree-like connected skeleton of all integer and non-integer MAT points which is traversed by the pruning procedure described earlier. The following algorithm selects one or two points from all nearby candidate MAT points (i.e. those points within chessboard distance ½ of the given MAT point).

Each MAT point's neighbors can be divided into two groups based on where a neighbor point lies with respect to the two vectors extending from the given MAT point to the boundary. A good linking scheme is one that selects one neighbor point from those in one of these regions and the second neighbor point from those in the other region. The rationale behind this rule is that this connection produces a local "straightest" possible MAT segment which is midway between two sides of the shape boundary.

To decide which neighbors to link to, each candidate neighbor will first be labeled as one of five types: 0-4. Before defining these classes, we introduce a few definitions. Let $p$ be a MAT point whose two associated boundary points are $p_1$ and $p_2$. Similarly, let $q$, $q_1$, $q_2$ denote a second MAT point and its two boundary points, respectively. MAT point $p$ is said to *include* MAT point $q$ if the triangular region defined by $q$, $q_1$, $q_2$ is contained in the triangular region defined by $p$, $p_1$, $p_2$. MAT point $p$ *excludes* MAT point $q$ if the triangular regions defined by $p$, $p_1$, $p_2$ and $q$, $q_1$, $q_2$ intersect in at most one point, at either $q_1$ or $q_2$. If MAT points $p$ and $q$ are distinct and the angles subtended by the two vectors of each point are 180 degrees and the vectors of $p$ are parallel to the vectors of $q$, then $p$ neither includes nor excludes $q$.

Using these definitions we now classify MAT neighbors into five types. Consider a MAT point $p$ and a neighboring (within chessboard distance one) MAT point $q$. If $p$ includes $q$, then $q$ is a type 1 neighbor of $p$. If $p$ does not include $q$ but $q$ includes $p$ then $q$ is a type 2 neighbor. If neither of these cases is true but $p$ excludes $q$ then $q$ is a type 3 neighbor. If $q$ is not of type 1-3 and $p$ neither includes nor excludes $q$, then $q$ is called a type 4 neighbor. Otherwise, $q$ is a type 0 neighbor.

A MAT point $p$ selects its two neighbors to link to as follows. If $p$ has at least one type 1 neighbor, then link $p$ to the nearest of these points. If $p$ has no type 1 neighbors, then link it to its closest type 4 neighbor. The second link is made to either a type 2, 3 or 4 neighbor: If there is at least one type 2 neighbor, then link $p$ to the closest one. If there are no type 2 neighbors but there are type 3 neighbors, then link $p$ to its closest type 3 neighbor. Otherwise, if $p$ has type 4 neighbors and $p$ is not already linked to a type 4 neighbor, then link $p$ to its closest type 4 neighbor. Finally, if $p$ has only type 4 neighbors then $p$ is already linked to its closest type 4 neighbor and we now link it to its second closest type 4 neighbor, if one exists.

Fig. 7 illustrates a few typical situations around MAT points. Fig. 7a illustrates a MAT point lying along a side branch. Here, the linking algorithm picks the closest type 1 neighbor and the closest type 2 neighbor. Fig. 7b shows a MAT point near a MAT junction. In this case the linking rule selects a type 1 and a type 3 neighbor. Fig. 7c illustrates a MAT point which is lying on a MAT main branch. The algorithm picks two type 4 neighbors as the point's best pair in this situation. It has been found empirically that this neighbor linking scheme is very robust in that it almost always constructs the "correct" skeleton topology.

## 4. Results

Figs. 8-12 show the effects of applying a range of pruning thresholds to several shapes. The figures show boundary pixels as small squares, MAT points as dots, MAT vectors as line segments connecting MAT points to boundary points, and MAT links as line segments connecting pairs of MAT points (the directions of these links are not shown). The results of pruning are shown with retained MAT points highlighted by thickened dots and links. The new smoothed boundary constructed from these remaining MAT points is also shown in these figures.

The shape in Fig. 8a is a rectangle with a small notch. The spur is removed at a threshold of 0.4; Fig. 8b shows the pruned MAT and Fig. 8c shows the reconstructed smooth shape. The corners of the rectangle are rounded at a threshold of 0.8 as shown in Figs. 8d and 8e.

The shape in Fig. 9 is similar to that of Fig. 8 except that it is larger and rotated approximately forty-five degrees. The smoothing effect is similar to that of Fig. 8 except that the "notch" is removed at a threshold of 0.3 because it is relatively smaller than the notch in Fig. 8. Fig. 9a shows the pruned MAT using a threshold of 0.3 and Fig. 9b shows the result using a threshold of 0.8. Figs. 8 and 9 illustrate the rotational and scale invariance of the MAT and the smoothing process.

Fig. 10a shows an "irregular" shape. The spurs of the MAT are pruned off at successive thresholds until finally the shape is smoothed into roughly a circle at threshold value 0.9 as shown in Figs. 10b and 10c.

Fig. 11a shows a noisy rectangle and the MAT points remaining using a threshold of 0.8. Notice that the spurs are removed and the reconstructed shape in Fig. 11b resembles the ovals in Figs. 8e and 9b.

An airplane shape is shown in Fig. 12a. The results of pruning the MAT using threshold 0.6 are also shown in this figure. The reconstructed shape shown in Fig. 12b captures the essential structure of the airplane (main body, wings and engines).

## 5. Concluding Remarks

We are currently investigating the properties of the pruned MAT as a function of threshold. This is of interest since the threshold is really a scale parameter for determining relative shape prominence. The number of MAT points as a function of threshold value is a monotonically non-increasing function which contains large "plateaus" between shape features of distinct scale. Thresholds selected anywhere along a given plateau will be equally good at removing shape features below this scale.

Fig. 13 shows a plot of the number of MAT points as a function of threshold for a rectangle with a noisy boundary near one corner. For each plateau the MAT and the resulting smoothed shape are also shown. This example illustrates how the method presented for detecting relative shape feature prominence can be used to rank order the spurs of the MAT and obtain successively coarser approximations to a shape.

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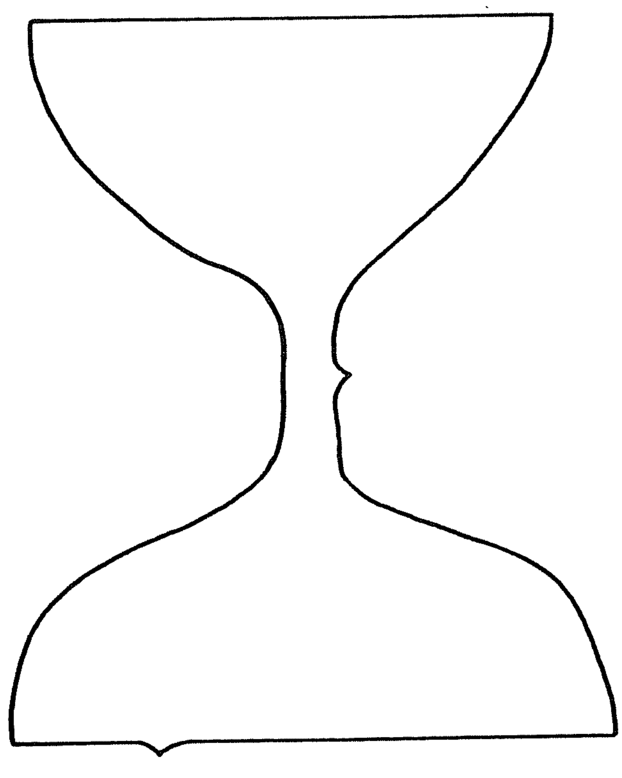


Fig. 1. An "hourglass" shape containing two "notches" of the same absolute size but appearing in two different contexts in the boundary.

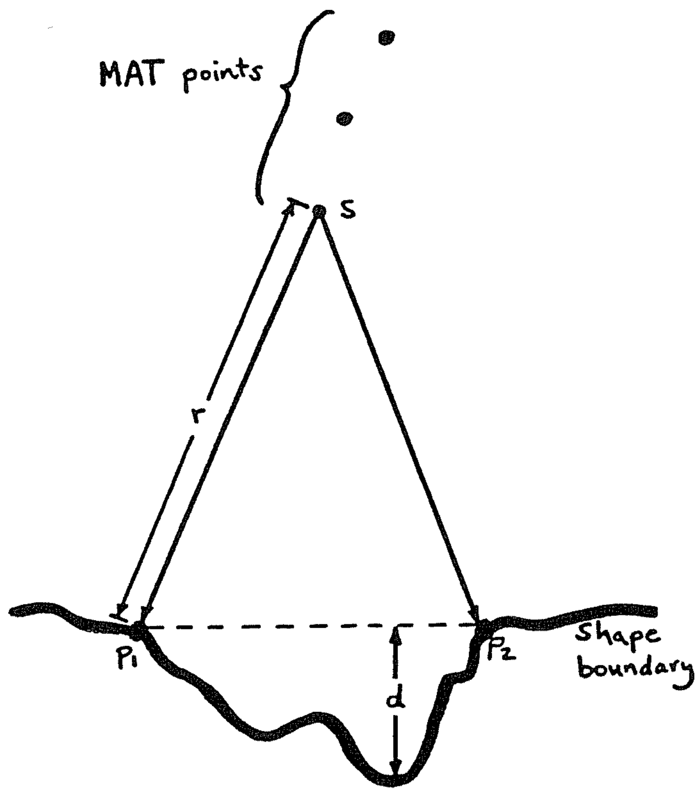



Fig. 2. The prominence of the shape boundary between points p1 and p2 is defined relative to MAT point s. The prominence is defined to be d; the relative prominence is d/r.

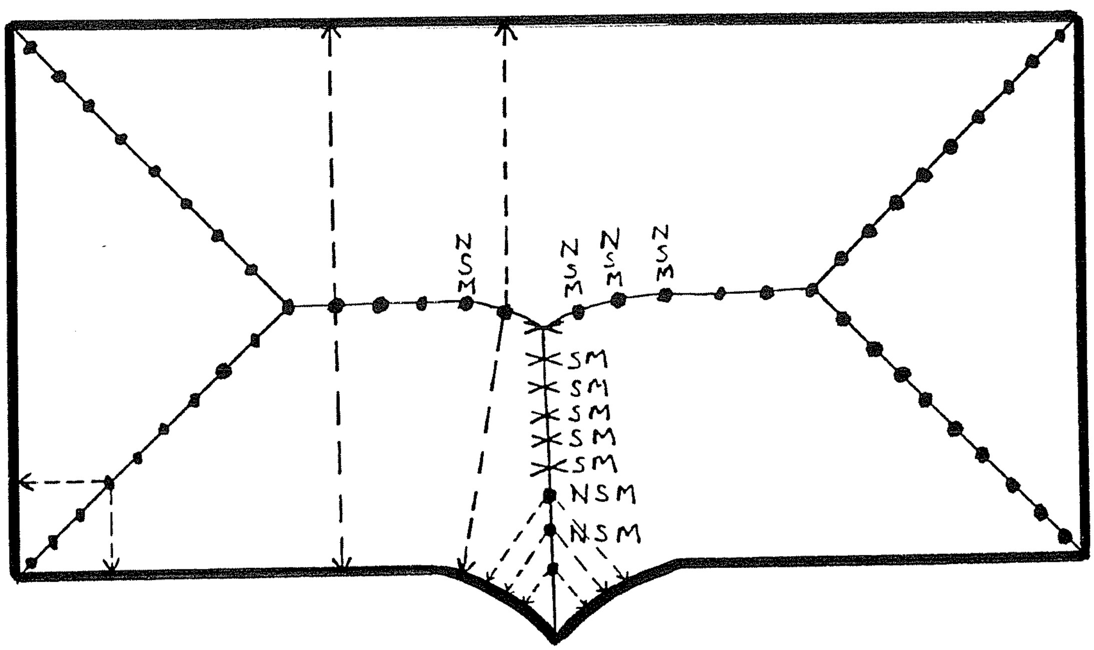



Fig. 3. MAT labeling procedure for a rectangle with notch. Threshold equal to 0.5. MAT points near notch are labeled NSM ( ● ); points on spur near main spine are labeled SM ( × ); points on main spine are labeled NSM. Dotted lines indicate the pair of boundary points associated with a given MAT point.

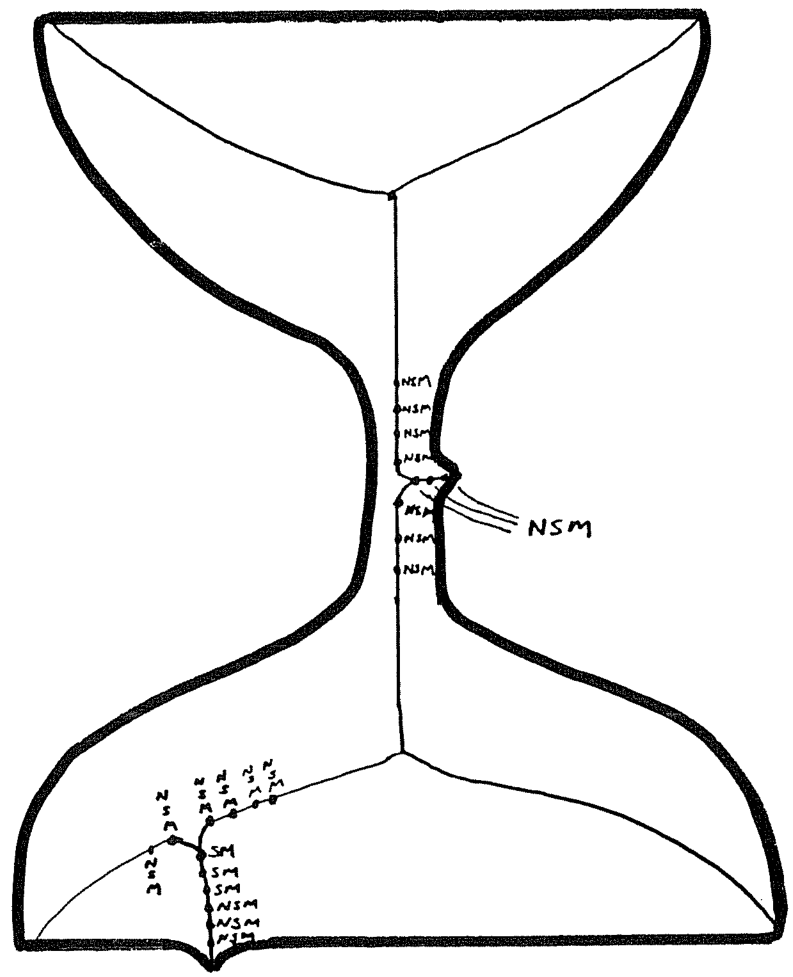



Fig. 4. MAT labeling of an hourglass shape. Threshold equal to 0.5. Note that the two notches, though locally identical, are determined to have different relative significance. The notch in the neck is marked as a prominent feature while the notch in the base is labeled noise and will be smoothed.

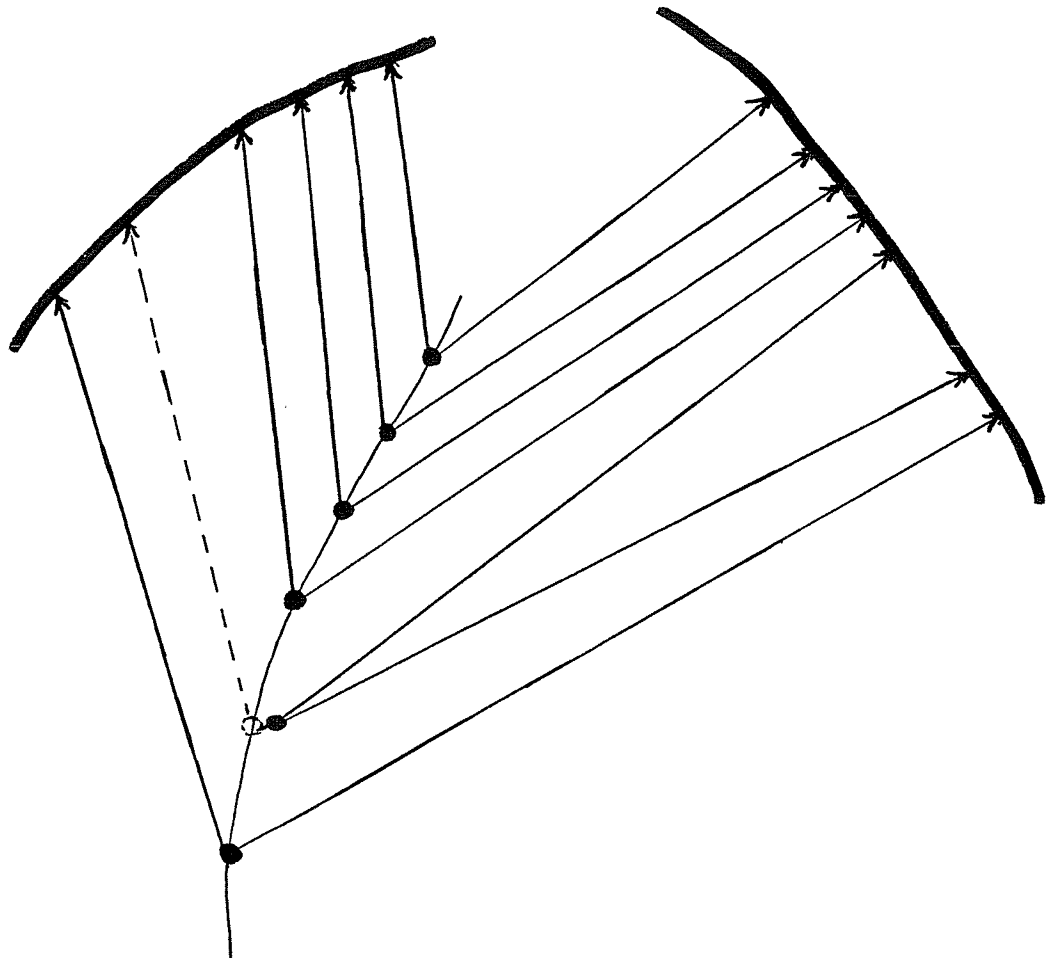


Fig. 5. Problems of forcing MAT points to lie only at integer coordinates. Continuous MAT passes through non-integer point shown as a dotted circle. Closest integer coordinates are shown as a solid circle. The pair of vectors associated with the solid point do not correctly represent the pair of vectors associated with the real MAT point (i.e. the dotted vector is lost).

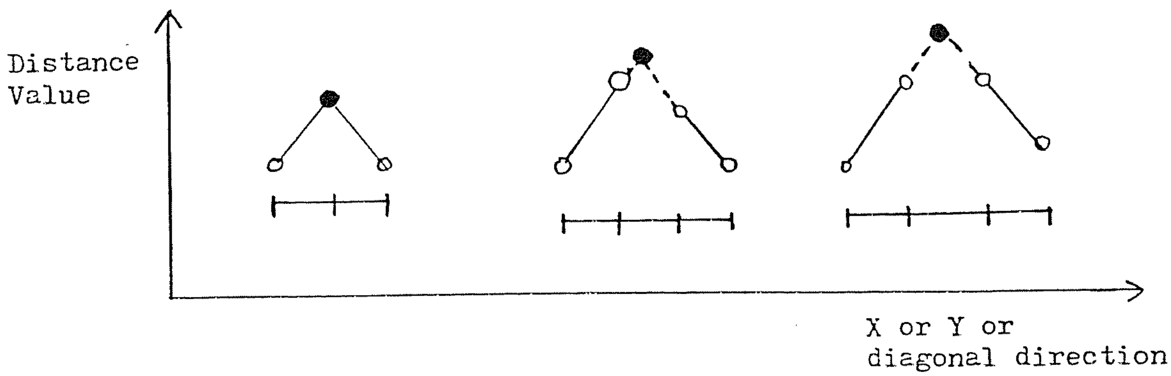



Fig. 6. The three possible distance value cross sections which are used to detect MAT points. Solid dots indicate MAT point locations. The first case implies the MAT point is at integer coordinates. The other two cases require that an interpolation procedure be used to locate a non-integer MAT point.

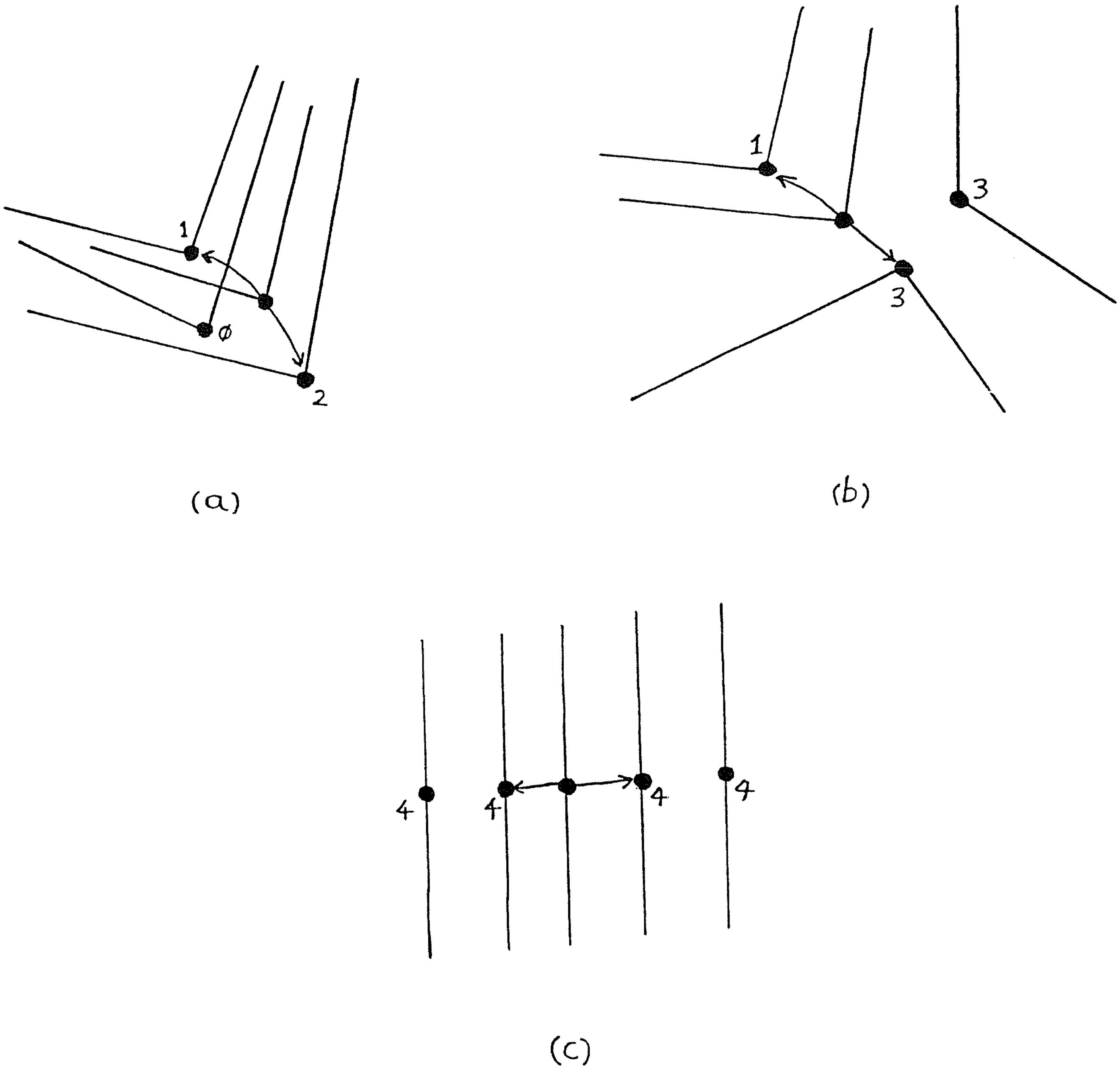



Fig. 7. Neighborhoods of MAT points used for MAT point linking. Three cases showing a MAT point, its labeled neighboring MAT points (numbered points), and the selected pair of links (arrows). Line segments associated with MAT points represent the points' vectors to the shape boundary.

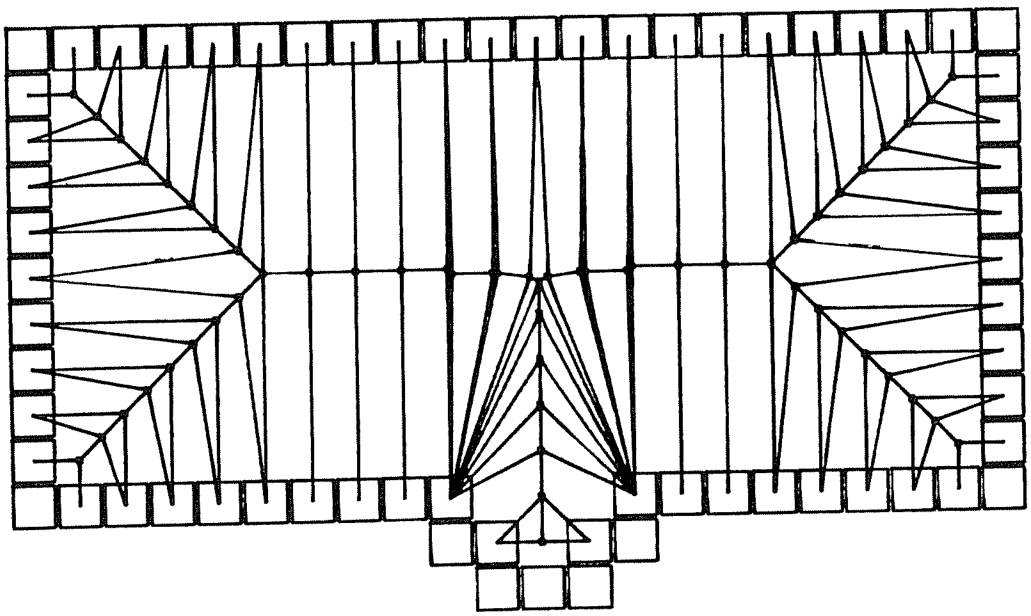
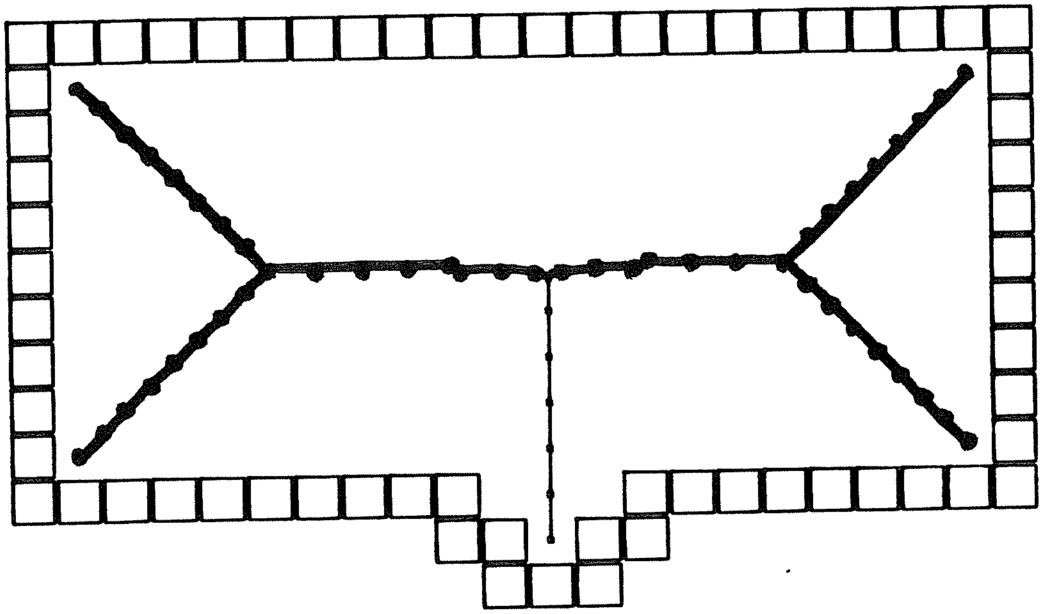
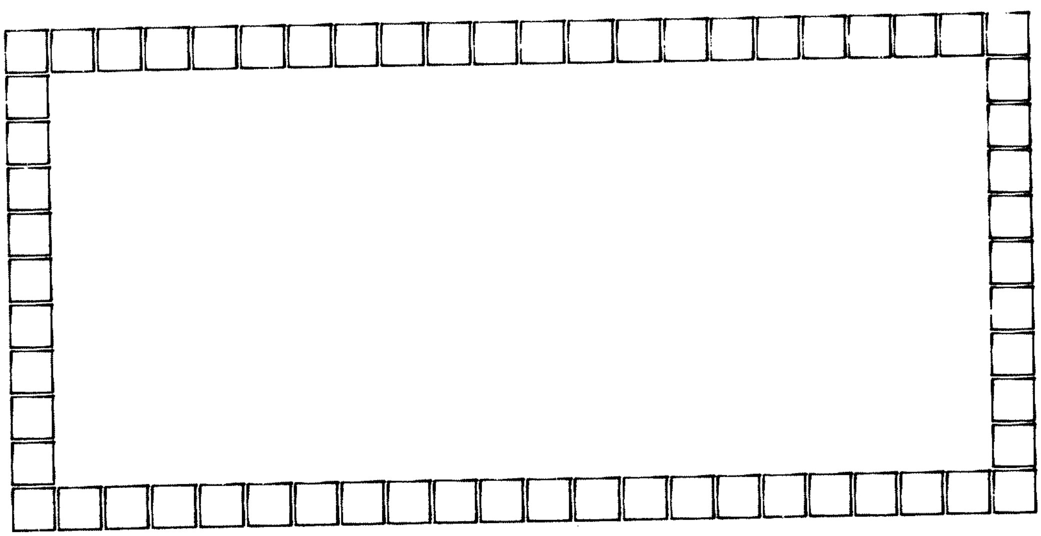


Fig. 8. (a) Rectangle with a notch. MAT points, vectors and links are also shown.

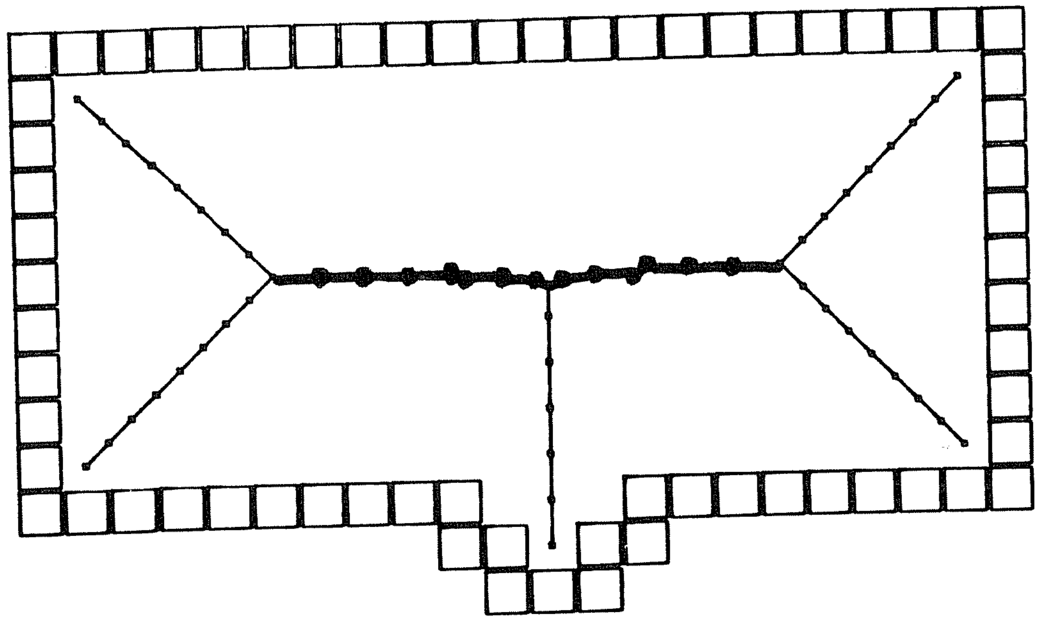


(b)

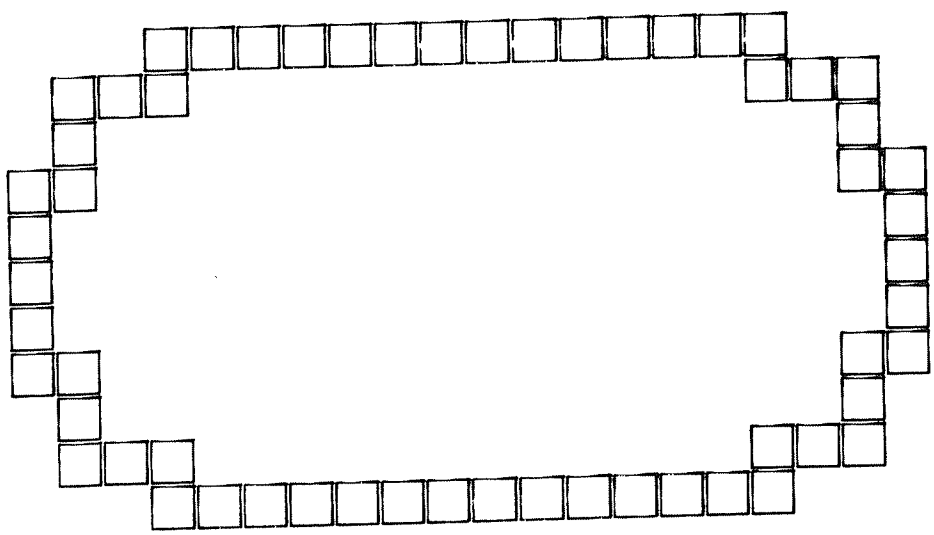


(c)

Fig. 8. (b) MAT points retained (thick lines) using threshold 0.4. (c) Reconstructed shape from MAT points remaining in (b).



(d)



(e)

Fig. 8. (d) MAT points retained using threshold 0.8. (e) Reconstructed MAT from (d).

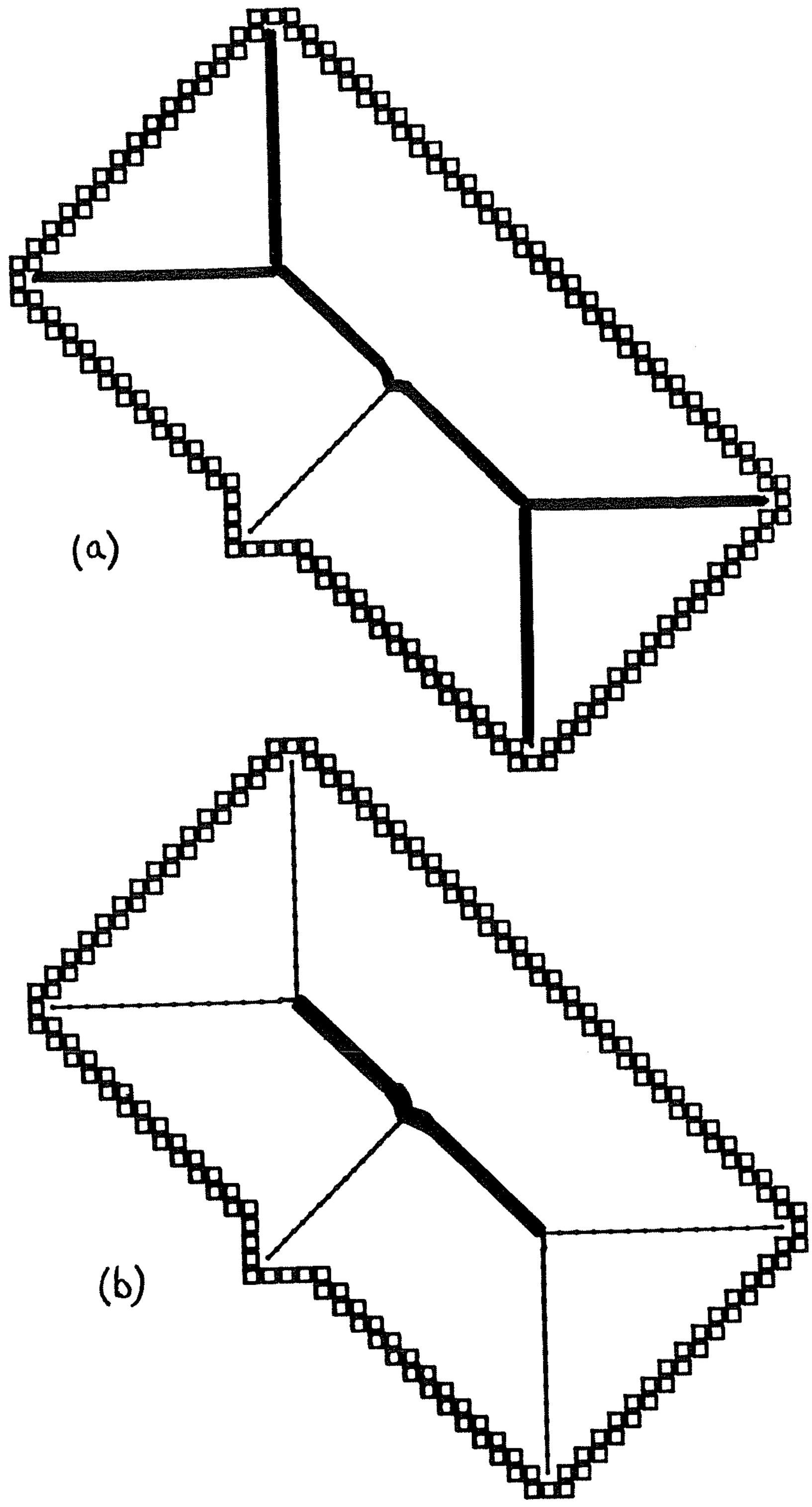



Fig. 9. Rotated rectangle with notch. (a) Threshold equal to 0.3. (b) Threshold equal to 0.8.

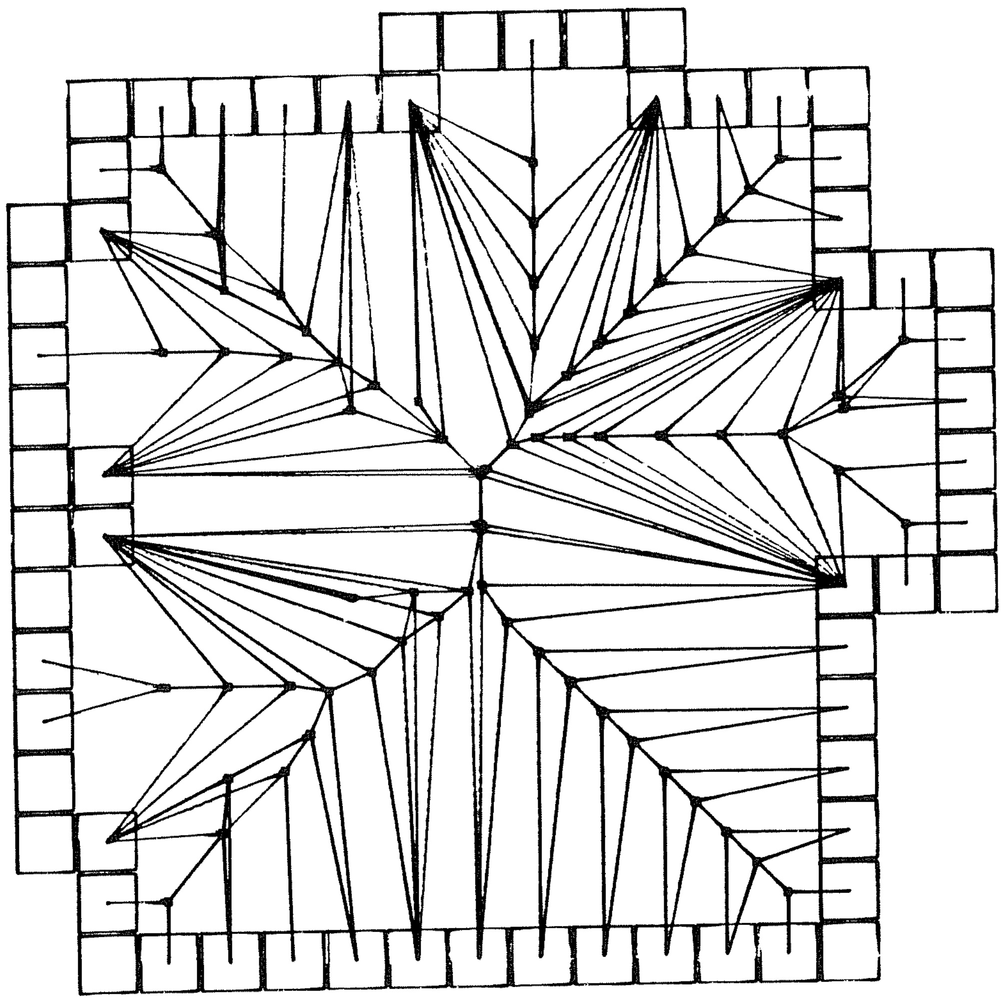


Fig. 10. (a) An irregular shape.

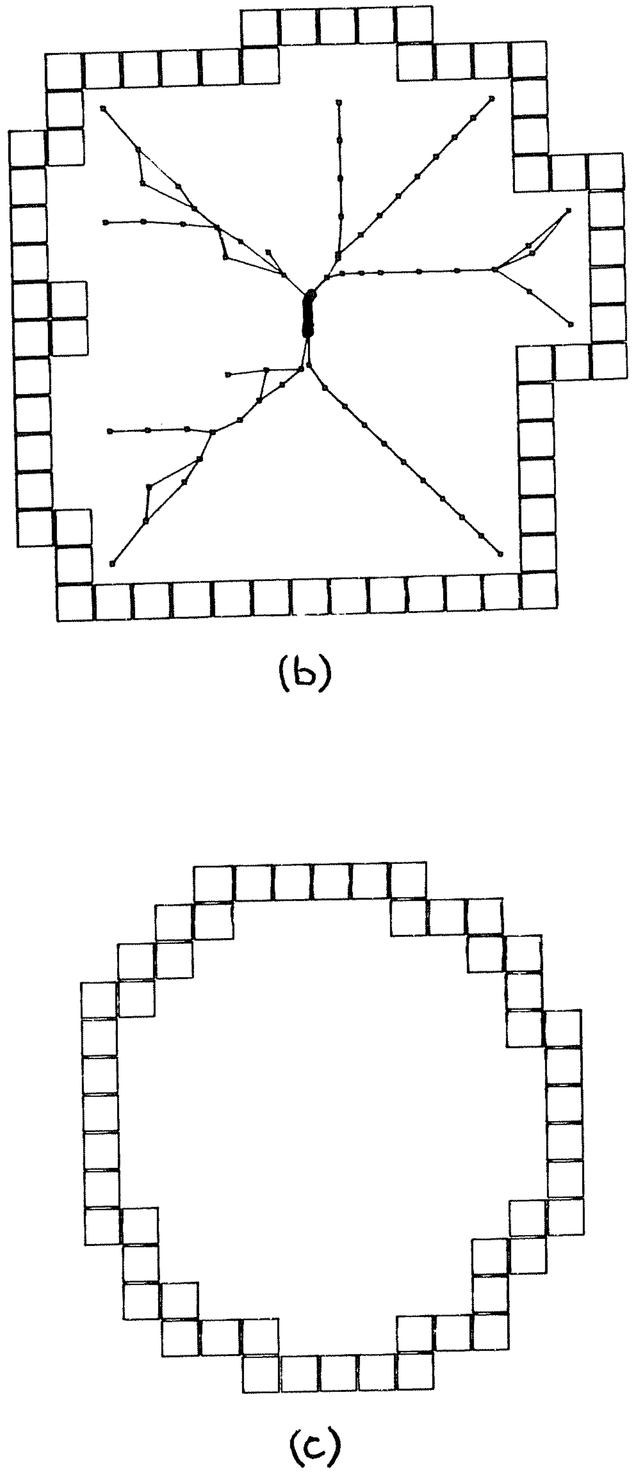


Fig. 10. (b) MAT points retained using threshold 0.9. (c) Reconstructed shape from remaining MAT points in (b).

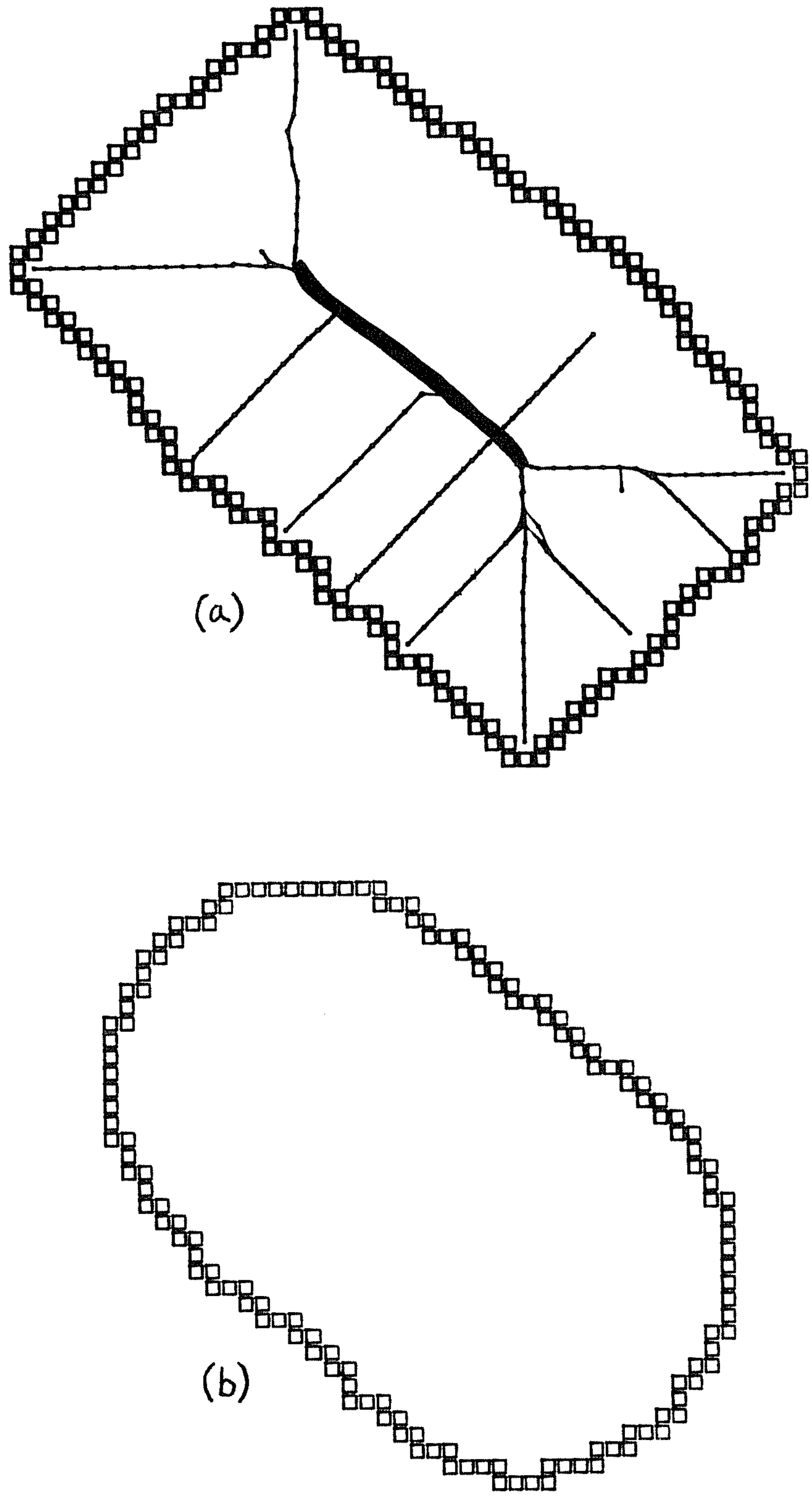



Fig. 11. (a) A noisy rectangle and MAT points saved at threshold 0.8. (b) Smoothed shape reconstructed from remaining points in (a).

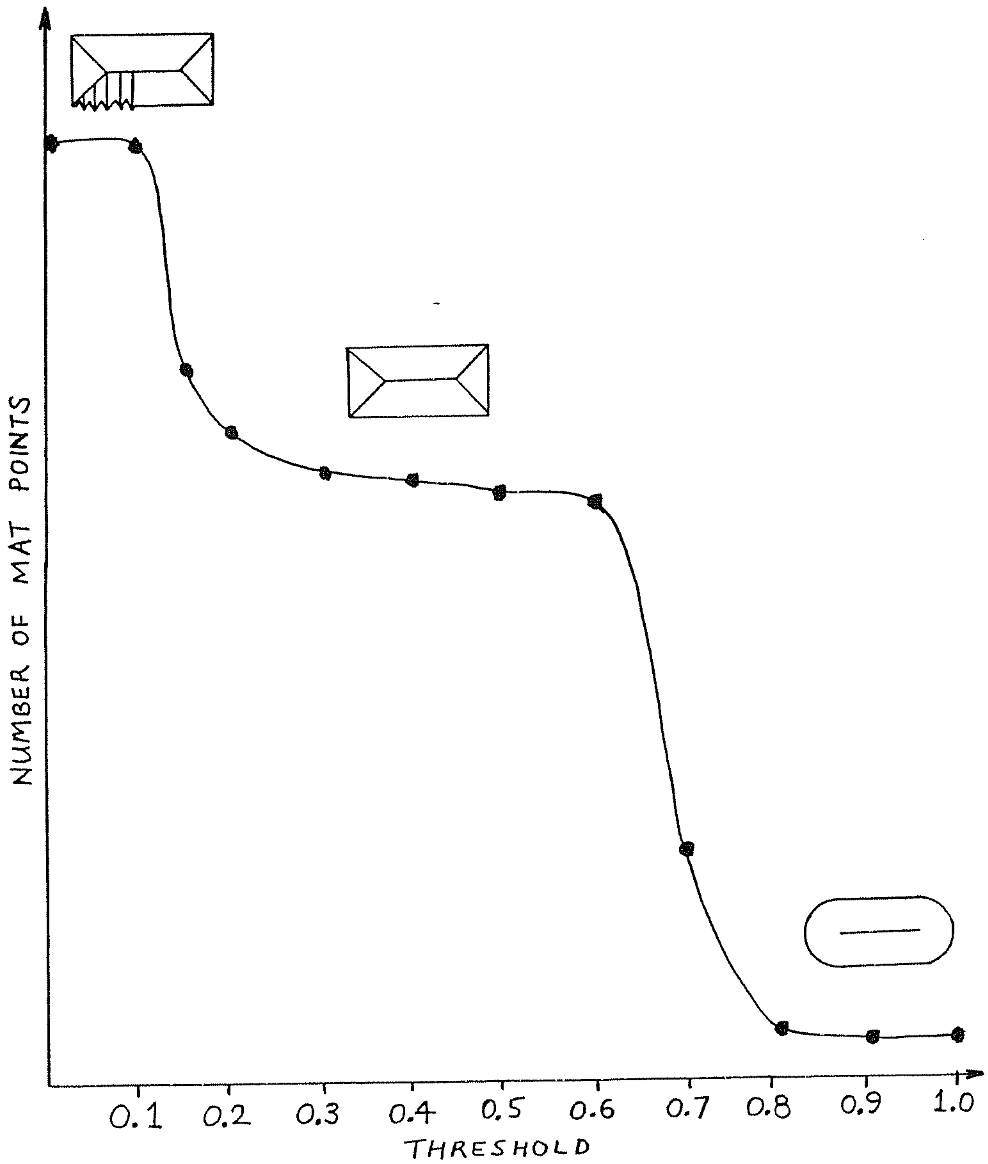



Fig. 13. The number of MAT points in the pruned MAT as a function of threshold for a rectangle with noisy boundary near one corner.

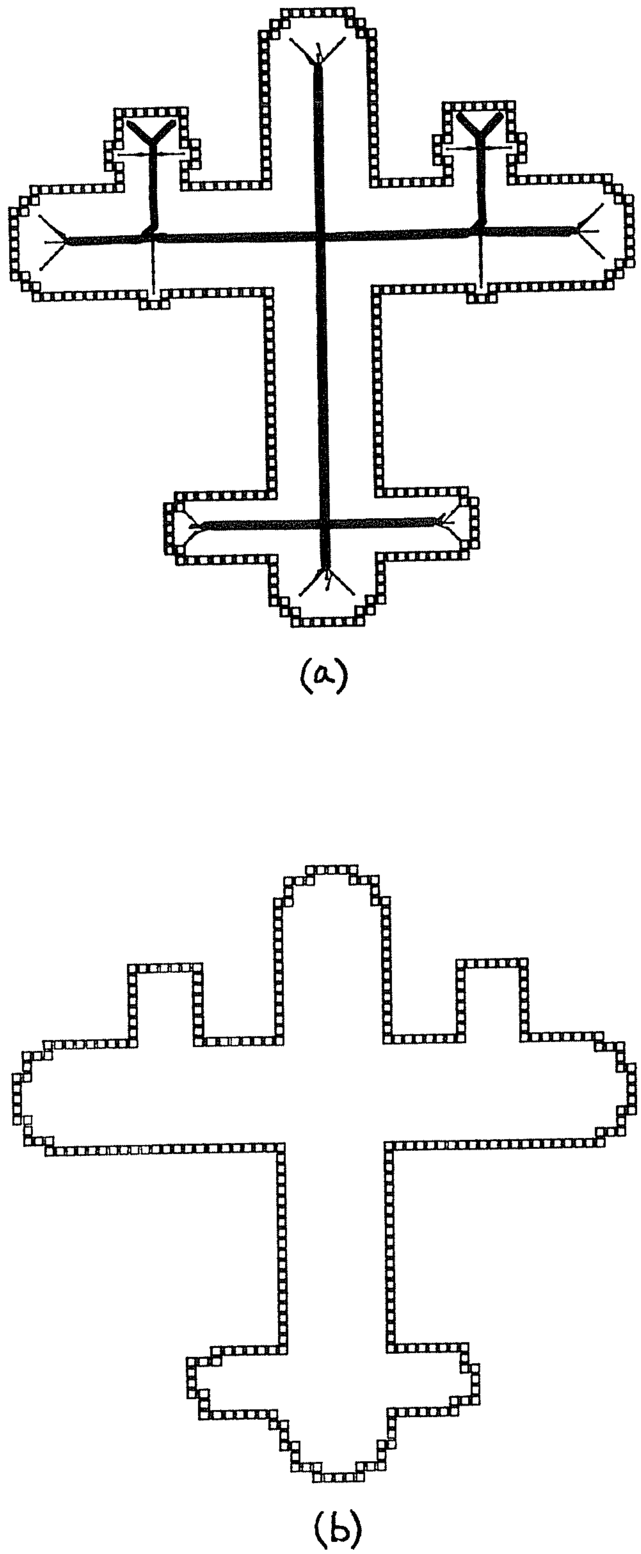



Fig. 12. (a) Airplane shape and its MAT using threshold 0.6. (b) Smoothed shape reconstructed from remaining points in (a).