SIMPLE COMPUTABLE BOUNDS FOR SOLUTIONS OF LINEAR COMPLEMENTARITY PROBLEMS AND LINEAR PROGRAMS

by

O. L. Mangasarian

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ABSTRACT

It is shown that each feasible point of a positive semidefinite linear complementarity problem which is not a solution of the problem provides simple numerical bounds for some or all components of all solution vectors. Consequently each pair of primal-dual feasible points of a linear program which are not optimal provide simple numerical bounds for some or all components of all primal-dual solution vectors. For example each feasible point, that is $(\hat{z},\hat{w}) \geqq 0$, of the linear complementarity problem $w = Mz + q \geqq 0$, $z \geqq 0$, $z^Tw = 0$, where M is positive semidefinite, provides the following simple bound for any solution $\bar{z}$ of the linear complementarity problem:

$$\sum_{i \in I} \bar{z}_i \leqq \hat{z}^T\hat{w}/\min \hat{w}_{i \in I}$$

where $I = \{i \mid \hat{w}_i > 0\}$. If $\hat{w} > 0$ then this inequality provides a bound on the 1-norm $\|\bar{z}\|_1$ of any solution point. Similarly each feasible point $(\hat{x},\hat{y}) \geqq 0$ of the primal linear program $\min c^Tx$ subject to $y = Ax - b \geqq 0$, $x \geqq 0$, and each feasible point $(\hat{u},\hat{v}) \geqq 0$ of the dual linear program $\max b^Tu$ subject to $v = -A^Tu + c \geqq 0$, $u \geqq 0$, provide the following simple bounds for any primal optimal solution $(\bar{x},\bar{y})$ and any dual optimal solution $(\bar{u},\bar{v})$:

$$\sum_{i \in J} \bar{x}_i \leqq (c^T\hat{x} - b^T\hat{u})/\min \hat{v}_{i \in J}, \quad \sum_{i \in I} \bar{u}_i \leqq (c^T\hat{x} - b^T\hat{u})/\min \hat{y}_{i \in I}$$

where $J = \{i \mid \hat{v}_i > 0\}$ and $I = \{i \mid \hat{y}_i > 0\}$. If $\hat{v} > 0$ we have a bound on $\|\bar{x}\|_1$, and if $\hat{y} > 0$ we have a bound on $\|\bar{u}\|_1$. In addition we show that the existence of such numerical bounds is not only sufficient but is also necessary for the boundedness of solution vector components for both the linear complementarity problem and the dual linear programs.

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1. Introduction

The linear complementarity problem of finding a (z,w) in the 2k-dimensional real space R^{2k} such that

$$w = Mz + q \geqq 0,\ z \geqq 0,\ z^T w = 0 \tag{1.1}$$

where M is a given $k \times k$ real matrix, q is a given $k \times 1$ real vector and $z^T w$ denotes the scalar product $\sum_{i=1}^{k} z_i w_i$, is a fundamental problem of mathematical programming which includes linear and quadratic programming problems, bimatrix games [2] and free boundary problems [3]. An important question of both theoretical and practical interest is the boundedness of the solution set of (1.1) which already has received attention in [9,4,7] in the form of necessary and/or sufficient conditions for this boundedness. In this work we provide simple numerical bounds for some or all components of any solution vector when M is positive semidefinite. In particular we show that each feasible point $(\hat{z},\hat{w})$, that is $(\hat{z},\hat{w}) \geqq 0$, which is not a solution of (1.1), contains information on the magnitude of some or all components of all solution points. For example Theorem 2.2 provides the following simple bounds for any solution $(\bar{z},\bar{w})$ of (1.1) in terms of any feasible point $(\hat{z},\hat{w})$ when M is positive semidefinite

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$$(1.2)\quad \begin{cases} \|\bar{z}_I\|_1 \leqq \hat{z}^T\hat{w} / \min \hat{w}_{i\in I} \\ \|\bar{w}_J\|_1 \leqq \hat{z}^T\hat{w} / \min \hat{z}_{i\in J} \\ \|\bar{z}_I, \bar{w}_J\|_1 \leqq \hat{z}^T\hat{w} / \min \{\hat{w}_{i\in I}, \hat{z}_{j\in J}\} \end{cases}$$

where $I = \{i \mid \hat{w}_i > 0\}$, $J = \{i \mid \hat{z}_i > 0\}$, $\bar{z}_I := \bar{z}_{i\in J}$ and $\|\cdot\|_1$ denotes the 1-norm. Note that since $(\hat{z},\hat{w})$ is not a solution of the linear complementarity problem (1.1) then both I and J cannot be empty. On the other hand if $\hat{w} > 0$, then $\bar{z}_I = \bar{z}$ and (1.2) provides a bound on the 1-norm $\|\bar{z}\|_1$ of any solution $(\bar{z},\bar{w})$ of (1.1). Similarly if $\hat{z} > 0$, then $\bar{w}_J = \bar{w}$ and (1.2) provides a bound on $\|\bar{w}\|_1$. Theorem 2.2 also characterizes the boundedness of the set of all $\bar{z}_I$, and the set of all $\bar{w}_J$ for $I, J \subset \{1,\ldots,k\}$ where $(\bar{z},\bar{w})$ is a solution of (1.1) and M is positive semidefinite. In particular it shows that the set of all $\bar{z}_I$ is bounded if and only if there exists a feasible point $(\hat{z},\hat{w}) \geqq 0$ such that $\hat{w}_I > 0$; the set of all $\bar{w}_J$ is bounded if and only if there exists a feasible point $(\hat{z},\hat{w}) \geqq 0$ such that $\hat{z}_J > 0$; and the set of all $(\bar{z}_I,\bar{w}_J)$ is bounded if and only if there exists a feasible point $(\hat{z},\hat{w}) \geqq 0$ such that $(\hat{z}_J, \hat{w}_I) > 0$. Theorem 2.2 can be used, as in Algorithm 2.6, to determine which components if any of the solution set are bounded, without solving the linear complementarity problem (1.1). Theorem 2.2 also provides necessary conditions for the boundedness of solution components of (1.1) when M is copositive plus, that is M satisfying (1.5)-(1.6) below. In Theorem 2.8 we give bounds for the unique solution of the positive definite linear complementarity problem.

Because a linear programming problem is a special case of the linear complementarity problem [2], the bounds of Section 2 can be used to obtain bounds for solutions of the dual linear programs

(1.3a) $$\min\ c^Tx \quad \text{s.t.}\quad y = Ax - b \geqq 0,\ x \geqq 0$$

(1.3b) $$\max\ b^Tu \quad \text{s.t.}\quad v = -A^Tu + c \geqq 0,\ u \geqq 0$$

where A is an $m \times n$ real matrix, c and b are $n \times 1$ and $m \times 1$ real vectors respectively. In [8] Robinson and in [6] this author both gave bounds for solutions of linear programs which involved a constant which was difficult to evaluate in general. By contrast in Section 3 we provide bounds for solutions of (1.3) which involve no constants or parameters. For example Theorem 3.1 provides the following simple bounds for any solution $(\bar{x},\bar{y}) - (\bar{u},\bar{v})$ of the dual linear programs (1.3) in terms of any pair $(\hat{x},\hat{y}) - (\hat{u},\hat{v})$ of primal-dual feasible points:

(1.4) $$\begin{cases} \|\bar{x}_{J_1}\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min\ \hat{v}_{i\in J_1} \\ \|\bar{y}_{I_1}\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min\ \hat{u}_{i\in I_1} \\ \|\bar{x}_{J_1}, \bar{y}_{I_1}\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min\ \{\hat{v}_{j\in J_1}, \hat{u}_{i\in I_1}\} \\ \|\bar{u}_{I_2}\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min\ \hat{y}_{i\in I_2} \\ \|\bar{v}_{J_2}\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min\ \hat{x}_{i\in J_2} \\ \|\bar{u}_{I_2}, \bar{v}_{J_2}\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min\ \{\hat{y}_{i\in I_2}, \hat{x}_{j\in J_2}\} \end{cases}$$

where $J_1 = \{i \mid \hat{v}_i > 0\}$, $J_2 = \{i \mid \hat{x}_i > 0\}$, $I_1 = \{i \mid \hat{u}_i > 0\}$ and $I_2 = \{i \mid \hat{y}_i > 0\}$. In Theorem 3.4 we consider a nonsymmetric dual linear programming pair and provide numerical bounds for its solution set.

We describe briefly now our notation. All vectors will be column vectors unless transposed to a row vector by a superscript T. For a vector x in the n-dimensional Euclidean space R^n, $\|x\|$ will denote an arbitrary but fixed norm and $\|x\|_p$ will denote the p-norm $\|x\|_p := (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ where $1 \leqq p < \infty$ and $\|x\|_\infty := \max_{1\leqq i\leqq n} |x_i|$. For an $m \times n$ real matrix A, A_i denotes the ith row and $A_{\cdot j}$ denotes the jth column, while $\|A\|_p$ denotes the matrix norm subordinate to the vector norm $\|\cdot\|_p$, that is $\|A\|_p = \max_{\|x\|_p=1} \|Ax\|_p$. The consistency condition $\|Ax\|_p \leqq \|A\|_p \|x\|_p$ follows immediately from this definition of a matrix norm. For a subset $J \subset \{1,\ldots,n\}$, x_J or $x_{i\in J}$, will denote those components x_i of the vector x in R^n such that $i \in J$. Similarly for $I \subset \{1,\ldots,m\}$, A_I will denote those rows A_i of A such that $i \in I$, while $A_{\cdot J}$ will denote those columns $A_{\cdot j}$ of A such that $j \in J$. A vector of ones in any real finite dimensional Euclidean space will be denoted by e. A $k \times k$ real (not necessarily symmetric) matrix M is said to be copositive [2] if

$$(1.5) \qquad z \geqq 0 \Rightarrow z^T M z \geqq 0$$

M is said to be copositive plus [2] if it is copositive and

$$(1.6) \qquad z \geqq 0,\ z^T M z = 0 \Rightarrow (M + M^T)z = 0$$

A $k \times k$ real (not necessarily symmetric) matrix M is said to be positive semidefinite (definite) if

$$z^T M z \geqq 0 \ (> 0) \quad \text{for all} \quad z \neq 0$$

Note that a positive definite matrix is also positive semidefnite, while a positive semidefnite matrix is also a copositive plus matrix.

2. Bounds for Solutions of Positive Semidefinite Linear Complementarity Problems

We begin by a simple but useful identity.

2.1 Lemma Let M be a $k \times k$ real matrix and let q be a $k \times 1$ real vector. Then for any z and $\bar{z}$ in R^k such that $\bar{z}^T(M\bar{z}+q) = 0$ it follows that

$$(2.1) \qquad z^T(Mz+q) = \bar{z}^T(Mz+q) + z^T(M\bar{z}+q) + (z-\bar{z})^T M(z-\bar{z})$$

Proof By direct algebraic verification. □

Before establishing the principal result of this section, we need to define some sets. Let I and J be subsets of $\{1,2,\ldots,k\}$. Define

$$(2.2) \quad \begin{cases} S := \{(z,w) \mid z \geqq 0,\ w = Mz+q \geqq 0\} \\ \bar{S} := \{(z,w) \mid (z,w) \in S,\ z^T w = 0\} \\ S_{IJ} := \{(z,w) \mid (z,w) \in S,\ (z_I, w_J) > 0\} \\ \bar{S}_{IJ} := \{(z_I, w_J) \mid (z,w) \in \bar{S}\} \\ Z_I := \{(z,w) \mid (z,w) \in S,\ z_I > 0\} \\ \bar{Z}_I := \{z_I \mid (z,w) \in \bar{S}\},\ \bar{Z} := \{z \mid (z,w) \in \bar{S}\} \\ W_I := \{(z,w) \mid (z,w) \in S,\ w_I > 0\} \\ \bar{W}_I := \{w_I \mid (z,w) \in \bar{S}\},\ \bar{W} := \{w \mid (z,w) \in \bar{S}\} \end{cases}$$

With these definitions we are able to characterize the boundedness of the set of solutions of linear complementarity problems and to give simple numerical bounds for those components of the solution set which are bounded.

2.2 Theorem Let M be a $k \times k$ copositive plus matrix, let $S \neq \phi$ and let I and J be subsets of $\{1,2,\ldots,k\}$. Then

(a) $W_I \neq \phi \Leftarrow \bar{Z}_I$ bounded

(b) $Z_I \neq \phi \Leftarrow \bar{W}_I$ bounded

(c) $S_{JI} \neq \phi \Leftarrow \bar{S}_{IJ}$ bounded

If in addition M is positive semidefinite then

(a′) (i) $W_I \neq \phi \Leftrightarrow$ (ii) $\bar{Z}_I$ bounded $\Leftrightarrow$ (iii) $\left\langle \begin{array}{l} W_I \neq \phi \text{ and} \\ \|\bar{z}_I\|_1 \leqq \hat{z}^T\hat{w}/\min \hat{w}_{i\in I} \\ \forall \bar{z}_I \in \bar{Z}_I,\ \forall(\hat{z},\hat{w}) \in W_I \end{array} \right.$

(b′) (i) $Z_I \neq \phi \Leftrightarrow$ (ii) $\bar{W}_I$ bounded $\Leftrightarrow$ (iii) $\left\langle \begin{array}{l} Z_I \neq \phi \text{ and} \\ \|\bar{w}_I\|_1 \leqq \hat{z}^T\hat{w}/\min \hat{z}_{i\in I} \\ \forall \bar{w}_I \in \bar{W}_I,\ \forall(\hat{z},\hat{w}) \in Z_I \end{array} \right.$

(c′) (i) $S_{JI} \neq \phi \Leftrightarrow$ (ii) $\bar{S}_{IJ}$ bounded $\Leftrightarrow$ (iii) $\left\langle \begin{array}{l} S_{JI} \neq \phi \text{ and} \\ \left\| \begin{matrix} \bar{z}_I \\ \bar{w}_J \end{matrix} \right\|_1 \leqq \hat{z}^T\hat{w}/\min \{\hat{z}_{i\in J},\hat{w}_{i\in I}\} \\ \forall(\bar{z}_I,\bar{w}_J) \in \bar{S}_{IJ},\ \forall(\hat{z},\hat{w}) \in S_{JI} \end{array} \right.$

Proof First by Lemke's algorithm [2], it follows that $\bar{S} \neq \phi$ since $S \neq \phi$.

(a) We shall prove the contrapositive implication.

$$W_I = \phi \iff Mz + q \geqq 0,\ z \geqq 0,\ M_I z + q_I > 0 \text{ has no solution}$$

$$\iff M^T u \leqq 0,\ u \geqq 0,\ 0 \neq \begin{pmatrix} u_I \\ -q^T u \end{pmatrix} \geqq 0 \text{ has solution}$$

(By Motzkin's theorem of the alternative [5])

$$\iff M^T u \leqq 0,\ u \geqq 0,\ q^T u < 0 \text{ has solution, or}$$

$$M^T u \leqq 0,\ u \geqq 0,\ q^T u = 0,\ 0 \neq u_I \geqq 0 \text{ has solution}$$

$$\iff M^T u \leqq 0,\ u \geqq 0,\ q^T u = 0,\ 0 \neq u_I \geqq 0 \text{ has solution}$$

($q^T u < 0$ alternative excluded by $S \neq \phi$)

$$\iff u^T M u = 0,\ M^T u \leqq 0,\ u \geqq 0,\ q^T u = 0,\ 0 \neq u_I \geqq 0 \text{ has solution}$$

(Since M is copositive)

$$\iff Mu = -M^T u \geqq 0,\ u \geqq 0,\ q^T u = 0,\ 0 \neq u_I \geqq 0 \text{ has solution}$$

(Since M is copositive-plus)

$$\Rightarrow \bar{z} + \lambda u \in \bar{Z} \text{ for any } (\bar{z},\bar{w}) \in \bar{S},\ \text{any } \lambda > 0 \text{ and } u \geqq 0,$$

$$Mu = -M^T u \geqq 0,\ q^T u = 0,\ 0 \neq u_I \geqq 0$$

$\Rightarrow \bar{Z}_I$ unbounded.

(b) We again prove the contrapositive implication.

$$Z_I = \phi \iff Mz + q \geqq 0,\ z \geqq 0,\ z_I > 0 \text{ has no solution}$$

$$\iff M^T u \leqq 0,\ u \geqq 0,\ 0 \neq \begin{pmatrix} (M^T u)_I \\ q^T u \end{pmatrix} \leqq 0 \text{ has solution}$$

(By Motzkin's theorem)

$$\iff M^T u \leqq 0,\ u \geqq 0,\ q^T u = 0,\ 0 \neq (M^T u)_I \leqq 0 \text{ has solution}$$

(Alternative $q^T u < 0$ is excluded by $S \neq \phi$)

$$\iff Mu = -M^T u \geqq 0,\ u \geqq 0,\ q^T u = 0,\ 0 \neq (Mu)_I = -(M^T u)_I \geqq 0 \text{ has solution}$$

(By copositivity plus of M)

$$\Rightarrow \bar{z} + \lambda u \in \bar{Z} \text{ for any } (\bar{z},\bar{w}) \in \bar{S},\ \text{any } \lambda > 0 \text{ and } u \geqq 0,$$

$$Mu = -M^T u \geqq 0,\ q^T u = 0,\ 0 \neq (Mu)_I \geqq 0$$

$\Rightarrow \bar{W}_I$ unbounded.

(c) $\bar{S}_{IJ}$ bounded implies $\bar{Z}_I$ bounded and $\bar{W}_J$ bounded. By (a) above it follows that $W_I \neq \phi$, and by (b) above it follows that $Z_J \neq \phi$. Let $(\hat{z},\hat{w}) \in W_I$ and let $(\tilde{z},\tilde{w}) \in Z_J$. Then

$$\left(\frac{\hat{z}+\tilde{z}}{2}, \frac{\hat{w}+\tilde{w}}{2}\right) \in Z_J \cap W_I = S_{JI}$$

(a′) The implication (i)$\Leftarrow$(ii) follows from (a) above. The implication (ii)$\Leftarrow$(iii) is evident. We now establish the implication (i)$\Rightarrow$(iii) by means of Lemma 2.1. Let $(\hat{z},\hat{w}) \in W_I$ and $\bar{z}_I \in \bar{Z}_I$. Then by Lemma 2.1 and the positive semidefiniteness of M we have

$$\hat{z}^T\hat{w} = \hat{z}^T(M\hat{z}+q) \geqq \bar{z}^T(M\hat{z}+q) + \hat{z}^T(M\bar{z}+q) \geqq \bar{z}_I^T(M\hat{z}+q)_I$$
$$\geqq \|\bar{z}_I\|_1 \min \hat{w}_{i\in I}$$

Hence

$$\|\bar{z}_I\|_1 \leqq \hat{z}^T\hat{w}/\min \hat{w}_{i\in I}$$

(b′) The implication (i)$\Leftarrow$(ii) follows from (b) above. The implication (ii)$\Leftarrow$(iii) is evident. We now establish (i)$\Rightarrow$(iii). Let $(\hat{z},\hat{w}) \in Z_I$ and let $\bar{w}_I \in \bar{W}_I$. By Lemma 2.1 and the positive semidefiniteness of M we have

$$\hat{z}^T\hat{w} \geqq \bar{z}^T\hat{w} + \hat{z}^T\bar{w} \geqq \hat{z}_I^T\bar{w}_I \geqq \|\bar{w}_I\|_1 \min \hat{z}_{i\in I}$$

Hence

$$\|\bar{w}_I\|_1 \leqq \hat{z}^T\hat{w}/\min \hat{z}_{i\in I}$$

(c′) Again the implication (i)$\Leftarrow$(ii) follows from (c) above. The implication (ii)$\Leftarrow$(iii) is evident. To establish (i)$\Rightarrow$(iii), let $(\hat{z},\hat{w}) \in S_{JI}$

and let $(\bar{z}_I, \bar{w}_J) \in \bar{S}_{IJ}$. Then by Lemma 2.1 and the positive semidefiniteness of M we have

$$\hat{z}^T\hat{w} \geqq \bar{z}^T\hat{w} + \hat{z}^T\bar{w} \geqq \bar{z}_I^T\hat{w}_I + \hat{z}_J^T\bar{w}_J \geqq \left\|\begin{matrix}\bar{z}_I \\ \bar{w}_J\end{matrix}\right\|_1 \min\{\hat{z}_{i\in J}, \hat{w}_{i\in I}\}$$

Hence

$$\left\|\begin{matrix}\bar{z}_I \\ \bar{w}_J\end{matrix}\right\|_1 \leqq \hat{z}^T\hat{w}/\min\{\hat{z}_{i\in J}, \hat{w}_{i\in I}\} \qquad \square$$

2.3 <u>Remark</u> The sets I and J of Theorem 2.2 above may be taken as singletons in which case the bounds in (a′), (b′) and (c′) simplify respectively to

$$\bar{z}_i \leqq \hat{z}^T\hat{w}/\hat{w}_i \quad \text{for} \quad \bar{z}_i \in \bar{Z}_i, \ (\hat{z}_i, \hat{w}_i) \in W_i$$

$$\bar{w}_i \leqq \hat{z}^T\hat{w}/\hat{z}_i \quad \text{for} \quad \bar{w}_i \in \bar{W}_i, \ (\hat{z}_i, \hat{w}_i) \in Z_i$$

$$\bar{z}_i + \bar{w}_j \leqq \hat{z}^T\hat{w}/\min\{\hat{z}_j, \hat{w}_i\} \quad \text{for} \quad (\bar{z}_i, \bar{w}_j) \in \bar{S}_{ij}, \ (\hat{z},\hat{w}) \in S_{ji}$$

2.4 <u>Remark</u> The positive semidefiniteness assumption plays an indispensible role in obtaining the numerical bounds of parts (a′), (b′) and (c′) of Theorem 2.2. It is unlikely that such numerical bounds can be obtained for the copositive plus case. Whether the forward implications of parts (a), (b) and (c) of Theorem 2.2 also hold under a copositive plus assumption is an open question. However when $I = \{1,2,\ldots,k\}$, the forward assumption of (a) does hold for a copositive plus M. See Theorem 2, (ii)$\Leftrightarrow$(ix) [7].

The following corollary which is a direct consequence of part (a′) of Theorem 2.2 provides a practical method for determining which components of the solution set are bounded and which are not without solving the linear complementarity problem (1.1).

<u>2.5 Corollary</u> Let M be a $k \times k$ positive semidefinite matrix and let $S \neq \phi$. There exists a partition $I \cup L$ of $\{1,2,\ldots,k\}$ such that

(2.3) $\bar{Z}_I$ is bounded, $\bar{Z}_L$ is unbounded

or equivalently such that

(2.4) $W_I \neq \phi, \quad W_L = \phi$

One way to determine the partition $I \cup L$ of the above corollary for a given linear complementarity problem is to solve at most $N(I)$ linear programs, where $N(I)$ is the number of elements in I, as in the following Algorithm 2.6. This algorithm determines the partition $I \cup L$ of $\{1,2,\ldots,k\}$ for a positive semidefinite linear complementarity problem (1.1) such that $\bar{Z}_I$ is bounded and $\bar{Z}_L$ is unbounded, by determining W_I such that $W_I \neq \phi$ and W_L such that $W_L = \phi$. The algorithm which does not solve the linear complementarity problem, solves at most $N(I)$ (but potentially considerably fewer) linear programs.

2.6 Algorithm (Determination of $I \cup L = \{1,2,\ldots,k\}$ such that $\bar{Z}_I$ is bounded, $\bar{Z}_L$ is unbounded, for a positive semidefinite M)

Step 0: Set $j = 0$, $I_0 = \phi$, $L_0 = \{1,2,\ldots,k\}$

Step 1: Solve the LP: $\max \sum_{j \in L_j} (Mz + q)_j$ s.t. $Mz + q \geqq 0$, $z \geqq 0$

If LP is infeasible, LCP (1.1) is infeasible. Stop.

If LP max = 0, set $I = I_j$, $L = \{1,2,\ldots,k\} \backslash I_j$. Stop.

If $0 < \text{LP max} < \infty$, set $z(\lambda) = \bar{z}$ where $\bar{z}$ is an LP solution.

If LP max $\to \infty$, set $z(\lambda) = \bar{z} + \lambda\bar{d}$ where $\bar{z} + \lambda\bar{d}$ is feasible for all $\lambda > 0$ and $\sum_{j \in L_j} M_j \bar{d} > 0$.

Set $I_{j+1} = I_j \cup \{i \mid M_i z(\lambda) + q_i > 0, \ \lambda \to \infty\}$

$L_{j+1} = \{1,2,\ldots,k\} \backslash I_{j+1}$

Step 2: $j + 1 \to j$

Step 3: Go to Step 1.

2.7 Remark The LP solutions of Algorithm 2.6 can be used in conjunction with Theorem 2.2 (a'iii) to give numerical bounds for $\|\bar{z}_I\|_1$, $\bar{z}_I \in \bar{Z}_I$.

In [1] Adler and Gale characterized the solution set of a positive semidefinite linear complementarity problem as the solution set of a system of linear inequalities. Writing these inequalities requires the knowledge of a solution to the complementarity problem. Determining which components of the solution set are bounded by using these inequalities may require the solution of as many as n linear programs in addition to solving the linear complementarity problem.

When M is positive definite, additional simple bounds can be obtained as follows.

2.8 Theorem Let M be a $k \times k$ positive definite matrix with $\alpha > 0$ being the smallest eigenvalue of $\frac{M + M^T}{2}$ and $\beta > 0$ the smallest eigenvalue of $\frac{M^{-1} + (M^{-1})^T}{2}$. Then the unique solution $(\bar{z}, \bar{w})$ of the linear complementarity (1.1) is bounded by

$$\text{(2.5a)} \qquad \max\{0, \|\hat{z}\|_2 - (\hat{z}^T\hat{w}/\alpha)^{\frac{1}{2}}\} \leqq \|\bar{z}\|_2 \leqq \|\hat{z}\|_2 + (\hat{z}^T\hat{w}/\alpha)^{\frac{1}{2}}$$

$$\text{(2.5b)} \qquad \max\{0, \|\hat{w}\|_2 - (\hat{w}^T\hat{z}/\beta)^{\frac{1}{2}}\} \leqq \|\bar{w}\|_2 \leqq \|\hat{w}\|_2 + (\hat{w}^T\hat{z}/\beta)^{\frac{1}{2}}$$

for any feasible $\hat{z} \geqq 0$, $\hat{w} = M\hat{z} + q \geqq 0$.

Proof By Lemma 2.1 we have that

$$\hat{z}^T\hat{w} \geqq (\hat{z} - \bar{z})^T M(\hat{z} - \bar{z}) \geqq \alpha\|\hat{z} - \bar{z}\|_2^2$$

Hence

$$\|\bar{z}\|_2 \leqq \|\hat{z}\|_2 + \|\hat{z} - \bar{z}\|_2 \leqq \|\hat{z}\|_2 + (\hat{z}^T\hat{w}/\alpha)^{\frac{1}{2}}$$

which gives the second inequality of (2.5a). The first inequality of (2.5a) follows from

$$\|\hat{z}\|_2 \leqq \|\bar{z}\|_2 + \|\hat{z} - \bar{z}\|_2 \leqq \|\bar{z}\|_2 + (\hat{z}^T\hat{w}/\alpha)^{\frac{1}{2}}$$

To obtain (2.5b) note first that since M is positive definite it is nonsingular and its inverse is positive definite. Consequently the linear complementarity problem (1.1) is equivalent to

(2.6) $$z = M^{-1}w - M^{-1}q,\ w \geqq 0,\ w^Tz = 0$$

Hence (2.5a) of this theorem applied to (2.6) yields (2.5b). □

We conclude this section by demonstrating, by means of a simple example, that our bounds can be tight.

2.9 Example

$$M = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},\quad q = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

For this problem $\bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the unique solution. By taking $\hat{w}_1 = \hat{w}_2 = \hat{w} > 0$, $\hat{x}_1 = -\hat{w} + 1 \geqq 0$, $\hat{x}_2 = 2\hat{w} + 1 \geqq 0$, we have by Theorem 2.2 that

$$2 = \|\bar{x}\|_1 \leqq \inf_{1 \geqq \hat{w} > 0} \frac{(-\hat{w}+1)\hat{w} + (2\hat{w}+1)\hat{w}}{\hat{w}} = \inf_{1 \geqq \hat{w} > 0} (2 + \hat{w}) = 2$$

3. Bounds for Solutions of Linear Programs

We begin this section with some results which are direct consequences of Section 2. These results follow by considering the pair of dual linear programs (1.3) as a linear complementarity problem with a skew-symmetric, and hence, positive semidefinite matrix. Later on in this section we shall obtain bounds for solutions of linear programs with explicit equality constraints.

By considering the dual linear programs (1.3) as a linear complementarity problem [2] defined by (1.1) and

$$(3.1) \qquad M:=\begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix},\ q:=\begin{pmatrix} c \\ -b \end{pmatrix},\ z:=\begin{pmatrix} x \\ u \end{pmatrix},\ w:=\begin{pmatrix} v \\ y \end{pmatrix}$$

the following theorem is a direct consequence of Theorem 2.2.

3.1 Theorem Assume that the dual linear programs (1.3) are both feasible and hence both solvable. Let caret variables $(\hat{x},\hat{y})$, $(\hat{u},\hat{v})$ denote primal and dual feasible vectors respectively, that is

$$(3.2) \qquad \hat{y} = A\hat{x} - b \geqq 0,\ \hat{x} \geqq 0,\ \hat{v} = -A^T\hat{u} + c \geqq 0,\ \hat{u} \geqq 0$$

and let bar variables $(\bar{x},\bar{y})$, $(\bar{u},\bar{v})$ denote primal and dual optimal vectors respectively, that is

$$(3.3) \qquad \bar{y} = A\bar{x} - b \geqq 0,\ \bar{x} \geqq 0,\ \bar{v} = -A^T\bar{u} + c \geqq 0,\ \bar{u} \geqq 0,\ c^T\bar{x} - b^T\bar{u} = 0$$

Let $J \subset \{1,2,\ldots,n\}$ and let $I \subset \{1,2,\ldots,m\}$. Then the following equivalences hold, where the notation "$\forall\bar{x}_J$" is defined as

$$\forall\bar{x}_J := \{\bar{x}_J \mid \bar{x} \text{ solves (1.3a)}\}$$

(a1) $\exists \hat{v}_J > 0 \Leftrightarrow \forall \bar{x}_J$ bounded $\Leftrightarrow \left\langle \begin{array}{l} \exists \hat{v}_J > 0;\ \forall(\bar{x},\hat{x}),\ \forall \hat{u}\ \text{ s.t. } \hat{v}_J > 0: \\ \qquad \|\bar{x}_J\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min\ \hat{v}_{j\in J} \end{array} \right.$

(a2) $\exists \hat{y}_I > 0 \Leftrightarrow \forall \bar{u}_I$ bounded $\Leftrightarrow \left\langle \begin{array}{l} \exists \hat{y}_I > 0;\ \forall(\bar{u},\hat{u}),\ \forall \hat{x}\ \text{ s.t. } \hat{y}_I > 0: \\ \qquad \|\bar{u}_I\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min\ \hat{y}_{i\in I} \end{array} \right.$

(b1) $\exists \hat{x}_J > 0 \Leftrightarrow \forall \bar{v}_J$ bounded $\Leftrightarrow \left\langle \begin{array}{l} \exists \hat{x}_J > 0;\ \forall(\bar{v},\hat{u}),\ \forall \hat{x}\ \text{ s.t. } \hat{x}_J > 0: \\ \qquad \|\bar{v}_J\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min\ \hat{x}_{j\in J} \end{array} \right.$

(b2) $\exists \hat{u}_I > 0 \Leftrightarrow \forall \bar{y}_I$ bounded $\Leftrightarrow \left\langle \begin{array}{l} \exists \hat{u}_I > 0;\ \forall(\bar{y},\hat{x}),\ \forall \hat{u}\ \text{ s.t. } \hat{u}_I > 0: \\ \qquad \|\bar{y}_I\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min\ \hat{u}_{i\in I} \end{array} \right.$

(c1) $\exists \hat{v}_J > 0, \hat{u}_I > 0 \Leftrightarrow \forall(\bar{x}_J, \bar{y}_I)$ bounded $\Leftrightarrow \left\langle \begin{array}{l} \exists \hat{v}_J>0,\ \hat{u}_I>0;\ \forall(\bar{x},\hat{x}),\ \forall \hat{u}\ \text{ s.t. } \hat{v}_J>0,\ \hat{u}_I>0: \\ \qquad \left\| \begin{matrix} \bar{x}_J \\ \bar{y}_I \end{matrix} \right\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min\ \{\hat{v}_{j\in J}, \hat{u}_{i\in I}\} \end{array} \right.$

(c2) $\exists \hat{y}_I > 0, \hat{x}_J > 0 \Leftrightarrow \forall(\bar{u}_I, \bar{v}_J)$ bounded $\Leftrightarrow \left\langle \begin{array}{l} \exists \hat{y}_I>0,\ \hat{x}_J>0;\ \forall(\bar{u},\hat{u}),\ \forall \hat{x}\ \text{ s.t. } \hat{y}_I>0,\ \hat{x}_J>0: \\ \qquad \left\| \begin{matrix} \bar{u}_I \\ \bar{v}_J \end{matrix} \right\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min\ \{\hat{y}_{i\in I}, \hat{x}_{j\in J}\} \end{array} \right.$

3.2 <u>Corollary</u> The quantity $c^T\hat{x}$ in parts (a1), (b2) and (c1) of Theorem 3.1 can be replaced by any upper bound α to $\min\ c^Tx$ s.t. $Ax \geqq b$, $x \geqq 0$, while the quantity $b^T\hat{u}$ in parts (a2), (b1) and (c2) of Theorem 3.1 can be replaced by any lower bound β to $\max\ b^Tu$ s.t. $A^Tu \leqq c$, $u \geqq 0$.

<u>Proof</u> To prove the first part, set $\hat{x}$ in (a1), (b2) and (c1) equal to a solution $\bar{x}$ of (1.3a) and note that $c^T\hat{x} = c^T\bar{x} \leqq \alpha$. To prove the second part, set $\hat{u}$ in (a2), (b1) and (c2) equal to a solution $\bar{u}$ of (1.3b) and note that $-b^T\hat{u} = -b^T\bar{u} \leqq -\beta$. $\square$

3.3 Remark When the index sets I, J are taken as singletons, the first equivalence in each of the statements (a1) to (c2) of Theorem 3.1 reduce to Theorem 3b of Williams [12]. In [11] Williams characterizes boundedness of components of feasible, but not optimal, points of linear constraint sets. In [10] Williams characterizes the boundedness of the totality of all the components (in contrast with individual components) of optimal points of linear programs. None of Williams' characterizations contain quantitative bounds like ours.

We turn our attention now to the nonsymmetric pair of dual linear programs

$$\text{(3.4a)} \qquad \min\ c^T x \quad \text{s.t. } Ax = b,\ x \geqq 0$$

$$\text{(3.4b)} \qquad \max\ b^T u \quad \text{s.t. } v = -A^T u + c \geqq 0$$

and establish the following bounds for their solutions.

3.4 Theorem Assume that the dual linear programs (3.4) are both feasible and hence both solvable. Let caret variables denote primal and dual feasible vectors, that is

$$\text{(3.5)} \qquad A\hat{x} = b,\ \hat{x} \geqq 0,\ \hat{v} = -A^T\hat{u} + c \geqq 0$$

and let bar variables denote primal and dual optimal vectors, that is

$$\text{(3.6)} \qquad A\bar{x} = b,\ \bar{x} \geqq 0,\ \bar{v} = -A^T\bar{u} + c \geqq 0,\ c^T\bar{x} - b^T\bar{u} = 0$$

Let $J \subset \{1,2,\ldots,n\}$ and $I \subset \{1,2,\ldots,m\}$. Then the following equivalences hold:

(a1) (i) $\exists \hat{v}_J > 0 \Leftrightarrow$ (ii) $\forall \bar{x}_J$ bounded $\Leftrightarrow$ (iii) $\left\langle \begin{array}{l} \exists \hat{v}_J > 0;\ \forall(\bar{x},\hat{x}),\ \forall \hat{u}\ \text{ s.t. } \hat{v}_J > 0: \\ \|\bar{x}_J\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min\ \hat{v}_{j\in J} \end{array} \right.$

(a2) (i) $\exists \hat{x} > 0$ and rows of A lin. indep. $\Leftrightarrow$ (ii) $\forall \bar{u}$ bounded

$\Leftrightarrow$ (iii) $\left\langle \begin{array}{l} \exists \hat{x} > 0 \text{ and rows of } A \text{ lin. indep; } \forall(\bar{u},\hat{u}),\ \forall \hat{x} > 0: \\ \|\bar{u}\|_1 \leqq \|(AA^T)^{-1}A\|_1 (\|c\|_1 + (c^T\hat{x} - b^T\hat{u})/\min_{1\leqq i\leqq n}\ \hat{x}_i) \end{array} \right.$

(b1) (i) $\exists \hat{x}_J > 0 \Leftrightarrow$ (ii) $\forall \bar{v}_J$ bounded $\Leftrightarrow$ (iii) $\left\langle \begin{array}{l} \exists \hat{x}_J > 0;\ \forall(\bar{v},\hat{u}),\ \forall \hat{x}\ \text{ s.t. } \hat{x}_J > 0: \\ \|\bar{v}_J\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min\ \hat{x}_{j\in J} \end{array} \right.$

<u>Proof</u> (a1): (ii) $\Leftarrow$ (iii): Evident.

(i) $\Leftarrow$ (ii): We establish the contrapositive implication.

$\nexists \hat{v}_J > 0 \Leftrightarrow -A^Tu + c\zeta \geqq 0,\ (-A^Tu + c\zeta)_J > 0,\ \zeta > 0$ has no solution

$\Leftrightarrow -Ax - A_{\cdot J}z_J = 0,\ c^Tx + c_J^Tz_J + \eta = 0,\ x \geqq 0,\ 0 \neq (z_J,\eta) \geqq 0$, has solution

(By Motzkin's theorem)

$\Leftrightarrow -Ax = 0,\ c^Tx + \eta = 0,\ x \geqq 0,\ 0 \neq (x_J,\eta) \geqq 0$ has solution

$\Rightarrow (\eta = 0)$ For each solution $\bar{x}$ of (3.4a), $\bar{x} + \lambda x$ is also a solution for any $\lambda > 0$, where $Ax = 0,\ c^Tx = 0,\ x \geqq 0,\ 0 \neq x_J \geqq 0$

($\eta > 0$ excluded, because it implies (3.4a) is unbounded below, which is ruled out by primal-dual feasibility assumption)

$\Rightarrow \exists$ unbounded $\bar{x}_J$

(i) $\Rightarrow$ (iii): $c^T\hat{x} \geqq c^T\bar{x} = b^T\bar{u} = b^T\bar{u} + \bar{x}^T\bar{v} \geqq b^T\hat{u} + \bar{x}^T\hat{v} \geqq b^T\hat{u} + \bar{x}_J^T\hat{v}_J$

$\geqq b^T\hat{u} + \|\bar{x}_J\|_1 \min\ \hat{v}_{j\in J}$

Hence

$$\|\bar{x}_J\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min\ \hat{v}_{j\in J}$$

(a2): (ii) $\Leftarrow$ (iii): Evident.

(i) $\Leftarrow$ (ii): We shall prove the contrapositive implication.

Rows of A lin. dep. or $\nexists \hat{x} > 0$ such that $A\hat{x} = b$

$\Longleftrightarrow$ Rows of A lin. dep. or $0 \neq \begin{pmatrix} -A^Tu \\ b^Tu \end{pmatrix} \geqq 0$ has solution

(By Motzkin's theorem)

$\Longleftrightarrow$ Rows of A lin. dep. or $0 \neq -A^Tu \geqq 0$, $b^Tu = 0$ has solution (Case of $-A^Tu \geqq 0$, $b^Tu > 0$, ruled out because it implies (3.4b) is unbounded above which is impossible by primal-dual feasibility assumption)

$\Rightarrow$ For each solution $\bar{u}$ of (3.4b), $\bar{u} + \lambda u$ is also a solution for any $\lambda > 0$ where either $b^Tu = 0$, $A^Tu = 0$, $u \neq 0$ or $b^Tu = 0$, $0 \neq -A^Tu \geqq 0$.

$\Rightarrow \exists$ unbounded $\bar{u}$

(i) $\Rightarrow$ (iii): Since $A^T\bar{u} = c - \bar{v}$ and rows of A are linearly independent it follows that $\bar{u} = (AA^T)^{-1}A(c - \bar{v})$ and hence

$$\|\bar{u}\|_1 \leqq \|(AA^T)^{-1}A\|_1(\|c\|_1 + \|\bar{v}\|_1)$$

But

$$b^T\hat{u} \leqq c^T\bar{x} = c^T\bar{x} - \bar{v}^T\bar{x} \leqq c^T\hat{x} - \bar{v}^T\hat{x} \leqq c^T\hat{x} - \|\bar{v}\|_1 \min_{1 \leqq i \leqq n} \hat{x}_i$$

Hence

$$\|\bar{v}\|_1 \leqq (c^T\hat{x} - b^T\hat{u}) \Big/ \min_{1 \leqq i \leqq n} \hat{x}_i$$

and consequently

$$\|\bar{u}\|_1 \leqq \|(AA^T)^{-1}A\|_1(\|c\|_1 + (c^T\hat{x} - b^T\hat{u}) \Big/ \min_{1 \leqq i \leqq n} \hat{x}_i)$$

(a3): (ii) $\Leftarrow$ (iii): Evident.

(i) $\Leftarrow$ (ii): We shall prove the contrapositive implication.

$\nexists \hat{x}$ such that $\hat{x}_J > 0$, $\hat{x} \geqq 0$ and $A\hat{x} = b$

$\Longleftrightarrow A^T u + v = 0$, $b^T u \geqq 0$, $v \geqq 0$, $\begin{pmatrix} v_J \\ b^T u \end{pmatrix} \neq 0$, has solution

(By Motzkin's theorem)

$\Longleftrightarrow A^T u + v = 0$, $b^T u = 0$, $v \geqq 0$, $v_J \neq 0$ has solution

(Case of $b^T u > 0$, $A^T u + v = 0$, $v \geqq 0$, ruled out because it implies (3.4b) is unbounded above which is impossible by primal-dual feasibility assumption)

$\Rightarrow$ For each solution $(\bar{u},\bar{v})$ of (3.4b), $(\bar{u} + \lambda u, \bar{v} + \lambda v)$ is also a solution for any $\lambda > 0$ where $A^T u + v = 0$, $b^T u = 0$, $v \geqq 0$, $v_J \neq 0$.

$\Rightarrow \exists$ unbounded $\bar{v}_J$

(i) $\Rightarrow$ (iii): $b^T\hat{u} \leqq c^T\bar{x} = c^T\bar{x} - \bar{v}^T\bar{x} \leqq c^T\hat{x} - \bar{v}^T\hat{x} \leqq c^T\hat{x} - \bar{v}_J\hat{x}_J$

$$\leqq c^T\hat{x} - \|\bar{v}_J\|_1 \min \hat{x}_{j\in J}$$

Hence

$$\|\bar{v}_J\|_1 \leqq (c^T\hat{x} - b^T\hat{u})/\min \hat{x}_{j\in J} \qquad \square$$

4. Conclusion

We have shown that every feasible point of a positive semidefinite linear complementarity problem contains numerical information on the size of some or all components of all solution vectors of the problem. Similarly each pair of primal-dual feasible points of a linear program was shown to contain information on the size of some or all components of all primal-dual solution vectors. Such bounds may be useful in obtaining information on where solutions lie without actually solving the problem. That such numerical bounds existed was not known, and the results presented here can also be thought of as a quantification of some of the duality relations that underly linear complementarity problems and linear programs.

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