ITERATIVE METHODS FOR DISCRETE

ELLIPTIC EQUATIONS

by

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ABSTRACT

We describe a basic theory for the estimation of the rates of convergence of iterative methods for the solution of the systems of linear equations which arise in the numerical solution of elliptic boundary value problems. This theory is then applied to finite-element equations solved via certain block or point iterative methods. There is a special emphasis on the "point" SOR iterative method.

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INTRODUCTION

Consider the system of linear algebraic equations

$$AU = F \tag{1}$$

which arises from the discretization of a boundary-value problem for an elliptic partial equation

$$Lu = f \quad \text{in} \quad \Omega, \qquad \beta u = 0 \quad \text{on} \quad \partial\Omega . \tag{1.1}$$

A direct iterative method for the solution of (1) is provided by a "splitting" $A = M - N$, where the matrix M has an inverse and it is not too difficult to solve problems of the general form $MX = Y$. Then, after choosing a guess U^0 one uses the splitting to construct successive iterates $\{U^k\}$ by the formula

$$MU^{k+1} = NU^k + F . \tag{1.2}$$

This iterative scheme leads one to study the eigenvalue problem

$$\lambda MU = NU . \tag{1.3}$$

It is well-known that this scheme is convergent iff

$$\rho := \max|\lambda| < 1 . \tag{1.4}$$

Moreover, smaller ρ implies faster convergence (see [17]).

This report is concerned with a method for determining the asymptotic behavior of $\rho = \rho(n)$, $n \to \infty$ where n is the order of the matrix A. This theory is of interest both for mathematical reasons and for practical reasons. At this time let us concentrate on the practical value of the theory.

A typical result obtained from this theory is of the form

$$\rho \approx 1 - \Lambda_0\left(\frac{1}{n}\right)^p , \tag{1.5}$$

where the exponent p is known exactly and the coefficient Λ_0 is given as $\Lambda_0 = \operatorname{Re} \Lambda_{min}$ where Λ_{min} is the "minimal" eigenvalue of an eigenvalue problem

$$Lu = \Lambda Qu \quad \text{in} \quad \Omega, \qquad \beta u = 0 \quad \text{on} \quad \partial\Omega . \tag{1.6}$$

In the formula (1.6), the operator Q is a differential operator of lower (than L) degree. Hence, p, the "order" (in $1/n$)) of the rate of convergence is well-determined. But, the coefficient Λ_0 is - in general - given implicitly as an eigenvalue of a problem which is just as hard as (if not harder) than the original problem (1.1).

Of course, a knowledge of the order p is very useful in comparing competitive schemes. Still, one can only make a complete comparison of the efficiency of these schemes if one knows the coefficients Λ_0. Nevertheless, even without exact knowledge of Λ_0, the theory is still useful.

(A) There are cases - model problems - in which one can compute the eigenvalues. For these model problems a precise comparison is possible. And, as is frequently the case, one can hope the qualitative nature of this comparison extends to more general problems.

(B) In many cases the operator Q is a zero'th order operator i.e. there is a function $q > 0$ and $Qu = qu$. In these circumstances there are many cases in which Λ_0 is monotone decreasing in q. That is, if

(1.7) $$q_1 \geqslant q_2, \forall x \in \Omega; \quad \text{then} \quad \Lambda_0(q_2) \geqslant \Lambda_0(q_1) .$$

Thus, qualitative information about Q provides qualitative information about Λ_0. This is the case for second order elliptic operators with nice boundary conditions (e.g. Dirichlet Conditions) and for all self-adjoint problems.

(C) There are cases where one is considering a "scale" of schemes, e.g., the "k-line schemes", and the parameter of the scale (say k) appears explicitly in the formula for $\Lambda_0(k)$. In these cases one can make useful comparisons of the schemes.

(D) Finally, there are cases where one is dealing with a continuous family of iterative schemes - say depending on a parameter ω (e.g., SOR type schemes). In such a case it is desirable to have qualitative information about the dependency of $\rho(\omega)$ on ω. This qualitative information may then be used to formulate adaptive schemes to optimize the choice of ω.

Such eigenvalue problems have been studied intensively - see [5-9,12,15,17-19]. While we cannot give a complete discussion of the earlier results, it is useful to recall some of this history and comment on some of the current activity along these lines.

One of the most useful ideas is the concept of schemes which satisfy "Property A" or "Block Property A" - see [1,19]. This theory enables one to connect the eigenvalues of the block Jacobi method with the associated block SOR methods. Thus, when this important condition holds, one can estimate SOR methods in terms of the Jacobi methods. More importantly, there is a simple algorithm for an adaptive method for determining the optimal ω (see [7]).

Garabedian [6] considered the case where L is a second order operator. Let

$$\varepsilon^k = U - U^k, \quad \text{then} \quad M(\varepsilon^{k+1} - \varepsilon^k) = -A\varepsilon^k \; . \tag{1.8}$$

Formally taking $\Delta t = \alpha h$, α = constant, Garabedian recognized - in the point SOR case - that (1.8) is a formal difference approximation to a time dependent equation of the form

$$b \frac{\partial \varepsilon}{\partial t} = L\varepsilon + b_1 \frac{\partial^2 \varepsilon}{\partial x \partial t} + b_2 \frac{\partial^2 \varepsilon}{\partial y \partial t} \; . \tag{1.9}$$

Thus, the separation of variables transformation

$$\varepsilon(x,y,t) = \varepsilon_0(x,y)e^{-\lambda t}$$

leads to the elliptic eigenvalue problem

$$L\varepsilon = \lambda Q \varepsilon \tag{1.10a}$$

where

$$Q = b - b_1 \frac{\partial}{\partial x} - b_2 \frac{\partial}{\partial y} \; . \tag{1.10b}$$

Garabedian then argues that the slowest rate of decay of $\varepsilon(x,y,t)$ yields the rate of convergence. This heuristic approach leads one to a "formula" for the asymptotic behavior of ρ as $n \to \infty$. While Garabedian never completed the details of a rigorous proof; in all the cases he studied, the results obtained by this method are correct. Garabedian then went on to use this "result" to obtain a formula for the optimal choice of ω.

Some years ago, Parter [8,9], developed a theory which was limited to self adjoint L, symmetric A and N. More recently Parter and Steuerwalt [12-14] have extended that theory to include non-self-adjoint L, non-self-adjoint splittings, parabolic problems and SOR methods. The treatment of SOR raises some interesting mathematical questions which we mention in Section 3.

In the last few years a group of active researchers at the Université Libre de Bruxelles under the guidance of R. Beauwens (see [2-4]) have been studying these problems in depth. They - and Parter and Steuerwalt [14] have discussed appropriate generalizations of the method of Garabedian. However, it seems that the Belgian school has not discussed many of the rigorous details.

In Section 2 we describe the general theory. In Section 3 we discuss the application to finite-element methods. In this section we describe the theoretical results of [13] for k-line block Jacobi, Gauss-Seidel and SOR and the results of [14] point SOR.

The results for finite-difference equations are contained in [12].

A GENERAL APPROACH

For simplicity we develop the basic ideas within the framework of the simplest finite-element approach to the boundary-value-problem (1.1), (1.2).

Consider the following "elliptic boundary value problem". Let Ω be a smooth domain in R^m and let $H_m(\Omega)$ be the usual space of functions with generalized L_2 derivatives of order m. Let

$$\tilde{H}_m \subset H_m(\Omega)$$

be a subspace and let $B(u,v)$ be a bilinear form which is continuous and coercive over $\tilde{H}_m$. We seek a function $u \in \tilde{H}_m$ which satisfies

$$B(u,\phi) = F(\phi), \ \forall\phi \in \tilde{H}_m \tag{2.1}$$

where $F(\phi)$ is a continuous linear functional defined on $\tilde{H}_m$.

Let $\{S_h\}$, $0 < h \leq h_0$, be a family of finite-dimensional subspaces of $\tilde{H}_m$. Let $\{\phi_1^h, \phi_2^h, \ldots, \phi_n^h\}$ be a basis for S_h. The discrete problem (1) arises from problem (2.1) restricted to S_h. That is, find $U^h \in S_h$ such that

$$B(U^h,\phi) = F(\phi) \quad \forall\phi \in S_h \ . \tag{2.2}$$

Setting

$$a_{ij} = B(\phi_j^h, \phi_i^h), \qquad f_j = F(\phi_j^h), \qquad U^h = \sum u_j \phi_j^h \tag{2.3}$$

the problem (2.2) takes the form (1) with

$$U = (u_1, u_2, \ldots, u_n)^T, \qquad F = (f_1, f_2, \ldots, f_n)^T \ . \tag{2.4}$$

Following the discussion in Section 1 we imagine a splitting (1.3) of the $n \times n$ matrix A. Our theory starts with the following heuristic approach:

Assumption A.1: Suppose there is an exponent $p > 0$ and an operator Q defined on $H_m(\Omega)$ such that $h^p N \sim Q$ in the following weak sense: for all sufficiently smooth $\phi, \psi \in \tilde{H}_m$ let Φ^h, Ψ^h be their projection into S_h and let $\hat{\Phi}$ and $\hat{\Psi}$ be the corresponding vectors of coefficients. Then, we assume that

$$h^p \hat{\Phi}^* N \hat{\Psi} \to \iint \bar{\phi}[Q\psi]\,dx, \qquad h \to 0 \ . \tag{2.5}$$

Remark: Given a splitting, the discovery of p and Q is part of the "art" of this method. We will have more to say about this subject later.

Having made this assumption, let $\lambda \neq 0$ be an eigenvalue of (1.5) with eigenvector U. Then $\lambda MU = NU$ and $\lambda(M - N)U = (1 - \lambda)NU$. Thus

$$AU = \frac{(1-\lambda)}{\lambda h^p} (h^p N)U \ .$$

That is, for every $\phi \in S_h$ we have

$$B(U^h,\phi) = \mu[\iint \bar{\phi}[NU^h]\,dx + E(\phi,U^h)] \ , \tag{2.6}$$

where

$$\mu = \frac{1-\lambda}{\lambda h^p}, \qquad U^h = \sum u_j \phi_j^h$$

and the "error" $E(\phi,u)$ is "small" if ϕ, U^h are smooth.

From this starting point we see that we need some further technical assumptions to complete the theory.

Assumption A.2: Consider the eigenvalue problem: Find Λ and $v \in \tilde{H}_m$ such that

(2.9) $$B(v,\phi) = \Lambda \iint \bar{\phi}[Qv]dx, \quad \forall\phi \in \tilde{H}_m .$$

We assume there is a minimal eigenvalue $\Lambda_m = \Lambda_0 + iT$. By minimal we mean that for any eigenvalue λ, it is true that

(2.8a) $$0 < \Lambda_0 \leq \mathrm{Re}\,\lambda ,$$

and, if $\mathrm{Re}\,\lambda = \Lambda_0$ then

(2.8b) $$|\lambda| \geq |\Lambda_m| .$$

Assumption A.3: The eigenvalues of the family of discrete problems (2.6) approximate the eigenvalues of the limit problem (2.9) and vice-versa.

Remark: It is usually not difficult to prove A.3. In many cases it follows from the standard theories of spectral approximation (see [8]). However, it is an essential point. In dealing with this condition one raises many technical questions about "spectral approximations" and estimates which are used to establish (A.1).

Finally, we require some information about the eigenvalues λ of (1.5) which satisfy $|\lambda| = \rho$.

Assumption A.4: Either

(2.9a) $\rho < 1$ and ρ itself is an eigenvalue,

or there is a constant $C_0 > 0$ and for every h, there is an eigenvalue λ with

(2.9b) $$|\lambda| = \rho \quad \text{and} \quad \left|\frac{1-\lambda}{h^p}\right| \leq C_0 .$$

Remarks: Both (2.9a) and (2.9b) have been used. In particular, (2.9b) was used in [13] to give a new convergence theorem - as well as an estimation of rates of convergence - for certain non-self-adjoint, nonpositive type problems.

The basic results are:

Theorem 1: Let (A.1), (A.2) and (A.3) hold. Then

$$\rho \geq 1 - \Lambda_0 h^p + o(h^p) \ .$$

If (A.4) also holds, then

$$\rho \approx 1 - \Lambda_0 h^p \ .$$

The proof is relatively straightforward. See [13].

EXAMPLES: THEORY

This approach has been used to estimate the rates of convergence for many block iterative schemes for elliptic difference equations - particularly by Parter [9,10]; Parter and Steuerwalt [12] and the Belgian school [2-4]. In this section we describe the application to certain finite-element equations.

Consider the Dirichlet problem for a second order elliptic operator defined on Ω, the unit square. That is

$$Lu = f, \quad (x,y) \in \Omega; \quad u(x,y) = 0 \quad \text{for} \quad (x,y) \in \partial\Omega \tag{3.1}$$

where

$$Lu := -[(au_x)_x + (bu_x)y + (bu_y)_x + (cu_y)_y] + d_1u_x + d_2u_y + d_0u$$

and L is uniformly elliptic with $d_0 \geq 0$.

Let $\Delta x, \Delta y$ be chosen and consider the finite-element subspace S_h of Tensor products of hermite cubic splines. That is, on each rectangle

$$x_k \leq x \leq x_{k+1}, \quad y_j \leq y \leq y_{j+1}$$

the elements of S_h are cubic polynomials in (x,y) which are completely determined by the 4-vectors

$$U_{kj} = (u_{kj}, h(u_x)_{kj}, h(u_y)_{kj}, h^2(u_{xy})_{kj}) \ . \tag{3.2}$$

Moreover, we choose the basis of S_h to be that basis for which the $\{U_{kj}\}$ are the correct interpolation conditions. We use these subspaces in a regular finite element approach to the approximate solution of the boundary-value-problem (3.1). In this way the equations (2.1) lead to a system of equations (1.).

Because the basic unknowns are described by the 4-vectors U_{jk} we choose to write the matrix A in a double subscript notation. That is: a typical term of AU has the form

$$\sum a^0_{jk,\sigma\mu} u_{\sigma\mu} + \sum a^1_{jk,\sigma\mu} (hu_x)_{\sigma\mu} + \sum \tilde{a}^1_{jk,\sigma\mu} (hu_y)_{\sigma\mu} + \sum a^2_{jk,\sigma\mu} (h^2 u_{xy})_{\sigma\mu} . \tag{3.3}$$

The matrix A is sparse, but nevertheless, in the general case each row of A contains 36 non-zero entries. This complexity in A is frequently reflected in N as well. Hence, it appears it is not too easy to study the bilinear form V^*NU and verify (A.1). However, in this case - and the discerning reader will appreciate that similar simplifications occurs in all "nodal" finite-element spaces - the basic nature of S_h yield estimates which enable one to ignore certain terms in V^*NU. Specifically we have:

<u>Theorem 3.1</u>: Let

$$h = \sqrt{\Delta x \Delta y} .$$

There is a constant $K > 0$ such that, for every $u,v \in S_h$ and every $\phi \in C^1(\bar{\Omega})$ we have

$$h^2 \sum_{i,j} |v_{ij}|^2 \leq K\|v\|^2_{L_2} , \tag{3.5a}$$

$$h^2 \sum_{i,j} u_{ij} v_{ij} \phi_{ij} = \iint uv\phi\, dxdy + \delta(u,v,\phi) , \tag{3.5b}$$

$$h^2 \sum_{i,j} [|v_{i,j} - v_{i+1,j}|^2 + |v_{ij} - v_{i,j+1}|^2] \leq Kh^2 \|\nabla v\|^2_{L_2} , \tag{3.5c}$$

$$h^2 \sum_{i,j} \sum_{|\alpha|=1}^{6} (\Delta x)^{\alpha_1} (\Delta y)^{\alpha_2} |(D^\alpha u)_{ij}|^2 \leq Kh^2 \|\nabla u\|^2_{L_2} , \tag{3.5d}$$

$$h^2 \sum_{i,j} \{ \sum_{|\alpha|+|\beta|\geq 1} (\Delta x)^{\alpha_1+\beta_1} (\Delta y)^{\alpha_2+\beta_2} |(D^\alpha u)_{ij} (D^\beta v)_{ij}| \} \leq K\eta(u,v,h) \tag{3.5e}$$

where

(3.6a) $$|\delta(u,v,\phi)| \leq K[1 + \|\nabla\phi\|_\infty]\eta(u + v, u - v, h)$$

with

(3.6b) $$\eta(u,v,h) = h[\|u\|_{L_2}\|\nabla v\|_{L_2} + \|v\|_{L_2}\|\nabla u\|_{L_2} + h\|\nabla u\|_{L_2}\|\nabla u\|_{L_2}] \ .$$

Proof: See Section 7 of [13].

Having established these estimates we now turn to a class of splittings which includes the block Jacobi methods and the point Gauss-Seidel methods but not the SOR methods.

We suppose our splitting satisfies

Property S:

(i) If $a_{ij} = 0$, then $n_{ij} = m_{ij} = 0$.

(ii) If $a_{ij} \neq 0$ and $n_{ij} \neq 0$, then $n_{ij} = -a_{ij}$.

For such schemes we consider the bilinear form $V^*(h^2N)U$.

As we attempt to find Q and verify Assumption (A.1) our task is simplified by the estimates of Theorem 3.1. Using (3.5c) we see that we need only consider the terms of the form

$$h^2 \sum_{i,j,\sigma,\mu} n_{ij,\sigma\mu}\bar{v}_{ij}u_{\sigma\mu} \ .$$

Then, using (3.5c) we may rewrite this expression as

(3.7) $$h^2 \sum_{i,j} \bar{v}_{ij}u_{ij}\left(\sum_{\sigma,\mu} n_{ij,\sigma\mu}\right) + o(1) \ .$$

Moreover (3.5b) suggests two basic facts:

(3.8) $$h^2 \sum_{i,j} \bar{v}_{ij}u_{ij}\left(\sum_{\sigma,u} n_{ij,\sigma u}\right) \approx \iint q(x,y)\bar{v}(x,y)u(x,y)\,dxdy$$

and, in order to determine the function q we need only a rough evaluation of the integrals which constitute the $\{n_{ij,\sigma\mu}\}$.

These arguments have been carried out in complete detail in [12]. If

$$\sum_{\sigma,\mu} n_{ij,\sigma\mu} = \bar{q}_{ij}(h)$$

converges to a function $q(x,y)$, then A.1 is satisfied where the phrase sufficiently smooth merely means $\phi,\psi \in \overset{\circ}{H}_1(\Omega)$. If $q(x,y) > 0$ then A.2 is satisfied. Because we only require $\phi,\psi \in H_1$, it is an easy matter to obtain A.3 from the standard theory of Spectral Approximation [8]. Finally, the verification A.4 rests on the fact that these block schemes satisfy block property A. Even so, in the non-self-adjoint case, it is not easy. The main results are

<u>Theorem 3.2</u>: Consider the k-line (horizontal block Jacobi scheme) applied to (3.3a). Let Γ_0 be the minimal eigenvalue of the elliptic eigenvalue problem

$$L\phi = \lambda c(x,y)\phi \quad \text{in} \quad \Omega, \quad \phi = 0 \quad \text{in} \quad \partial\Omega \ . \tag{3.9}$$

Then, the radius of convergence of this iterative scheme is given by

$$\rho_J(k) \approx 1 - \frac{5}{12} k\Gamma_0(\Delta y)^2 \ . \tag{3.10a}$$

Since these schemes satisfy block property A the theory of Young [18] gives estimates

$$\rho_{GS}(k) \approx 1 - \frac{5}{6} k\Gamma_0(\Delta_y)^2, \quad \rho(k,\omega_b) \approx 1 - 2\left(\frac{5}{6} k\Gamma_0\right)^{1/2}\Delta y \ .$$

<u>Theorem 3.3</u>: Consider the "point" Gauss-Seidel iterative scheme. Let $r := \Delta y/\Delta x$ and let

$$q(x,y) := \frac{156}{175} [ra(x,y) + \frac{1}{r} c(x,y)] \ .$$

Then

$$\rho \geqq 1 - \Lambda_0 h^2 + o(h^2)$$

where Λ_0 is the minimal eigenvalue of the problem

$$Lu = \lambda qu \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega \ .$$

The "point" SOR splitting does not satisfy property S. Moreover, this scheme does not satisfy property A. To illustrate the application of the theory in this case we consider the simplest case, i.e.

$$Lu = -(u_{xx} + u_{yy}) \ . \tag{3.11}$$

We write the matrix A as $A = D - W - W^*$ where D is the diagonal of A and W is strictly lower triangular. With $\omega = 2 - Ch$, $C > 0$, the SOR iteration is described by

$$\frac{1}{\omega}(D - \omega W)U^{k+1} = \frac{1}{\omega}[\omega W^* + (1-\omega)D]U^k + F .$$

Then

$$hN = h[W^* - \frac{1}{2}D] + \frac{Ch^2}{4}D + 0(h^3)D . \tag{3.12}$$

Using the estimates of Theorem 3.1 we find that

$$\begin{aligned} V^*(hN)U &= h^2 \sum_{i,j} C\frac{156}{175}u_{ij}\bar{v}_{ij} + h^2 \sum_{i,j} \bar{v}_{ij}\left[\frac{210}{175}\,\frac{u_{i,j+1} - u_{i,j}}{h}\right] \\ &+ h^2 \sum_{i,j} \bar{v}_{ij}\left[\frac{102}{175}\,\frac{u_{i+1,j} - u_{i,j}}{h}\right] + \frac{h^2}{10}\sum_{i,j}[(\bar{v}_y)_{ij}u_{ij} - (u_y)_{ij}\bar{v}_{ij}] + \varepsilon(u,v) \end{aligned} \tag{3.13}$$

where $\varepsilon(u,v)$ is $o([\|u\|_{H_1} + \|v\|_{H_1}]^2)$. Thus (A.1) holds with

$$Qu = C\frac{156}{175}u + \frac{102}{175}u_x + u_y . \tag{3.14}$$

Unfortunately, in this case the phrase "sufficiently smooth" would seem to require more than $\phi,\psi \in \overset{\circ}{H}_1(\Omega)$. This fact also makes the proof of A.3 more difficult. Nevertheless it is true that A.3 holds. As for (A.4), we do not know. Nevertheless we may apply Theorem I and obtain an inequality. Then, assuming that equality actually holds we use the Garabedian substitution

$$u = v\exp\{-\frac{\Lambda}{2}(\frac{102}{175}x + y)\}$$

to obtain an "exact" formula for $\rho(\omega)$ provided that $\omega = 2 - Ch > 0$. For each ω, let $\tilde{\rho}(\omega)$ denote the value so obtained. Then $\tilde{\rho}(\omega)$ is a lower bound (and is probably equal to) the true value of $\rho(\omega)$. Using this formula we obtain a candidate $\tilde{\omega}_b$ for the optimal ω. This approach yields

$$\tilde{\omega}_b = 2 - c_b h, \quad c_b = \frac{\sqrt{2}\,\pi}{156}[41029]^{1/2} \tag{3.15a}$$

$$\rho(\tilde{\omega}_b) \geqslant 1 - \frac{2\sqrt{2}\,\pi(175)}{[4109]^{1/2}}h = \tilde{\rho}(\tilde{\omega}_b) . \tag{3.15b}$$

It is both more interesting and more convenient to take $c = (2 - \omega)/h$ as the independent variable and consider the function

$$\Lambda_0(c) = \frac{1 - \tilde{\rho}(\omega)}{h} . \tag{3.16}$$

The results are qualitatively like the results obtained by Young in the case of (block) property A. We have

<u>Case 1</u>: $0 < c < c_b$. Then

$$\Lambda_0(c) = \frac{2\sqrt{2}\ \pi(175)}{[41029]^{1/2}} \frac{c}{c_b} . \tag{3.17}$$

<u>Notice</u>, (3.17) asserts that in the range $\tilde{\omega}_b \leq \omega \leq 2$, the quantity $\tilde{\rho}(\omega)$ is asymptotically linear in ω.

<u>Case 2</u>: $c_b < c < 2/h$. Then

$$\Lambda_0(c) = \frac{2\sqrt{2}\ \pi(175)}{[41029]^{1/2}} \frac{c}{c_b} - \frac{1}{[41029]} [(156c)^2 - 2\pi^2(41029)]^{1/2} . \tag{3.18}$$

<u>Observe</u> that

$$\frac{d}{dc} \Lambda_0(c)\Big|_{c=c_b+} = -\infty . \tag{3.19}$$

Thus, just as in the case when property A applies, it is better to over-estimate $\tilde{\omega}_b$ than to underestimate $\tilde{\omega}_b$. Moreover, the function is linear for $\omega > \tilde{\omega}_b$ and decreases sharply for $\omega < \tilde{\omega}_b$. Finally, we emphasize that $\tilde{\rho}(\omega)$ has a unique minimum. Hence, if we could believe that $\rho(\omega) = \tilde{\rho}(\omega)$ we could use these facts to develop an adaptive approach for the determination of $\tilde{\omega}_b$.

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