ITERATIVE LINEAR PROGRAMMING FOR LINEAR COMPLEMENTARITY AND RELATED PROBLEMS

by

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Tzong-Huei Shiau

Under the supervision of Professor Olvi L. Mangasarian

ABSTRACT

This thesis is principally concerned with methods for solving the linear complementarity problem (LCP) of finding a vector x in R^n such that $x \geq 0$, $Mx+q \geq 0$, and $x^T(Mx+q)=0$ where M is a given n-by-n real matrix and q is a given n-vector. A geometric characterization is given for the vector p of the linear program: minimize p^Tx subject to $x \geq 0$, $Mx+q \geq 0$, each solution of which solves the (LCP). It is shown that for some positive (semi) definite matrix M and some positive matrix M, there is no single fixed p such that each solution of the linear program solves the (LCP). This result suggests that solving the (LCP) by a single linear program may be very difficult in certain cases. Hence an iterative linear programming (ILP) method is proposed in which the vector p is updated. The method guesses at a vector p, then iteratively updates it while

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applying the simplex method to the linear program. The method terminates, in a finite number of iterations, for M positive (semi) definite, for M with positive principal minors, and for some other matrices M for which current algorithms may fail. Numerical examples indicate that the method may be more robust than Lemke's method. An extension of the method is given for solving linearly constrained convex programs. We also give an SOR-based ILP method which is suitable for very large scale LCP problems with a sparse matrix M. The class of matrices for which the SOR-based ILP converges includes the class of nonsymmetric positive semi-definite matrices. Finally we give a comprehensive theory for quasi-diagonally dominant matrices in relation to the linear complementarity problem.

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CHAPTER I

INTRODUCTION

This thesis is principally concerned with methods for solving the linear complementarity problem. These methods of solutions are discussed in the first four chapters of this thesis. In addition we also give, in Chapter 3, an extension of one of the computational algorithms to solving linearly constrained convex programming problems. In Chapter 5 we present a comprehensive theory for quasi-diagonally dominant matrices as related to the linear complementarity problem.

1.1 THE LINEAR COMPLEMENTARITY PROBLEM

We consider the linear complementarity problem (LCP) of finding z in R^n such that

$$w = Mz + q \geq 0, \quad z \geq 0, \quad z^T w = 0, \tag{I.1}$$

where M is a given real $n \times n$ matrix and q is a given vector in R^n. Linear programming, quadratic programming and bimatrix games can be solved as LCP's [Lemke 65, Cottle and Dantzig 68]. One of the most popular methods for nonlinear programming is the iterative quadratic programming method [Wilson 63, Garcia and Mangasarian 76, Han 76,77, Powell 78] where a quadratic program is solved, usually by an LCP, at each iteration. LCP's arise in other fields, e.g. in finite difference schemes for free boundary problems [Cryer 71] and electronic circuit simulation [van Bokhoven 80].

The general LCP is NP-complete [Chung and Murty 81] and therefore very difficult to solve even for moderate n. However in many applications the matrix M has some nice properties, e.g. M is positive semi-definite or all principal minors of M are positive (i.e. M is a P-matrix [Murty 72]), and the problem can be solved by some direct pivoting methods, e.g. the principal pivoting method [Dantzig and Cottle 67] or Lemke's algorithm [Lemke 68]. Since the polynomial algorithm for linear programming was proposed [Hacijan 79], people have tried to modify it for solving LCP and have succeeded in showing that for M positive definite or positive semi-definite, (LCP) is polynomial-time solvable [e.g. Chung and

Murty 81, Kozlov, Taransov and Hacijan 79]. But the polynomial algorithms are very slow. Since the LCP has a vertex solution provided it has a solution [Mangasarian 78], Mangasarian tried to reduce the LCP to the following linear program (LP)

$$\textit{minimize } p^T z, \textit{ subject to } Mz+q\geq 0,\ z\geq 0, \tag{I.2}$$

by finding an appropriate cost vector p [Mangasarian 76,78,79]. An algebraic characterization of p was given, and for some cases the vector p can be obtained without great effort. The approach has many advantages. (1) The simplex method [Dantzig 63] can be started at any basis, hence in applications where parametric families of LCP's are to be solved, we can use prior solutions as starting points to the next LCP. (2) The simplex method is more robust than other direct methods for solving the LCP in dealing with numerical errors , such as loss of feasibility or near singularity of the basis. Unlike Lemke's algorithm in which the variable to enter the basis is uniquely determined, the simplex method can choose, among those nonbasic variables with negative reduced cost, to avoid these problems. (3) The simplex method is very popular and well-documented reliable LP codes are available at most computer installations. On the other hand a disadvantage of the LP approach is the difficulty in finding p in some important cases e.g. M is positive (semi)definite (see Chapter 2). Cottle and Pang have utilized a least-element geometric interpretation of the LP approach [Cottle and Pang 78].

In Chapter 3 we give an alternative approach, the iterative linear programming (ILP) method. The ILP method works like a single LP and has most of the advantages of the LP-approach. It converges to a vertex solution (or indicates that no solution exists) for M positive semidefinite. The method also works for M with positive principal minors and for some other matrices for which current algorithms may fail.

In many applications where n is very large and M is sparse, the pivoting methods become impractical because sparsity can quickly be lost after a few pivots and hence we need to store and process all $n \times n$ entries. Iterative methods that can preserve sparsity [Ahn 81, Cheng 81, Cottle, Golub and Sacher 78, Cryer 71, Mangasarian 77, Pang 81,82] are more suitable. A sparsity-preserving SOR-based ILP algorithm based on an successive overrelaxation method (SOR) [Agmon 54] is given in Chapter 4. The algorithm steps are simple and fast and the method converges for a wide range of matrices compared to other SOR-based methods.

1.2 ILP FOR LINEARLY CONSTRAINED MINIMIZATION PROBLEMS

At the end of Chapter 3 we extend the ILP method to solve linearly constrained minimization problems:

$$\textit{minimize } f(x) \textit{ subject to } Ax=b,\ x\geq 0, \tag{I.3}$$

where f is convex and has a Lipschitz continuous gradient. The algorithm can employ any pivoting method to solve the LP in each iteration of the ILP. Hence it is very attractive for problems with special structure, e.g. network flow problems, where very efficient pivoting methods exist.

1.3 QUASI-DIAGONALLY DOMINANT MATRICES

In Chapter 5 we give a comprehensive complementarity theory for quasi-diagonally dominant matrices, i.e. $n\times n$ matrices M such that there exists a positive vector d in R^n, such that

$$M_{ii}d_i \geq \sum_{j\neq i} |M_{ij}|\, d_j,\ i=1,\ldots,n. \tag{I.4}$$

This class of matrices contains H-matrices with positive diagonals [Pang 79]. A quasi-diagonally dominant matrix is not necessarily a P-matrix since we do not require strict inequality in (I.4) (see Chapter 5). We show that if M is quasi-diagonally dominant, then (LCP) has a solution for all q for which (LCP) is feasible, i.e. there exists x satisfying $x \geq 0$, $Mx+q \geq 0$. We also give a characterization for the feasibility of (LCP). If the transpose of M is quasi-diagonally dominant, we give a complete characterization of the set of all solutions of (LCP), and the solution set can be easily computed. We also give a characterization of the uniqueness of the solution.

1.4 NOTATION

All matrices, vectors and scalars are real. The following notation and reference system are used.

1. The n-dimensional Euclidean space is denoted by R^n.
2. Colomn vectors are denoted by lower case italic letters: $a, b, c, \ldots,$ or s, t, u, v, w, x, y, z. x_i denotes the $i-th$ component of x. $x>0$ denotes $x_i>0$ for all i, while $x \geq 0$ denotes $x_i \geq 0$ for all i.
3. Superscripts are used to denote different vectors, e.g. x^1, x^k and x^{k+1}, but the superscript T denotes the transpose, e.g. x^T, M^T. In the inner product $x^T y$ of two vectors x, y, the superscript T is usually suppressed.
4. Matrices are denoted by upper case italic $A, B, C, \ldots, M, N, \ldots, Z$. A_i and A_{ij} denote respectively the $i-th$ row and $ij-th$ element of A, and M^j denotes the $j-th$ column vector of M. I is the $n \times n$ identity matrix.

5. Real functions defined on subsets of R^n are denoted by $f, g, h, \ldots$, while $\nabla f(x)$ and $\nabla^2 f(x)$ denote respectively the gradient and the Hessian of f at x if $f: R^n \to R$, while $\nabla g(x)$ denotes the $m \times n$ Jacobian if $g: R^n \to R^m$.

6. Lower case Greek letters denote scalars, e.g. α, λ.

7. The Euclidean norm $(x^T x)^{1/2}$ of a vector x in R^n is denoted simply by $|x|$.

8. References are referred to by author's name and year (e.g. [Mangasarian 79]) and are in alphabetical order in the bibliography.

CHAPTER II

SOLVABILITY BY A SINGLE LINEAR PROGRAM

2.1 INTRODUCTION

It was shown that the LCP(I.1) is solvable if and only if there exists a p in R^n satisfying certain conditions such that the LP(I.2) is solvable. Furthermore for such p, each solution of the LP is a solution of the LCP [Mangasarian 76,78]. Algebraic characterizations of the vector p have been given [Mangasarian 76,78,79]. Based on the characterizations, the vector p can be easily determined for a number of special cases [Mangasarian 78,Table 1]. However for some important cases, e.g. M being positive semi-definite or $M>0$, it is not easy to find an appropriate p. Because for all the explicit cases for which p is given [Mangasarian 78, Theorem 4], the choice of p is independent of the vector q in (1.1), i.e. given the matrix M we can find a p which works for every q, the following question is important. *Given a positive semidefinite matrix M (or given $M>0$), can we find a vector p which works for every q?* We shall answer this question in the negative in this chapter. It should be pointed out that Cottle and Pang have given a least-element geometric interpretation of Mangasarian's results [Cottle and Pang 78].

In this chapter a geometric characterization of p is given, which may not be as useful in finding p as Mangasarian's characterization, however it does lead to a greater understanding of the LP approach. Consequently we are able to show that for some positive definite matrices M and some $M>0$, the choice of p must be dependent on the vector q.

(2.1.1) Notation From now on we use the symbol (M,q) to denote the LCP

$$Mx+q\geq 0,\ x\geq 0,\ x^T(Mx+q)=0, \qquad \text{(M,q)}$$

and use (M,q,p) to denote the LP

$$\min p^T x \quad \textit{subject to } Mx+q\geq 0,\ x\geq 0 \qquad \text{(M,q,p)}$$

2.2 NECESSARY AND SUFFICIENT CONDITIONS

(2.2.1) Definition *(M,q) is LP–solvable by a given fixed $p \neq 0$* iff the following statement holds,

(M,q) has a solution if and only if (1) (M,q,p) is solvable and (2) every solution of (M,q,p) solves (M,q).

In short, we can either get a solution of (M,q) or conclude that (M,q) is not solvable by solving (M,q,p). Note that (2) of the definition is required only when (M,q) has a solution. Hence we know that (M,q) has no solution either when (M,q,p) has no solution or when we get a solution of (M,q,p) which is not a solution to (M,q).

We say that *M is LP–solvable by a given fixed p* if (M,q) is LP-solvable by p for all q. Note that in this case, p is independent of q. ∎

Here we review some linear programming terminology which will be frequently used in the sequel. By introducing slack variables, the system

$$Mx+q \geq 0,\ x \geq 0,$$

can be written as

$$Iw-Mx = q,\ x \geq 0,\ w \geq 0. \tag{II.1}$$

(2.2.2) Definition A *basis* is a set of n linearly independent column vectors of I and $-M$ in (II.1), a variable x_i or w_i is *basic* (with respect to the basis) if its corresponding column vector is in the basis. A *basic solution* is the unique one found by solving for the values of basic variables when the nonbasic variables are set to zero. A basis is *feasible* if the corresponding basic solution is nonnegative, hence it satisfies (II.1), in which case the basic solution is called a BFS (*Basic Feasible Solution*). A basis is *complementary* if exactly one of x_i or w_i is basic for $i=1,2,\ldots,n$, hence $x^T w=0$, and the complementary basic solution is a solution to (M,q) iff it is feasible.

(2.2.3) Definition Let M be an $n \times n$ matrix. Let S be a subset of the index set $J=\{1,2,\ldots,n\}$, let $\bar{S}=J-S$, define $P_M(S)$ and $Q_M(S)$ as follows,

$$
\begin{aligned}
P_M(S) &:= \{\, p \mid p = \sum_{i\in S} s_i M_i + \sum_{j\in \bar{S}} s_j I_j,\ s\geq 0\} \\
&= \textit{convex cone generated by } \{M_i \mid i\in S\}\cup\{I_j \mid j\in\bar{S}\}
\end{aligned}
$$

$$
Q_M(S) := \textit{convex cone generated by } \{-M^j \mid j\in S\}\cup\{I^i \mid i\in\bar{S}\}
$$

Note that vectors in $P_M(S)$ are row vectors. $Q_M(S)$'s are called complementary cones [Murty 72]. For significance of $P_M(S)$ see Remarks (2.2.5).

(2.2.4) Lemma

$q \in Q_M(S)$ iff (M,q) has a solution x such that $x_i = 0$ for $i \in \overline{S}$ and $(Mx+q)_j = 0$ for $j \in S$.

Proof. Let $q \in Q_M(S)$, i.e. there exists $s \geq 0$ such that $q = \sum_{j \in S} -s_j M^j + \sum_{j \in \overline{S}} s_j I^j$. Hence

$$q = -Mx + Iw, \textit{ where} \tag{II.2}$$

$$x_j := s_j \geq 0,\ w_j := 0 \text{ for } j \in S, \text{ and} \tag{II.3}$$

$$x_j := 0,\ w_j := s_j \geq 0 \text{ for } j \in \overline{S}. \tag{II.4}$$

Hence x is a solution of (M,q) with the asserted properties.

The other part of the Lemma can easily be proved by reversing the argument. ∎

(2.2.5) Remarks

(i) From now on throughout this chapter, we consider M is given and fixed, and suppress the subscripts of $P_M(S)$ and $Q_M(S)$.

(ii) Given subset S of J, we can define a *complementary* basis $B := \{-M^j \mid j \in S\} \cup \{I^j \mid j \in \overline{S}\}$, provided B is linearly independent. From now on when we say basis S, we are referring to the complementary basis B.

(iii) Given basis S, $Q(S)$ is the set of all q-vectors for which S is *feasible* [see (2.2.2)], so the BFS is a solution of (M,q). On the other hand, $P(S)$ is the set of all p-vectors for which S is optimal for the linear program (M,q,p). Hence if $q \in Q(S)$ and $p \in P(S)$ then the BFS is a solution to both (M,q) and (M,q,p). However, this does not imply that ***every*** solution to (M,q,p) is a solution to (M,q). To handle this, we introduce $P^o(S)$ which is the relative interior of $P(S)$.

(2.2.6) Definition Let S and $\overline{S}$ be as in (2.2.2), define

$$P^o(S) := \{ p \mid p = \sum_{i \in S} s_i M_i + \sum_{j \in \overline{S}} s_j I_j, \; s > 0 \}$$

(2.2.7) Lemma Let $q \in Q(S), p \in P^o(S)$. Then (M,q,p) has a solution and every solution is a solution to (M,q). Moreover, if S is a basis, i.e. the corresponding n column vectors are linearly independent, then the solution to (M,q,p) is unique.

Proof Since $q \in Q(S)$, as in Lemma (2.2.4), there exist x and w satisfying (II.2,3,4). Hence x is feasible in (M,q,p). Since $p \in P^o(S)$,

$$p = \sum_{i \in S} t_i M_i + \sum_{j \in \overline{S}} t_j I_j, \; t > 0.$$

Hence for all feasible z,

$$p(x-z) = \sum_{i \in S} t_i M_i (x-z) + \sum_{j \in \overline{S}} t_j (x_j - z_j) \tag{II.5}$$

$$= \sum_{i\in S} t_i(-q_i - M_i z) + \sum_{j\in \bar{S}} t_j(-z_j) \le 0 \tag{II.6}$$

since $M_i x = w_i - q_i = q_i$ for $i \in S$ (II.2,3), and $x_j = 0$ for $j \in \bar{S}$ (II.4). Hence x is optimal for (M,q,p).

Let z be any solution to (M,q,p), let $v := Mz + q$. Then $p(x-z)=0$. By (II.5,6), note that $t_i > 0$ for $i = 1,\ldots,n$, we have $v_i \equiv M_i z + q_i = 0$, $z_j = 0$ for $i \in S$, $j \in \bar{S}$. Hence z is also a solution to (M,q). If, in addition, that S is a basis, then since

$$I(w-v) - M(x-z) = (w - Mx) - (v - Mz) = q - q = 0$$

and the n column vectors $\{I^j \mid j \in \bar{S}\} \cup \{-M^j \mid j \in S\}$ are linearly independent, we have $w = v$ and $z = x$. Hence the solution to (M,q,p) is unique. ■

(2.2.8) Theorem (Sufficient Condition) For a given p, let $A^o(p) := \cup\{Q(S) \mid p \in P^o(S)\}$. If $Q(T) \subseteq A^o(p)$ for every *index subset* T, then *M is LP-solvable by p*.

Proof Given any q such that (M,q) has a solution, we have $q \in Q(T)$ for some index set T, hence $q \in A^o(p)$, i.e. $q \in Q(S)$ for some S such that $p \in P^o(S)$. The theorem then follows by Lemma(2.2.7). ■

(2.2.9) Theorem (Necessary Condition) If *M is LP-solvable by p* ($p \neq 0$), then $Q(T) \subseteq A(p)$ for every *basis* T, where $A(p) := \cup\{Q(S) \mid p \in P(S)\}$.

Proof Let *M be LP−solvable by p*. We suppose there is a ***basis*** T such that $Q(T)$ is not contained in $A(p)$, and shall show a contradiction. Note that the (topological) interior of $Q(T)$, denoted by $Q(T)^o$, is nonempty (an open cone generated by *n linearly independent* vectors) and not contained in $A(p)$ (since $A(p)$ is closed and the closure of $Q(T)^o$ is $Q(T)$). Hence $Q(T)^o - A(p)$ is a nonempty open set, which contains some open ball, say, $B(q, \delta)$ with center q and radius δ. If (M,q,p) has no solution, we are done since the basic solution of T is a solution to (M,q). So we assume (M,q,p) has an optimal BFS (x, w) with basis B. For any complementary basis S which is feasible, i.e. $q \in Q(S)$, we have $p \notin P(S)$ (for $q \notin A(p)$), so S is not optimal in (M,q,p) and hence can not be the basis B. Therefore B is not complementary. If necessary we perturb the q vector a little (keeping it in the open ball) to make B nondegenerate, e.g. let $(\bar{x}, \bar{w}, \bar{q})$ be defined as follows:

$$\bar{x}_i := x_i + \varepsilon,\ \bar{w}_j := w_j + \varepsilon \quad \text{for } \textbf{\textit{basic variables}}\ x_i \text{ and } w_j,$$

the rest components of $\bar{x}$ and $\bar{w}$ remain zero, and

$$\bar{q} := I\bar{w} - M\bar{x},$$

where $\varepsilon > 0$ is sufficiently small so that $\bar{q} \in B(q, \delta)$. Note that after the perturbation, it is still true that the basic solution with basis T is a solution to $(M, \bar{q})$. Since B is nondegenerate and is not complementary, $\bar{x}^T\bar{w}\neq 0$. Hence $(\bar{x},\bar{w})$, which is a solution to (M,$\bar{q}$,p), is not a solution to (M,$\bar{q}$), which contradicts the assumption that M is LP-solvable by p. ■

(2.2.10) Remarks

(i) Note that the $A^o(p)$ in Theorem (2.2.8) is slightly different from the $A(p)$ in (2.2.9).

(ii) Also note that we use the phrase ***index set*** in Theorem (2.2.8) but, in (2.2.9) we can only prove $Q(S)\subseteq A(p)$ for ***basis*** S. If, in (2.2.9), we assume that for every $q\notin A(p)$, (M,q,p) has a ***nondegenerate*** optimal BFS whenever it has a solution, then we need no perturbation in the proof and we can have the stronger conclusion that $Q(S)\subseteq A(p)$ for every ***index set*** S.

(iii) The stronger conclusion in (ii) is also true if we simply change the definition of being ***LP−solvable*** in (2.2.1) as follows.

> (M,q) is (strongly) ***LP−solvable by p*** iff (1) (M,q) has a solution if and only if (M,q,p) has, and (2) every solution of (M,q,p) is a solution of (M,q).

Note that in Definition (2.2.1), (2) is required only when (M,q) has solutions.

Proof of (iii). With the strong version, i.e. assuming (1) and (2) hold for all vector q, let T be any index set (not necessarily a basis) such that $Q(T)-A(p)$ is not empty. We shall show a contradiction. Let $q \in Q(T)-A(p)$. Then by Lemma (2.2.4), (M,q) has a solution. By (1) (M,q,p) must also have a solution, hence has one which is a BFS with basis B, say. By the same reason as before B is not complementary. Again, we perturb q to $\bar{q}$ so that B is not degenerate, and we have a solution to (M,q̄,p) which is not a solution to (M,q̄), contradicting to (2). {Note that (M,q̄) may have no solution, so we can not have any contradiction to the old Definition (2.2.1).} ∎

2.3 APPLICATIONS

(2.3.1) Example

$$M := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

It is clear from fig. 1 that $\{Q(\phi), Q(\{1\})\}$ is a covering of the union of all $Q(S)$'s. It follows by (2.2.8) that M is LP-solvable by any vector

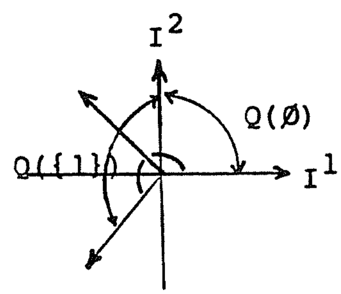

Figure 1.

$$p \in P^o(\phi) \cap P^o(\{1\}) = \{\alpha[1,1]+\beta[0,1] \mid \alpha,\beta>0\}$$

(2.3.2) Example (of a positive definite matrix not solvable by any p-vector)

$$M := \begin{bmatrix} 1 & 1 & -1 \\ \frac{1}{2} & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

is positive definite, hence (1) for any index set S, S is a basis and $Q(S)$ has nonempty interior and (2) the class of all $Q(S)$ is a partition of R^n [Samelson et. al. 58]. Therefore, $Q(T)$ is not contained in $\bigcup\{Q(S) \mid S \neq T\}$ for any T. On the other hand, it is straightforward to see that $P(\{1\}) \cap P(\{2,3\}) \cap P(\phi) = \{0\}$. Hence for any $p \neq 0$, there exists T such that $p \notin P(T)$. So it follows by (2.2.9) that M is not LP-solvable by p. Thus for example, let

$$q^1 := -M^1 + I^2 + I^3 = \begin{bmatrix} -1 \\ \frac{1}{2} \\ 0 \end{bmatrix} \in Q(\{1\})^o,$$

$$q^2 := I^1 - M^2 - M^3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \in Q(\{2,3\})^o,$$

$$q^3 := I^1 + I^2 + I^3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in Q(\phi)^o,$$

then for (M,q^1), p must be in $P(\{1\})$, but for (M,q^2), p must be in $P(\{2,3\})$, and for (M,q^3), p must be in $P(\phi)$, so no single p can solve all of (M,q^1), (M,q^2) and (M,q^3). ▪

(2.3.3) Example (of a positive matrix not solvable by any p-vector)

$$M := \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

is a positive matrix. It is easy to see that $P(\{1\}) \cap P(\{2\}) = \{0\}$ and neither $Q(\{1\})$ nor $Q(\{2\})$ is covered by the union of those $Q(S)$ other than itself. So by the same argument, note that both $\{1\}$ and $\{2\}$ are basis, M is not LP-solvable by any $p \neq 0$. Thus for example, let

$$q^1 := -M^1 + 2I^2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \in Q(\{1\})^o,$$
$$q^2 := 2I^1 - M^2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \in Q(\{2\})^o,$$

then no single p can solve both (M,q^1) and (M,q^2).

2.4 SUMMARY

We considered the problem of reducing the linear complementarity problem (M,q) to a linear program (M,q,p) by choosing p appropriately. In Mangasarian's results, we observed that p is independent of q in all cases where p can easily be found. So we concentrated on the problem of finding a single p, given M, so that solving (M,q,p) will give answer to (M,q) for *all* q. We gave necessary and sufficient conditions for the existence of such a p. Based on the results, we gave examples showing that there exists no such p for some positive definite matrices M and some positive matrices M.

So it seems difficult, in general, to find a p in one shot for solving (M,q) when M is positive (semi-) definite or positive. In the next chapter, we give an algorithm that *guesses* a p, and then modifies it along the way when (M,q,p) is being solved.

CHAPTER III

THE ITERATIVE LINEAR PROGRAMMING METHOD

3.1 INTRODUCTION

Although the LP approach has many nice properties, it is in general very difficult to find the cost vector p. This is the case, as example (2.3.2) suggests, even when M is positive (semi-) definite. In this chapter we give another approach, called iterative linear programming (ILP), which allows p to vary along the way while solving the linear program (M,q,p). In general the method does not solve a sequence of linear programs as its name suggests, but rather it solves a single linear program with a variable cost vector p which is changed whenever a certain criterion is met.

In this chapter we give a simplex based ILP method which uses the primal simplex method to solve (M,q,p). Starting at a basic feasible solution (BFS), we apply the primal simplex method to (M,q,p) until either (1) the current BFS is a solution to the LCP or (2) the objective value is lower than certain level which is easily calculated at the beginning of each iteration, or (3) the current BFS is a solution to (M,q,p). In cases (2) and (3), we update the cost vector p and the level, then begin the next iteration. The algorithm is a generalization to a nonconvex linear complementarity problem of a finite algorithm by Frank and Wolfe for solving convex quadratic programs [Frank and Wolfe 56]. This should not be confused with the potentially slowly convergent Frank-Wolfe algorithm for solving linearly constrained convex programs. In fact, if (M,q) is formed by the Karush-Kuhn-Tucker conditions for a convex quadratic program then the two algorithms are equivalent. However for the nonconvex case both our algorithm and the convergence proof are different from the Frank-Wolfe algorithm with a cutting plane. We will derive the algorithm directly for LCP (M,q) for which the quadratic function $x^T Mx$ is convex at some solution, and then generalize it for a general M without any convexity assumption. It is shown that the algorithm converges for matrices with positive principal minors, for positive semidefinite matrices and for some other matrices for which current algorithms may fail. The convergence is finite and very fast. According to our computational experiments, the number of iterations is around 5 for n=40~50 and M being positive semi-definite. Some numerical results for general M are given in Section (3.5) which indicate that our method can solve problems which cannot be solved by Lemke's method [Lemke 68].

In Section 3.6, the simplex based ILP algorithm is modified for solving linearly constrained convex programming problems. Since the algorithm does not change the original linear constraints, it is very attractive for problems with specially structured constraints for whose linear programming approximation very efficient simplex codes exist.

3.2 THE BASIC IDEA OF THE ILP METHOD

Let us first concentrate on problems where M is positive semi-definite. Consider the following quadratic programming problem,

$$\begin{aligned} \min f(x) &:= x^T(Mx+q) \\ s.t. \;\; Mx+q &\geq 0, \\ x &\geq 0. \end{aligned} \tag{QP}$$

It is obvious that the linear complementarity problem (M,q) has a solution iff (QP) has a solution with objective value 0, in which case the two solution sets are equal. (QP) is considered an easy problem because of the following.

(3.2.1) Remarks Assume that M is positive semi-definite and that (M,q) has a solution $\bar{x}$. Then (QP) has the following properties.

(i) f is convex.

(ii) The optimal value is known to be 0.

(iii) (QP) has a vertex solution since (M,q) does.

When solving a problem with iterative methods, the convergence is usually much faster if the problem has a vertex solution. In fact, (iii) enables us to show that the simplex based ILP algorithm terminates in finite number of iterations. (i) and (ii) give us a hyperplane that separates a given feasible but not complementary point from the solution set, as the next lemma shows.

(3.2.2) Lemma Let x^k be feasible but not complementary, i.e. $x^k \geq 0$, $Mx^k+q \geq 0$ but $f(x^k) > 0$. Then the linear inequality

$$f(x^k)+\nabla f(x^k)(x-x^k) \leq 0 \tag{III.1}$$

which is violated by x^k is satisfied by every solution $\bar{x}$ of (M,q) satisfying

$$(x^k-\bar{x})^T M(x^k-\bar{x}) \geq 0 \tag{III.2}$$

Proof It is obvious that x^k violates (III.1) because $f(x^k)>0$. Let $\bar{x}$ be any solution of (M,q) satisfying (III.2).

Let $v^k := Mx^k+q$, $\bar{v} := M\bar{x}+q$, we have

$$f(x^k) + \nabla f(x^k)(\bar{x}-x^k)$$

$$= v^k x^k + (v^k+M^T x^k)^T(\bar{x}-x^k)$$

$$= v^k x^k + v^k(\bar{x}-x^k) + x^k M(\bar{x}-x^k)$$

$$= v^k \bar{x} + x^k(M\bar{x}+q-Mx^k-q)$$

$$= (v^k-\bar{v})\bar{x} + x^k(\bar{v}-v^k) \qquad (\text{* since } \bar{v}\bar{x}=f(\bar{x})=0 \text{ *})$$

$$= (\bar{x}-x^k)(v^k-\bar{v})$$

$$= -(x^k-\bar{x})(Mx^k+q-M\bar{x}-q)$$

$$= -(x^k-\bar{x})M(x^k-\bar{x}) \leq 0 \qquad (\textit{* by III.2 *})$$

So $\bar{x}$ satisfies (III.1). ∎

Note that (III.2) is automatically satisfied if M is positive semi-definite.

So we have a simple algorithm —the cutting plane method:

(3.2.3) The Cutting Plane Method

(1) Start at any feasible point x^o, (2) Given x^k, add the constraint (III.1) and solve for x^{k+1}. ∎

It is easy to establish that any accumulation point of $\{x^k\}$ is a solution.

Convergence Theorem of The Cutting Plane Method

Let $\{x^k\}$ be generated by the cutting plane method defined above. Then any accumulation point $\bar{x}$ of $\{x^k\}$ is a solution of (M,q).

Proof. Let $\{x^{k_j}\}$ be a subsequence converging to $\bar{x}$. Since $x^{k_{j+1}}$ satisfies all previous cuts, i.e. all the cuts (III.1) for which $k<k_{j+1}$, we have

$$f(x^{k_j})+\nabla f(x^{k_j})(x^{k_{j+1}}-x^{k_j}) \leq 0$$

Taking the limit as j approaches infinity, we get $f(\bar{x})\leq 0$. On the other hand, it is obvious that $\bar{x}$ is feasible. So $f(\bar{x})\geq 0$. ∎

However, this algorithm, like other cutting plane methods, suffers from the following diffculties. As the number of contraints increases, the system becomes bigger and the number of vertices increases exponentially. Therefore it not only takes more computing time and storage space at each iteration, it may very well slow down the speed of convergence. Consequently more iterations are needed and more cuts (constraints) are added and the system becomes even bigger.

Fortunately, we have found a solution in our ILP method. By simply doing a line search, we will show that none of previous cuts is needed. Even the current cut, that is (III.1) in the k-th iteration, is not needed if we use the following LP to find the next point,

$$\textit{minimize } \nabla f(x^k)(x-x^k)+f(x^k) \qquad \text{(LPk)}$$

$$\textit{subject to } Mx+q \geq 0, x \geq 0.$$

By considering the cut as an objective function to be minimized, we keep the feasible set fixed. Hence any vertex we get, when we solve (LPk) by the primal simplex method, is a vertex of the original LCP. This is desirable because the LCP has a vertex solution. Since all we really expect at each iteration is the satisfaction of (III.1), we need not completely solve (LPk) and get an optimum of (LPk). Instead we can stop as soon as current BFS satisfies (III.1). This is bound to occur if there is any vertex satisfying (III.1) since (LPk) is bounded (see Lemma (3.2.4)) and the left hand side of (III.1) is the objective function to be minimized. Note that should (LPk) be unbounded, the simplex method might terminate at a unbounded ray without finding any vertex satisfying (III.1).

(3.2.4) Lemma Let x^k be feasible. Then (LPk) is bounded (without any convexity assumption).

Proof. Let d be any direction of the feasible region, i.e. there is a vector y such that $y+\lambda d$ is feasible for all $\lambda \geq 0$. Then $d>0$ and $Md>0$. Hence

$$\nabla f(x^k)d = (Mx^k+q)^T d + (Md)^T x^k \geq 0$$

Therefore any direction of the feasible region is not a descent direction, so (LPk) is bounded. ∎

There is another reason why we do not want to waste time going all the way to an optimum. We can view (LPk) as the linearization of (QP) at x^k, and usually a linearization is a good approximation only locally. So there is no point going to an optimum of (LPk) which may be a poor approximation of the objective of (QP).

We are now ready to give the formal definition of the simplex based ILP algorithm and its convergence proof for the convex case.

3.3 THE SIMPLEX BASED ILP ALGORITHM—THE CONVEX CASE

(3.3.1) Definition Of The Algorithm

(0) Find any BFS, say x^o (e.g. by phase one of simplex method). Repeat next two steps until x^k is complementary.

(1) Given x^k, use the primal simplex method and partially solve (i.e. take pivot steps)

$\min \nabla f(x^k)^T y$

$s.t.\ My+q \geq 0,\ y \geq 0.$

until

$$\nabla f(x^k)^T y \leq \nabla f(x^k) x^k - f(x^k) \tag{III.3}$$

Let y^k be the BFS obtained satisfying (III.3).

(2) Line-search along the direction $p^k := y^k - x^k$ using the minimization stepsize $x^{k+1} := x^k + t_k p^k$ where

$$t_k := arg\ \underset{0 \leq t \leq 1}{minimum}\ f(x^k + tp^k)$$

In the following lemma we do not assume any convexity of f.

(3.3.2) Lemma Assume p^k is a descent direction, i.e.

$$\nabla f(x^k) p^k < 0 \tag{III.4}$$

Let x^{k+1} and t_k be defined as in (2) of Algorithm (3.3.1).

Then

$$t_k = \begin{cases} -\nabla f(x^k) p^k / 2p^k M p^k =: \bar{t} & \text{if } 0<\bar{t}<1 \\ 1 & otherwise \end{cases} \tag{III.5}$$

and

$$f(x^{k+1})-f(x^k)\le \tfrac{1}{2}t_k \nabla f(x^k)p^k \tag{III.6}$$

Proof. Since the function $\psi(t):=f(x^k+tp^k)$, $t\in R$, is quadratic, its Taylor expansion at $t=0$ up to second order terms has residual equal to zero. Hence

$$\begin{aligned}\varphi(t) \equiv f(x^k+tp^k) &= f(x^k)+t\nabla f(x^k)p^k+t^2p^kMp^k \\ &= \alpha+\beta t+\gamma t^2\end{aligned} \tag{III.7}$$

where

$$\alpha:=f(x^k)>0,\ \beta:=\nabla f(x^k)p^k,\ \gamma:=p^kMp^k.$$

Case1. $\gamma\le 0$.

Since $\beta<0$ by assumption (III.4), $\varphi(t)$ is strictly decreasing on the unit interval. So $t_k=1$ and $f(x^{k+1})=\varphi(1)$. So

$$f(x^{k+1})-f(x^k) = \beta+\gamma\le\beta\le \tfrac{1}{2}t_k\beta$$

which is (III.6). Note that $\bar{t}=-\beta/2\gamma<0$ is not in the open interval $(0,1)$ which results in the second case of (III.5).

Case2. $\gamma>0$ and $\gamma\le -\tfrac{1}{2}\beta$.

So $\bar{t}=-\beta/2\gamma\ge 1$ is the unique root of $\varphi'(t)=0$. Since $\varphi(t)$ is strictly convex, $\bar{t}$ is the global minimum and $\varphi(t)$ is strictly decreasing on interval $[0,\bar{t}]$. Hence $t_k=1$, and

$$f(x^{k+1})-f(x^k) = \beta+\gamma\le\beta-\tfrac{1}{2}\beta\le \tfrac{1}{2}t_k\beta$$

proving (III.6).

Case3. $\gamma > 0$ and $\gamma > -\frac{1}{2}\beta$.

So $0<\bar{t}<1$ and $t_k=\bar{t}$ because $\bar{t}$ is again the global minimum of $\varphi(t)$. Since

$$t_k\gamma = \bar{t}\gamma = -\beta/2 \tag{III.8}$$

we have

$$f(x^{k+1})-f(x^k) = t_k(\beta+t_k\gamma) = t_k(\beta-\tfrac{1}{2}\beta) = \tfrac{1}{2}t_k\beta$$

and the proof is complete. ■

(3.3.3) Theorem (Convergence of Simplex Based ILP Algorithm)
Assume that there exists a solution, say $\bar{x}$, of the LCP (M,q) such that

$$(y-\bar{x})^T M(y-\bar{x}) \geq 0 \quad \text{for all feasible y.} \tag{III.9}$$

Then for each $\varepsilon>0$ there exists $N\geq 0$, depending on ε, such that

$$f(x^0) > f(x^1) > \cdots > f(x^N), \text{ and}$$

$$0 \leq f(x^N) < \varepsilon$$

where $\{x^k\}$ is the sequence generated by Algorithm (3.3.1).

(We did not say $\lim_{k\to\infty} f(x^k)=0$ because, as the next theorem shows, only finite terms of $\{x^k\}$ are generated before we get a solution.)

Proof. First, let us show that each iteration , i.e. steps (1) and (2), of Algorithm (3.3.1) will successfully generate x^{k+1}, given $f(x^k)>0$. By Lemma (3.2.2) $\bar{x}$ satisfies (III.3). Since the LP in step (2) of Algorithm (3.3.1) is bounded by Lemma (3.2.4), and since we are minimizing the left hand side of (III.3), we will find y^k satisfying (III.3) by the simplex method. So $p^k := y^k - x^k$ is a descent direction, therefore x^{k+1} will be generated successfully and

$$f(x^k) > f(x^{k+1}). \tag{III.10}$$

Next, we show that $f(x^j)$ can be arbitrarily small provided j is sufficiently large. Given $\varepsilon>0$, suppose that

$$f(x^k) \geq \varepsilon > 0, \text{ for all } k, \tag{III.11}$$

and we shall exhibit a contradiction. Indeed, by (III.6) and (III.3),

$$f(x^{k+1}) - f(x^k) \leq \tfrac{1}{2} t_k(-f(x^k)) \tag{III.12}$$

It is easy to see, by induction, that x^k is a convex combination of x^o and vertices of (M,q). Since y^k is also a vertex of (M,q), $p^k := y^k - x^k$ remains in a compact set, hence $p^k M p^k$ is bounded above by a number, say V, which is independent of k and j. So by (III.5) and (III.3), if $t_k<1$, we have

$$t_k \geq -\nabla f(x^k)p^k / 2V \geq f(x^k)/2V \tag{III.13}$$

which, togeher with (III.11), implies

$$t_k \geq \min[1, f(x^k)/2V] \geq \min[1, \varepsilon/2V] =: \delta \qquad \text{(III.14)}$$

Summing up (III.12) for k from 0 to j, then applying (III.14), we have

$$f(x^{j+1}) - f(x^o) \leq -\tfrac{1}{2}\delta \sum_{k=0}^{j} f(x^k) \qquad \text{(III.15)}$$

$$\leq -\tfrac{1}{2}\delta(j+1)\varepsilon \qquad (\text{* by } III.\,11 \text{ *})$$

Dropping the term $f(x^{j+1})$, since it is nonnegative, we have

$$f(x^o) \geq \tfrac{1}{2}(j+1)\delta\varepsilon, \text{ for all } j \qquad \text{(III.16)}$$

which is impossible since, as j approaches to infinity, the right hand side becomes arbitrarily large while the left hand side remains fixed. ■

The above theorem says that Algorithm (3.3.1) either (1) finds a solution in a finite number of iterations or (2) generates a sequence $\{x^k\}$ such that $\{f(x^k)\}$ decreases to zero. The next theorem shows that only (1) can occur. The proof employed here is similar to that of [Frank and Wolfe 56] which shows the finiteness of the Frank-Wolfe algorithm when applied to the complementary condition of a convex quadratic program.

(3.3.4) Theorem (Finite Termination of Simplex Based ILP Algorithm) Under the same assumption as Theorem (3.3.3), i.e. there exists a solution $\bar{x}$ of the LCP (M,q) satisfying (III.9). Then there exists $N \geq 0$ such that

$$f(x^0) > f(x^1) > \cdots > f(x^N) = 0.$$

where $\{x^k\}$ are generated by Algorithm (3.3.1).

Proof. Let

U := { z | z is a vertex of (M,q), but not a solution }

V := { z | z is a vertex solution of (M,q) }

S := { x | x is a solution of (M,q) }

Since U and V are finite, their convex hull, denoted by conv(U) and conv(V) respectively, are compact.

Claim (1) : if $x \in$conv(U) then x is not a solution.

Indeed, suppose $x = \sum s_k z^k$ were a solution, where $z^k \in$U, $s_k > 0$, $\sum s_k = 1$. Then for $i = 1,\ldots,n$, either $x_i = 0$ or/and $w_i = 0$, where $w := Mx + q$. If $x_i = 0$, then

$$0 = x_i = \sum s_k z_i^k \geq 0,$$

so $z_i^k = 0$ for all k. Similarly, if $w_i = 0$,

$$0 = w_i = (M \sum s_k z^k + q)_i = \sum s_k (Mz^k + q)_i \geq 0,$$

so $(Mz^k + q)_i = 0$ for all k. Therefore for $i = 1,\ldots,n$, either $z_i^k = 0$ or/and $(Mz^k + q)_i = 0$ for all k. So we have that z^k is a vertex solution, which contradicts $z^k \in$U.

Hence $f(x)>0$ for all $x \in \mathrm{conv(U)}$. Therefore

$$\delta := \min \{ f(x) \mid x \in \mathrm{conv(U)} \} > 0$$

since conv(U) is compact. Let $\{x^k, y^k\}$ be defined by Algorithm (3.3.1). By Theorem (3.3.3), there exists $N \geq 0$ such that $f(x^N) < \delta$. Excluding the trivial case that $f(x^o)=0$, we can assume $\delta \leq f(x^{N-1})$ without loss of generality. Clearly from the Algorithm, if any of y^k, $0<k<N$, were a vertex solution, then x^{k+1} would be a solution. So we have $y^k \in U$ for $0<k<N$.

Claim (2) : y^{N-1} is itself a solution of (M,q).

Otherwise $y^{N-1} \in \mathrm{U}$. It is easy to see, by induction, that x^N is a convex combination of vertices x^o and y^k, $0 \leq k < N$, which are all in U. So $x^N \in \mathrm{conv(U)}$ and therefore $f(x^N) > \delta$, which is a contradiction.

Now that y^{N-1} is a solution, x^N is a solution because by the line search of Agorithm (3.3.1), $f(x^N) \leq f(y^{N-1}) = 0$. ∎

(3.3.5) Remarks

(i) The only assumption for Algorithm (3.3.1) to find a solution in a finite number of iterations is that (III.9) holds for *one* solution $\bar{x}$, i.e. M be positive semi-definite on the tangent cone of the feasible region at $\bar{x}$.

(ii) Since

$$f(\lambda y+(1-\lambda)\bar{x})$$

$$= \lambda f(y)+(1-\lambda)f(\bar{x})-\lambda(1-\lambda)(y-\bar{x})^T M(y-\bar{x}), \qquad \text{(III.17)}$$

(which is easily shown by considering both sides as quadratic polynomials of y and comparing their zero, first and second order dirivatives at $y=\bar{x}$,) (III.9) is equivalent to that f is convex at a solution w.r.t. the feasible region.

(iii) It is clear from the proof that Theorem (3.3.4) can be generalized to prove finite termination of other algorithms. The next theorem is one of the generalizations which will be used later.

(3.3.6) Finite Termination Theorem

Let (A) be any iterative method for solving (QP) that starts at a feasible point, say x^o, and at iteration $k\geq 1$, it generates x^k which is a convex combination of x^{k-1} and some vertex y^{k-1}.

If the algorithm converges, i.e. for all $\varepsilon>0$ there exists $N\geq 0$ such that

$$0 \leq f(x^N) < \varepsilon$$

then $f(y^k)=0$ for some $k\geq 0$.

Proof. Similar to the proof of (3.3.4). ∎

3.4 THE SIMPLEX BASED ILP ALGORITHM—THE GENERAL CASE

Without the assumption (III.9), i.e. $f(x)$ is convex at some solution $\bar{x}$, (III.3) may never be satisfied by any feasible point in step (1) of Algorithm (3.3.1). In this case, if we still apply Algorithm (3.3.1) and attempt to find a BFS satisfying (III.3), we will solve the LP completely, get an optimal BFS and fail. However, if we let y^k be the optimal BFS we still have a descent direction $p^k := y^k - x^k$. In this section we show that with this modification the simplex based ILP method will converges to a point, say $\bar{x}$, which together with some $\bar{u}$ and $\bar{v}$ satisfies the Karush- Kuhn-Tucker necessary optimality condition of (QP):

$$(M\bar{x}+q+M^T\bar{x})-M^T\bar{u}-\bar{v} = 0 \tag{III.18}$$

$$\bar{u}^T(M\bar{x}+q) = \bar{v}^T\bar{x} = 0 \tag{III.19}$$

$$\bar{u},\, \bar{v}, M\bar{x}+q,\, \bar{x} \geq 0 \tag{III.20}$$

We call $\bar{x}$ a KKT point. Hence instead of finding a global minimum of (QP) we find a KKT point. When solving practical minimization problems an algorithm which generates a KKT point is considered acceptable because (1) no known method can find global minima in reasonable time for general non-convex problems, and (2) for convex problems a KKT point is a global minimum. However, since our original problem is (M,q), a KKT point or even a local minimum of (QP) is useless unless it happens to be a global minimum. Hence we try to find the class of matrices M for which a KKT point of (QP) is a global minimum. It is known that when M is a P-matrix, i.e. all its principal minors are positive, (QP) has a unique KKT point which is the unique solution of (M,q) [Cheng]. In Chapter V we shall show that when M is quasi-diagonally dominant, which is a P_0-matrix (i.e. all its principal minors are nonnegative), any KKT point of (QP) solves (M,q).

(3.4.1) Generalized Simplex Based ILP Algorithm

(0) Find any BFS, say x^0 (e.g. by phase one of simplex method). Repeat next two steps until x^k is complementary or $\{x^k\}$ converges.

(1) Given x^k, use the primal simplex method to partially solve

$$\min \nabla f(x^k)^T y$$

$$s.t.\ My+q \geq 0,\ \ y \geq 0.$$

until either (i) current BFS y satisfies (III.3) or (ii) current BFS y is optimal, whichever happens first, and let y^k be the

BFS.

(2) Same as (2) of Algorithm (3.3.1).

For simplicity of reference we define $x^k := x^N$ for all $k > N$ if the algorithm finds a complementary solution x^N. Hence we can talk about the convergence of the sequence $\{x^k\}$ even when the algorithm terminates in finite steps.

(3.4.2) Lemma Let $\bar{x}$ be any vector, then $\bar{x}$ is a KKT point of (QP) iff $\bar{x}$ is an optimal solution of the LP :

$$\min \nabla f(\bar{x})x \quad s.t. \;\; Mx+q \geq 0, \; x \geq 0$$

Proof. $\bar{x}$ is an optimal of the LP iff it satisfies the KKT conditions of the LP which is precisely (III.18-20), the conditions for $\bar{x}$ to be a KKT point of (QP). ∎

(3.4.3) Lemma At iteration k of Algorithm (3.4.1), if $f(x^k) > 0$, then

(i) $\nabla f(x^k)p^k \leq 0$, and

(ii) equality holds in (i) iff x^k is a KKT ponit of (QP).

Proof.

(i) Either y^k satisfies (III.3) or y^k is an optimal of the LP. If (III.3) holds then $\nabla f(x^k)p^k < 0$, while if y^k is an optimum then

$$\nabla f(x^k)p^k = \nabla f(x^k)y^k - \nabla f(x^k)x^k \leq 0 \tag{III.21}$$

where equality holds iff x^k itself is also an optimum.

(ii) Since equality in (III.21) holds iff both y^k and x^k are optima of (LPk), the assertion follows by Lemma (3.4.2). ■

(3.4.4) Convergence Theorem of The Generalized Algorithm

Let (M,q) be feasible. Let $\{x^k\}$ be defined by Algorithm (3.4.1), then

(i) $f(x^k)$ is nonincreasing, and

$$\lim_{k\to\infty} \nabla f(x^k) p^k = 0 \tag{III.22}$$

(ii) $\{x^k\}$ is bounded, and any accumulation point $\bar{x}$ of $\{x^k\}$ is a KKT point of (QP) and $f(\bar{x})=\alpha:=\inf f(x^k)\geq 0$.

(iii) If $\alpha=0$ then x^N is complementary for some N, and the algorithm terminates finitely.

(iv) If M is a P-matrix or a quasi-diagonally dominant then (iii) is true.

(v) If there is a solution $\bar{x}$ satisfying (III.9), in particular if M is semi-definite, then (iii) is true.

Proof.

(i) If x^k is not a KKT point of (QP) then $f(x^k)>0$ and (III.4) follows by Lemma (3.4.3). Therefore we can apply Lemma (3.3.2). Hence by (III.6) and (III.5),

$f(x^k)-f(x^{k+1})$

$$\geq -\tfrac{1}{2}t_k \nabla f(x^k)p^k$$

$$\geq \min\{|\tfrac{1}{2}\nabla f(x^k)p^k|,\ |(\nabla f(x^k)p^k)^2/2p^k Mp^k|\}$$

$$\geq \min\{|\tfrac{1}{2}\nabla f(x^k)p^k|,\ |(\nabla f(x^k)p^k)^2/2V|\} =: \delta_k \qquad \text{(III.23)}$$

where $V := \sup_{j\geq 0} |p^j Mp^j| < \infty$ for the same reason as in the proof of Theorem (3.3.3).

(III.23) still holds when x^k is a KKT point, since the right hand side becomes 0 by Lemma (3.4.3). Summing up (III.23) for $k=0,\ldots,N$, we have

$$f(x^o) - f(x^{N+1}) \geq \sum_{k=0}^{N} \delta_k \qquad \text{(III.24)}$$

The positive series $\sum_{k=0}^{\infty} \delta_k$ converges since the left hand side of (III.24) is bounded as N approaches infinity. Hence $\lim_{k\to\infty} \delta_k = 0$. It follows, then, by the definition of δ_k in (III.23) that

$$\lim_{k\to\infty} \sup |\nabla f(x^k)p^k| = 0$$

which implies (III.22).

(ii) Since the feasible set of (M,q) has finite number of vertices, the convex hull generated by the vertices is compact. Since x^k is a convex combination of the vertices, the sequence

$\{x^k\}$ is bounded. Let $\bar{x}$ be an accumulation point of the sequence, then there exists a subsequence, say $\{z^k\}$, of $\{x^k\}$ which converges to $\bar{x}$. By (III.22), let $\{v^k\}$ be the corresponding subsequence $\{v^k\}$ of $\{y^k\}$, we have

$$\lim_{k \to \infty} \nabla f(z^k)(v^k - z^k) = 0 \tag{III.25}$$

Since there are only finitely many vertices, there is a vertex, say y, such that $y = v^k$ for infinitely many k. By (III.25), it follows that

$$\nabla f(\bar{x})(y - \bar{x}) = 0 \tag{III.26}$$

On the other hand, by the definition of Algorithm (3.4.1), $y = v^k$ only when y satisfies the cut

$$\nabla f(z^k)(y - z^k) \leq -f(z^k) \tag{III.27}$$

or y is optimal in the LP and hence

$$\nabla f(z^k)(y - z^k) \leq \nabla f(z^k)(x - z^k) \text{ for } \textit{all feasible } x \tag{III.28}$$

Since $y = v^k$ infinitely often, either (III.27) holds infinitely often or/and (III.28) does. In either case, by taking the limit of the appropriate subsequences, we have that at least one of the following two inequalities holds:

$$\nabla f(\bar{x})(y - \bar{x}) \leq -f(\bar{x}) \tag{III.29}$$

$$\nabla f(\bar{x})(y-\bar{x}) \le \nabla f(\bar{x})(x-\bar{x}) \text{ for } \mathit{all\ feasible}\ x \qquad \text{(III.30)}$$

Together with (III.26), (III.29) implies that $f(\bar{x}) \le 0$ so $\bar{x}$ is a global minimum of (QP) which is clearly a KKT point while (III.30) implies that $\bar{x}$ is an optimal solution of the LP

$$\min \nabla f(\bar{x})x \quad s.t. \ \ Mx+q \ge 0,\ x \ge 0$$

so $\bar{x}$ is a KKT point of (QP) by Lemma (3.4.2).

Let $\alpha := \inf_k f(x^k)$. It is obvious that $f(\bar{x}) = \alpha \ge 0$.

(iii) If $\alpha=0$ then Finite Termination Theorem (3.3.6) applies.

(iv) Since, by (ii), $\{x^k\}$ is bounded there is an accumulation point, say $\bar{x}$, which is a KKT point of (QP). If M is a P-matrix then (QP) has a unique KKT point which is also the unique solution of (M,q) [Cheng 81]. If M is quasy-diagonally dominant then by Theorem (5.5.1), which will be proved independently in Chapter V, every KKT point of (QP) is a solution of (M,q). Hence $\alpha = f(\bar{x}) = 0$.

(v) Since the cut (III.3) is satisfied at every iteration, Algorithm (3.4.1) generates exactly the same sequence as Algorithm (3.3.1) does, provided that the same simplex code is used. Hence $\alpha=0$ by Theorem (3.3.3). ∎

3.5 NUMERICAL RESULTS

In this section we present computational experience with Algorithm (3.4.1) for randomly generated problems. For the purpose of comparison we also test Lemke's Algorithm for these problems. Recall that Algorithm (3.4.1) may converge to a KKT point of (QP) which is not a complementary solution, while the Lemke's Algorithm may terminate at a *ray* and may fail to give a solution. In fact, since the general LCP is NP-complete, there is no known algorithm that is guaranteed to solve any LCP without essentially enumerating an exponential number of possible cases which is impossible to do even for moderate n. We wrote programs in C [Kernighan and Ritchie 78] and tested them on a VAX-11/780 computer under the virtual UNIX operating system. The reason for using the programming language C instead of FORTRAN is a matter of personal preference. All floating point computations are in double precision which provide about 16 figure decimal accuracy. Six problem sets were tested; each has 20 individual problems. The dimension n is fixed in each set and varies among sets. A random number generator is provided by the system and the test problems are generated as follows.

(3.5.1) Generation of Test Problems

(i) Generate the matrix M with each entry being a random number uniformly distributed on interval [-1, 1].

(ii) For each index i, randomly choose one of x_i and w_i to be zero, the other to be a random (floating) number in interval $[0, 1000]$.

(iii) Let $q := w - Mx$. ■

(3.5.2) Outputs And Interpretations

For the simplex based ILP, the iteration number and the pivot number are different. The former is the number of updatings of the cost vector of the LP while the latter is the total number of pivots of all iterations. Since updating the cost vector is easier than one pivot operation, and the number of updatings is significantly smaller than the number of pivots, we feel that the latter is a good measure of computational time. For Lemke's Algorithm the two numbers are the same. The comparison between pivot numbers of these two algorithms gives us a good estimate of the relative speed since each pivot takes roughly the same time in either algorithm. The results are shown in Table 3.1. The ILP method starts at the origin and uses the phase one of the simplex method to find the first BFS. The number of pivots given in Table 3.1 includes the pivot steps taken in phase one.

Table 3.1 Computational Results of ILP Versus Lemke's Method

Dimension of Problem ($=n$)		7	15	23	31	40	50
Average Number Of Iterations of ILP (= Number of cost updatings)		1.05	3.50	7.20	7.25	10.50	12.15
Number of Problems Solved Out Of 20 Problems	ILP	17	11	11	9	7	6
	Lemke	13	7	2	2	1	0
Average Number of Pivots For Solved Problems	ILP	4	15	35	45	90	144
	Lemke	5	21	40	231	205	-
Average Number of Pivots For Unsolved Problems	ILP	2	16	44	55	98	136
	Lemke	7	14	39	102	231	514
Maximum Number of Pivots Over the 20 Problems	ILP	8	29	75	101	129	234
	Lemke	17	47	91	493	1000*	1000*
Minimum Number of Pivots Over the 20 Problems	ILP	1	0	16	29	73	85
	Lemke	2	0	8	1	2	11
Ratio Of Max/Min of Pivots	ILP	8.0	-	4.7	3.5	1.8	2.8
	Lemke	8.5	-	11.4	493	500*	90.9*

* (quit after 1000 iterations, so the true value should be bigger)

We make the following observations regarding this table.

(i) Although Lemke's Algorithm is very efficient for M being a P-matrix or positive semi-definite, it rarely gives a solution for a general M when $n \geq 30$. Moreover, it becomes very slow no matter whether it gives a solution or not. Hence the so called ***almost complementary path*** has a very long average length.

(ii) The number of problems solved in a set decreases rapidly as n increases. This is not surprising since the problem is NP-complete.

(iii) The ratio of the maximum number to the minimum number of pivots of a set, given in the bottom row of each column, remains small for ILP but varies considerably for Lemke's method. Hence the almost complementary path can be very short as well as very long. It also indicates that the ILP is more robust.

There are some issues not covered in the experiment. Since ILP can start at any BFS, it will be much faster if we have a BFS to begin with or each probem set is a parametric LCP in which case ILP can start at the solution basis of the previous problem. It should be noted, however, that some generalizations of Lemke's method can start at an arbitrary point under certain assumptions [Eaves 78, Talman and Van der Heyden 81].

3.6 EXTENSION TO MINIMIZATION PROBLEMS

In this section we generalize the simplex based ILP to solve minimization problems with linear constraints and convex objective function, i.e.

$$\text{minimize } f(x) \;\; s.t. \;\; Ax \geq b \,,\; x \geq 0 \qquad \text{(MP)}$$

where f is a convex differentiable function on R^n and A is an $m \times n$ real matrix. Although (MP) and (QP) look the same, there are some significant differences.

(i) The constraints of (MP) are arbitrary relative to the objective function. In (QP) the objective function is a quadratic function constructed from the linear constraints. Hence minimization stepsize is much more difficult for (MP).

(ii) The minimum value of (QP) for a solvable LCP is zero but that of the (MP) is unknown. Hence, for (MP), we do not have a cut like (III.1) that is guaranteed to separate solutions from current point.

(iii) (MP) may have solutions but no vertex solution. Hence it is more difficult to solve than (QP). Moreover, the Finite Termination Theorem (3.3.4) cannot apply and the convergence may be slower.

Although there is nothing we can do about the nonexistence of a vertex solution, we can handle the difficulties in (i) and (ii). For (ii) we can use a lower bound of the minimum value which usually is easy to get, and in case the lower bound is not tight enough and ***overcut*** may take place which can be handled as in (3.4.1). Note that in (3.4.1) an overcut occurs for different reason, that is, the nonconvexity of f. For difficulty (i), if f is not quadratic we use another stepsize, the Armijo stepsize described in the next section, which is not hard to compute.

(3.6.1) Armijo Stepsize [Armijo 66]. Let $f(x), x \in R^n$ be differentiable. Let p be a descent direction at x^k, i.e. $\nabla f(x^k)p < 0$. Given $\delta \in (0,1)$, define the stepsize t_k by

$$t_k := \max_{i=0,1,2,\ldots} 2^{-i} \quad s.t. \quad f(x^k + 2^{-i}p) - f(x^k) \le 2^{-i}\delta \nabla f(x^k)p \qquad \text{(III.31)}$$

t_k is called the Armijo stepsize (w.r.t. δ). Maximization in (III.31) is well-defined since any sufficiently large i is feasible and we want such an i which is minimum.

(3.6.2) ILP Minimization Algorithm

(0) Let x^o be any feasible point, α_o be any lower bound to the minimum value of (MP). Choose $\delta \in (0,1)$. Repeat the next two steps until x^k converges.

(1) Given x^k, α_k, use the primal simplex method partially solve

$$\min \nabla f(x^k)^T y$$

$$s.t.\ Ay+q \geq 0,\ y \geq 0.$$

until any one of the following occurs: (i) Current BFS y satisfies the *cut*,

$$\nabla f(x^k)(y-x^k)+f(x^k) \leq \alpha_k \tag{III.32}$$

(ii) Current BFS y is optimal, (iii) Current BFS y is at the end of a unbounded ray with direction d. Let y^k be the BFS in case of (i) or (ii), or let $y^k := y+\lambda d$ where λ is the smallest positive number for which y^k satisfies the cut (III.32).

Let $\alpha_{k+1} := \alpha_k$ if (i) or (iii) holds, otherwise let $\alpha_{k+1} := \nabla f(x^k)(y-x^k)+f(x^k)$.

(2) Line-search along direction $p^k := y^k - x^k$ using the Armijo stepsize or the minimization stepsize if f is quadratic. That is, let $x^{k+1} := x^k + t_k x^k$, where t_k is defined either by (III.31) or by (III.5) if $f(x) := xMx+qx$ is quadratic.

(3.6.3) Theorem

Let f be a convex function with a locally Lipschitz continuous gradient, i.e. for any compact set F in R^n, there is a positive number K such that

$$|\nabla f(x) - \nabla f(y)| \leq K|x-y| \quad \text{for } x, y \in F \tag{III.33}$$

Let x^k, y^k, p^k, t_k be defined as in Algorithm (3.6.2). Let $\bar{x}$ be a solution of (MP) then

(i)

$$\alpha_o \leq \cdots \leq \alpha_k \leq \alpha_{k+1} \leq \cdots \leq f(\bar{x})$$

$$f(\bar{x}) \leq \cdots \leq f(x^{k+1}) \leq f(x^k) \leq \cdots \leq f(x^o)$$

(ii) If $\{y^k\}$ is bounded then $\{x^k\}$ is bounded and

$$\lim_{k\to\infty} \alpha_k = f(\bar{x}) = \lim_{k\to\infty} f(x_k)$$

hence $\{x^k\}$ has a accumulation point and any accumulation point is a solution of (MP).

Proof.

(i) Since (3.6.2) is clearly a descent algorithm, $\{f(x^k)\}$ is decreasing.

By (3.6.2) step (1), $\alpha_{k+1} \neq \alpha_k$ only when an overcut occurs, in which case y^k violates the cut (III.30) hence

$$\alpha_{k+1} := \nabla f(x^k)(y^k - x^k) + f(x^k) > \alpha_k$$

and, by convexity of f and the fact that y^k is optimal for the LP,

$$f(\bar{x}) \geq \nabla f(x^k)(\bar{x} - x^k) + f(x^k) \geq \nabla f(x^k)(y^k - x^k) + f(x^k) \equiv \alpha_{k+1}$$

(ii) Assume $\{y^k\}$ is bounded. Since x^k is a convex combination of y^j, $0 \leq j < k$, the sequence $\{x^k\}$ is also bounded. Hence there exist $L > 0$ s.t.

$$|x^k| < L, \; |y^k| < L, \; |p^k| < L, \quad \textit{for all } k \tag{III.34}$$

Let F be the closed ball at the origin with radius L, let K be the Lipchitz constant s.t. (III.33) holds. Then for $0 \leq \lambda \leq 1$, we have

$$f(x^k + \lambda p^k) - f(x^k) = \int_0^\lambda \frac{d}{dt} f(x^k + tp^k)\,dt$$

$$= \int_0^\lambda [\nabla f(x^k + tp^k)p^k - \nabla f(x^k)p^k]\,dt + \lambda \nabla f(x^k)p^k$$

$$\leq \int_0^\lambda |\nabla f(x^k + tp^k) - \nabla f(x^k)|\,dt \;\; |p^k| + \lambda \nabla f(x^k)p^k$$

$$\leq \int_0^\lambda tK|p^k|\,dt \;\; |p^k| + \lambda \nabla f(x^k)p^k \quad (\textit{by III.}\,33)$$

$$= \tfrac{1}{2}\lambda^2 K|p^k|^2 + \lambda \nabla f(x^k)p^k$$

$$\leq \frac{\lambda^2 KL^2}{2} + \lambda \nabla f(x^k) p^k \quad (by\ III.34)$$

$$= \lambda[\frac{\lambda KL^2}{2} + (1-\delta)\nabla f(x^k)p^k] + \lambda\delta\nabla f(x^k)p^k$$

$$\leq \lambda\delta f(x^k)p^k \quad \text{if} \quad (1-\delta)\nabla f(x^k)p^k + \frac{\lambda KL^2}{2} \leq 0$$

i.e., if $\lambda \leq \frac{2(1-\delta)}{KL^2}(-\nabla f(x^k)p^k =: \lambda_k$. So all i such that $2^{-i} \leq \lambda_k$ are feasible in the maximization of (III.31). Hence the Armijo stepsize t_k is either equal to 1 or greater than $\frac{1}{2}\lambda_k$. Therefore,

$$t_k \geq \min\{1, \tfrac{1}{2}\lambda_k\} = \min\{1, \frac{1-\delta}{V}(-\nabla f(x^k)p^k)\} \qquad \text{(III.35)}$$

where $V := KL^2$ is a fixed positive number. It is easy to see that when f is quadratic and minimization stepsize is used, (III.35) still holds with $\delta = \frac{1}{2}$ for some positive V which is independent of k [see III.5]. By changing sign of (III.31) [or III.6 if f is quadratic], we have

$$f(x^k) - f(x^{k+1}) \geq -t_k \delta \nabla f(x^k)p^k$$

$$\geq -\delta\nabla f(x^k)p^k \times \min\{1, \frac{1-\delta}{V}(-\nabla f(x^k)p^k)\} \quad (by\ III.35)$$

$$= \delta \times \min\{-\nabla f(x^k)p^k, \frac{1-\delta}{V}(\nabla f(x^k)p^k)^2\} =: \delta_k \qquad \text{(III.36)}$$

Summing up (III.36) for $k=0,\dots,N$, we have precisely the same inequality as (III.24). Hence it follows by similar argument that

$$\lim_{k\to\infty}(-\nabla f(x^k)p^k) = 0 \tag{III.37}$$

On the other hand, by the definition of α_{k+1} in (3.6.2) (1), we have

$$-\nabla f(x^k)p^k \geq f(x^k)-\alpha_{k+1}$$

So $\lim_{k\to\infty} f(x^k)-\alpha_{k+1} = 0$ by (III.37) and therefore, by (i),

$$\lim_{k\to\infty} f(x^k) = f(\bar{x}) = \lim_{k\to\infty}\alpha_k$$

which completes the proof of (ii). ▪

(3.6.4) Remarks

(i) In (3.6.2)(1), y^k is a vertex unless the LP is unbounded and case (iii) occurs. So it is very likely that $\{y^k\}$ is bounded. When the feasible region is bounded, e.g. when $l \leq x \leq u$ is part of the linear constraints of (MP), then $\{y^k\}$ is bounded and the algorithm converges by part (ii) of Theorem (3.6.3).

(ii) The lower bound α_o is not essential. We can pick up α_o arbitrarily and in step (1) if (i) or (iii) occurs, i.e. the cut is not too deep, then decrease α_k by a positive constant. Since the (MP) is bounded, an overcut will occur after a finite number of iterations. Then α_k is updated according to Step (1) case (ii) and we get a lower bound α_{k+1}.

(iii) At each iteration, the constraints of LP is the same as that of (MP). Hence the algorithm is very attractive for problems having specially structured linear constraints, e.g. non-linear network flow problems, for which very effective simplex methods exist [e.g. see Griffith and Hsu 79]. It should be noted that there are many other algorithms having this property [see Kamesam 82 for a brief survey].

(iv) In the very extreme case that an overcut occurs at every iteration, we solve every LP completely and the algorithm "degenerates" to the Frank-Wolfe method. However, this rarely occurred when the ILP method was used to solve the LCP.

3.7 SUMMARY

In this Chapter we gave an iterative linear programming method for solving linear complementarity problems. The algorithm has most of the advantages of the single LP approach discussed in the previous chapter. The single LP approach tries to find a *correct* vector p in the sense that a single LP (M,q,p) will solve the LCP (M,q). By the contrast the iterative programming method starts with a p which may not be correct. However, after a number of primal simplex pivot steps for the LP (M,q,p) which decreases the objective level below a certain precalculated value, called a cut, p is corrected. It was shown that after a finite number of corrections of p, a vertex solution of (M,q) can be obtained if certain convexity of x^TMx holds, e.g. M being positive semi-definite. Without any convexity assumption, the algorithm will either find a vertex solution in a finite number of pivots or converge to a KKT point of (QP). Hence, in cases where a KKT point of (QP) is always a global solution, e.g. M is diagonally dominant or M is a P-matrix, a vertex solution can be obtained in a finite number of pivots. Numerical results for randomly generated problems were presented. Finally we extended the method to solve linearly constrained convex programming problems.

CHAPTER IV

THE SOR-BASED ILP METHOD

The purpose of this chapter is to present an SOR-based method which can solve very large systems of LCP's and can take full advantage of any sparsity by preserving the original structure of the problems.

4.1 INTRODUCTION

Although the simplex based ILP, or any other direct algorithm using pivoting, gives a very accurate solution in a finite number of iterations, it is not suitable when the problem size is very large. The reason is at least twofold. First, as the dimension n of M increases, the number of vertices increases exponentially. Hence it is very likely that the number of pivots will grow rapidly, which is the case shown in Table 1 of Section (3.5). Second, the amount of storage for the simplex tablau is of order n^2. Hence it is impossible to keep the whole tablau in the main memory of a computer as n becomes large, say 1000. Hence some kind of *swapping*, either through programmer's effort or automatically by the system, must take place between the secondary storage and main memory. Since the speed of the secondary storage is much slower (e.g. 1000 times slower) than the main storage, swapping time will almost surely dominate other computing time. Since swapping is needed for each pivot, the computer will end up as doing nothing but swapping and therefore can not find a solution in reasonable amount of time [Mangasarian 83, Table 1].

In this chapter we present an SOR-based method which works only on the original input data, and stores and processes only nonzero entries. Hence it is suitable for very large scale problems having a sparse matrix M. Previous SOR based algorithms were developed primarily for symmetric matrix M with some kind of strict positivity [Cryer 71, Mangasarian 77]. For example, Mangasarian's algorithm converges when M is symmetric and strictly copositive, or copositive-plus and there exists an x s.t. $Mx+q>0$. We shall prove that our algorithm converges for M

positive semi-definite, among others, no matter whether M is symmetric or not.

The basic SOR iteration we use is different from the conventional SOR. The latter moves the point along a direction parallel to each coordinate axis to relax, i.e. to satisfy, each constraint in a fixed order, while the former moves the point along the direction orthogonal to each constraint hyperplane. We give more details in the next section.

4.2 SOR METHODS FOR LINEAR SYSTEMS

(4.2.1) SOR Methods For Linear Equations

First, let us review the conventional SOR method for systems of linear *equations*, e.g.

$$Mx+q = 0 \tag{IV.1}$$

where M is again a n by n matrix. We assume all diagonal entries of M are nonzero. The method for (IV.1) starts at an arbitrary point x^0 and repeats the SOR iteration, where the SOR iteration is defined by:

$$\text{For } i:=1 \text{ to } n \text{ do } x_i^{k+1} := x_i^k + \mu\Delta_i^k$$

where $\mu\in(0,2)$ and Δ_i^k is the correction that should be made for the

i-th component of the point in order to satisfy the i-th equation of (IV.1). Formally, Δ_i^k is defined by

$$\Delta_i^k := \frac{-1}{M_{ii}}\left(\sum_{j=1}^{i} M_{ij}x_j^{k+1} + \sum_{j=i+1}^{n} M_{ij}x_j^k + q_i\right) \qquad \text{(IV.2)}$$

Note that when the i-th component is updated, the first $i-1$ components have been updated and the new values are used. If $\mu=1$ the method reduces to the Gauss-Seidel method, if $1<\mu<2$ ($0<\mu<1$) the iteration is an *overrelaxation* (*underrelaxation*). The SOR iteration can be rearranged into matrix form and after easy manipulation it may be written as

$$x^{k+1} := (D+\mu L)^{-1}[(1-\mu)D-\mu U]x^k - \mu(D+\mu L)^{-1}q =: H_\mu x^k + q_\mu \qquad \text{(IV.3)}$$

where D, L and U are respectively the diagonal, strictly lower triangular part, and strictly upper triangular part of M. Under certain conditions, e.g. M being symmetric positive definite, the transformation definied in (IV.3) is a contraction mapping and hence $\{x^k\}$ converges to the unique fixed point which is clearly a solution of (IV.1) [see Varga 62 or Ortega 72 for more details]. By projecting the transformation on the nonnegative orthant we get the projected SOR method for the LCP (M,q) which converges if M is symmetric positive definite [Cryer 71]. Mangasarian gave a very generalized form of the projected SOR and its convergence theorem [Mangasarian 77].

Now let us review another SOR method which does not require symmetry of M. The method is a generalization of successive projection [Brégman 65]. Again we start at any x^0 and then systematically choose one equation from (IV.1), project the point on the hyperplane defined by the equation, i.e. we move the point along the direction orthogonal to the hyperplane until the equation is satisfied. Let the displacement be Δx. Then the SOR iteration is

$$x^{k+1} := x^k + \mu \Delta x, \quad 0<\mu<2$$

Again $1<\mu<2$ ($0<\mu<1$) is overrelaxation (underrelaxation), while $\mu=1$ is the successive projection. It is not difficult to show that the method converges to a solution provided there is one [Herman, Lent and Lutz 75]. Note that when the i-th equation of (IV.1) is being ***relaxed***, then Δx is a multiple of i-th row vector of M. Let m_i be the number of nonzero entries of the i-th row of M, then only m_i components of x^k need be changed. But for the conventional SOR iteration, all of the n components of x^k are updated. When M is a large sparse matrix, m_i is much much smaller than n. On the other hand, it is easy to see that for conventional SOR, if it is a contraction with fixed point $\bar{x}$, that

$$|x^{k+1} - \bar{x}| \le \rho(H_\mu) |x^k - \bar{x}|$$

where $\rho(H_\mu)<1$ is the spectral radius of the matrix H_μ defined in

(IV.3). But for the orthogonal SOR, we only have

$$|x^{k+1}-\bar{x}| < |x^k-\bar{x}|$$

for *any* solution $\bar{x}$. So we can conclude roughly that (1) The conventional SOR takes more time in each iteration but need less iterations to converge. (2) The orthogonal SOR converges for any M provided the system is consistent while the conventional SOR converges only for certain classes of matrices.

It is easy to extend the orthogonal SOR to general linear systems where the number of constraints may be different from that of variables, and the system may contains inequality constraints as well. Here we give a formal definition of the extension for solving systems of linear inequalities, which will be used to define the SOR based ILP method.

(4.2.2) Orthogonal SOR For Linear Inequalities [Agmon 54]

Problem: Find $x \in R^n$ s.t. $Ax \geq b$, where A is an $m \times n$ matrix, b is an m-vector. Hence the problem is the find a vector x satisfying m linear inequalities $A_i x \geq b_i$ for $1 \leq i \leq m$, where A_i is the i-th row vector of A. Without loss of generality we assume all A_i are nonzero vectors.

Method:

(0) Let x^o be arbitrary and let $i_o := 1$. Choose $\mu \in (0,2)$. Repeat the next step until $\{x^k\}$ converges.

(1) Given x^k, i_k. Define i_{k+1} by

$$i_{k+1} := \text{if } i_k < m \quad i_k + 1 \quad \text{else} \quad 1$$

(2) If x^k satisfies the i_{k+1}-th constraint then $x^{k+1} = x^k$. Otherwise define

$$\beta := \frac{b_{i_{k+1}} - A_{i_{k+1}} x^k}{A_{i_{k+1}}^T A_{i_{k+1}}} \neq 0 \tag{IV.4}$$

$$x^{k+1} := x^k + \mu \beta A_{i_{k+1}} \tag{IV.5}$$

(4.2.3) Convergence Theorem For Orthogonal SOR [Agmon 54]

Let the problem in (4.2.2) be consistent. Let $\bar{x}$ be any solution. Let the sequence $\{x^k\}$ be defined as in (4.2.2). Then

(i) $|x^{k+1} - \bar{x}| \leq |x^k - \bar{x}|$ (IV.6)

where equality holds only when $x^{k+1} = x^k$.

(ii) $\{x^k\}$ converges to a solution.

Proof.

(i) Suppose $x^{k+1} \neq x^k$. For simplicity of notation, let $j := i_{k+1}$. Then x^k must violate the j-th constraint. Hence,

$$A_j x^k - b_j < 0, \quad \beta > 0 \tag{IV.7}$$

where β is defined by (IV.4). Since $\bar{x}$ is a solution, we also have

$$A_j \bar{x} \geq b_j \tag{IV.8}$$

Define the function $\varphi(\lambda) := ½|x^k + \lambda\beta A_j - \bar{x}|^2$. The Taylor expansion of $\varphi(\lambda)$ at the origin up to second order terms has zero residual because $\varphi(\lambda)$ is quadratic. Hence

$$½|x^{k+1} - \bar{x}|^2 - ½|x^k - \bar{x}|^2 \equiv \varphi(\mu) - \varphi(0)$$

$$= \mu\varphi'(0) + \frac{1}{2}\mu^2\varphi''(0)$$

$$= \mu\beta A_j(x^k - \bar{x}) + \frac{1}{2}\mu^2\beta^2 A_j^T A_j$$

$$\leq \mu\beta(A_j x^k - b_j) + \frac{1}{2}\mu^2\beta(b_j - A_j x^k) \quad (\text{by } IV.\ 8,7\text{and}4)$$

$$= \mu\beta(1 - \frac{1}{2}\mu)(A_j x^k - b_j) < 0 \ (\text{by } IV.\ 7)$$

(ii) See references [Agmon 54, Motzkin and Schoenberg 54]. ▪

In all the SOR methods we discussed so far, μ is fixed. In fact, in many cases we can use different μ for each iteration as long as the μ's are bounded away from 0 and 2. That is, suppose μ_k is used in iteration k,

$$0 < \inf_k \mu_k \leq \sup_k \mu_k < 2 \tag{IV.9}$$

In Algorithm (4.2.2) if we allow μ to vary for each iteration, it can be shown that Theorem (4.2.3) still holds provided that (IV.9) is satisfied [Herman,Lent,Lutz 75]. The choice of μ in each iteration may considerably affect the speed of convergence. Usually $\mu>1$ is prefered. If the solution set contains an interior point, i.e. $Ax>b$ for some x, then by choosing $\mu=2$ the algorithm will find a solution in a finite number of iterations [Motzkin and Schoenberg 54].

4.3 THE SOR BASED ILP METHOD FOR LCP

In Chapter III we proposed an algorithm for solving (M,q) by iteratively finding a *vertex* satisfying a cut followed by a line-search. In this section we show how the orthogonal SOR (4.2.2) can be used in each iteration to find a *point* satisfying the cut. We will also show that with the same line-search the method converges under the same convexity assumption as in Section (3.3).

Fisrt let us rewrite Theorem (3.3.3) in a more general form.

(4.3.1) Theorem

Let $\{x^k, y^k\}$ satisfy the following assumptions:

(o) x^o is feasible for (M,q).

(i) For all k, y^k satisfies the cut (III.1) defined by x^k, i.e.

$$My^k+q \geq 0,\ y^k \geq 0,\ f(x^k)+\nabla f(x^k)(y^k-x^k) \leq 0 \qquad \text{(IV.10)}$$

(ii) $x^{k+1}:=x^k+t_kp^k$ where $p^k:=y^k-x^k$ and t_k is defined by (III.5).

(iii) y^k is bounded.

Then $\lim\limits_{k\to\infty} f(x^k) = 0$.

Proof. By (iii) there is a number α such that $|y^k|<\alpha$ for all k. By choosing α sufficiently large, we can assume $|x^o|<\alpha$. Hence

$$|p^kMp^k| \leq (|x^k|+|y^k|)\rho(M)(|x^k|+|y^k|) \leq 4\alpha^2\rho(M) =: V$$

where $\rho(M)$ is the spectral radius of M. The rest of the proof is exactly the same as that of Theorem (3.3.3). ∎

(4.3.2) LCPSOR Algorithm For (M,q)

(0) Find any feasible point x^o [e.g. by SOR (4.2.2)]. Repeat the following two steps until $\{x^k\}$ converges.

(1) Given x^k, use SOR (4.2.2) *starting at* x^k to solve

$$My+q \geq 0,\ y \geq 0,\ f(x^k)+\nabla f(x^k)(y-x^k) \leq 0, \qquad \text{(IV.11)}$$

Let the solution found be y^k.

(2) Line-search along the direction $p^k:=y^k-x^k$ using the minimization stepsize [see (3.3.1)(2)].

(4.3.3) Theorem Assume that there exists a solution $\bar{x}$ to the LCP (M,q) such that

$$(y-\bar{x})^T M(y-\bar{x}) \geq 0 \quad \text{for all feasible } y$$

Let $\{x^k, y^k\}$ be defined by (4.3.2). Then

(i) $\{x^k, y^k\}$ are well-defined and bounded. The sequence $\{|x^k-\bar{x}|\}$ is decreasing.

(ii) $\lim_{k\to\infty} f(x^k) = 0$

Hence any accumulation point of $\{x^k\}$ is a solution of (M,q).

Proof.

(i) Given x^k feasible, $\bar{x}$ satisfies (IV.11) by Lemma (3.2.2). Hence by Theorem (4.2.3), y^k can be found successfully and $|y^k-\bar{x}|<|x^k-\bar{x}|$ since x^k is the starting point of the SOR. Moreover, let t_k be the minimum stepsize, we have

$$|x^{k+1}-\bar{x}| \equiv |(1-t_k)x^k+t_k y^k-\bar{x}| \leq (1-t_k)|x^k-\bar{x}|+t_k|y^k-\bar{x}|$$

$$< (1-t_k)|x^k-\bar{x}|+t_k|x^k-\bar{x}| = |x^k-\bar{x}|$$

Hence the proof is complete by simple induction on k.

(ii)

(4.3.4) Computational Details

(1) When we solve (IV.11) using SOR, the n nonnegative constraints $x \geq 0$ can be *relaxed* easily by replacing the negative components by zero, or by small positive numbers if overrelaxation is preferred. Hence they can be relaxed as often as possible, i.e. every time any component becomes negative while relaxing one of the other $n+1$ constraints that component can be set to zero or a sufficiently small positive number.

(2) Since SOR is an infinite procedure, we may not solve y^k exactly. However we can get a point feasible to within a small tolerance in a finite number of SOR iterations. Note that each SOR iteration moves the point closer to $\bar{x}$ by (4.2.3)(i). Also note that the SOR starts at x^k which already satisfies all the constraints in (IV.11) except the cut.

(3) In general we may not have finite termination. However, if after $f(x^k)$ is sufficiently small, we search for a vertex satisfying (IV.11) at each iteration, then by the Finite Termination Theorem (3.3.6) we will get a vertex solution in a finite number of steps. That is, if the y^k found by the SOR procedure is not a vertex, then we move y^k along a direction orthogonal to the gradients of all active constraints until one more constraint is active, and repeat the process until y^k is a vertex of (IV.11). At that point if the cut is not active then y^k is a vertex of (M,q) as

we want. Otherwise move y^k along a direction orthogonal to gradients of all active constraints except the cut until one more constraint is active, then the point obtained is a vertex of (M,q). Intuitively, it is better off to choose the projection of $y^k + \nabla f(y^k)$ on the intersection of all active constraints as the direction along which y^k is to be moved while searching for a vertex. For we may very well decrease $f(y^k)$ in the process.

4.4 SUMMARY

We reviewed SOR methods for systems of linear equations and linear inequailities, then gave an algorithm LCPSOR which used an SOR method instead of the simplex method to find a point satisfying the cut described in Chapter III. Hence it is suitable for very large scale problems having sparse input data. LCPSOR converges to a solution of the linear complementarity problem (M,q) when M is positive semi-definite but not necessarily symmetric.

CHAPTER V

COMPLEMENTARITY THEORY FOR QUASI-DIAGONALLY DOMINANT MATRICES

5.1 INTRODUCTION

A square matrix is (strictly) diagonally row dominant if every diagonal entry is no less than (greater than) the sum of absolute values of all other entries in the same row. Diagonal column dominance is similarly defined. We study the existence and uniqueness of solutions to (M,q) for M being a quasi-diagonally row (column) dominant matrix, i.e. a matrix that can be made diagonally row (column) dominant by multiplying each column (row) by a positive number (see definition below). Diagonal row dominance and column dominance are so different that they are sometimes considered two independent properties. By the generalization to the quasi-diagonally dominance, we find they are closely related, e.g. a strictly quasi-diagonally row dominant matrix is strictly quasi-diagonally column dominant, and vice versa. We also find, because of the generalization to quasi-dominance, a comprehensive uniqueness and existence theory for solving (M,q) for M in those classes can be given. Thus, for example, if M is quasi-diagonally row dominant (not necessarily strictly dominant) then any KKT point of (QP) is a solution of (M,q), and therefore $M \in Q_0$, i.e. (M,q) has a solution for all q for which (M,q) is feasible. The latter result was also proved by Moré and Aganagic utilizing Lemke's method [Moré 74, Aganagic 81]. In addition, a characterization is given for the condition $M \in Q$, i.e. (M,q) is solvable for all q, for a quasi-diagonally dominant matrix M. Unlike Aganagic's characterization [Aganagic 81], our characterization is very easy to check. When M is quasi-diagonally column dominant, we give a sufficient and necessary condition for the uniqueness of solution to (M,q) which is very easy to verify. Moreover we also give a characterization of the solution set and a computational method to find the set.

5.2 UNIQUENESS FOR DIAGONALLY DOMINANT—IRREDUCIBLE MATRICES

(5.2.1) Definitions

(i) M is quasi-diagonally row dominant, denoted $M \in \mathrm{QD}_o$, iff for some n-vector $d>0$,

$$M_{ii} d_i \geq \sum_{j \neq i} |M_{ij}| d_j, \quad i=1,\ldots,n. \tag{V.1}$$

(ii) $M \in \mathrm{QD}_{o+}$ iff in addition to (i), the strict inequality holds for at least one i in (V.1).

(iii) M is strictly quasi-diagonally row dominant, denoted $M \in \mathrm{QD}_+$, iff in addition to (i), the strict inequality holds for all i.

(5.2.2) Propositions

(i) $M \in \mathrm{QD}_o$, QD_{o+}, or QD_+ iff $C(M) \in \mathrm{QD}_o$, QD_{o+}, or QD_+ respectively, where $C(M)$, called the comparison matrix of M, is defined by

$$C(M)_{ij} := \begin{cases} -|M_{ij}| & \text{if } i \neq j \\ M_{ii} & \text{if } i = j \end{cases} \tag{V.2}$$

(For convenience, this is a slight variance of the definition that is given in the literature in which M_{ii} of (V.2) is replaced $|M_{ii}|$.)

(ii) $QD_+ \subseteq P$, $QD_o \subseteq P_o$ where P (P_o) is the class of matrices having positive (nonnegative) principal minors.

(iii) $QD_+ := \{M \mid C(M)^{-1}$ exists and $C(M)^{-1} \geq 0\}$.

Proof.

(i) Trivial.

(ii) Let $M \in QD_+$ ($M \in QD_o$), then every principal submatrices of M is in QD_+ (QD_o). Hence it suffices to show that for any $A \in QD_+$ ($A \in QD_o$) the determinant of A, denoted by $det(A)$, is positive (nonnegative). Since $det(A)$ equals the product of the eigenvalues of A and since the complex eigenvalues appear in conjugate pairs and the product of each pair is positive, it suffices to show that every ***real*** eigenvalue of A is positive (nonnegative). Let λ be any real eigenvalue of A with eigenvector v, so

$$(A-\lambda I)v = 0, \quad v \neq 0. \tag{V.3}$$

By Definition (5.2.1) there exists $d>0$ such that (V.1) holds, in which the strict inequality holds for $i=1,\ldots,n$ if $M \in QD_+$. Let i be the index that

$$\delta := \frac{|v_i|}{d_i} := \underset{1 \leq k \leq n}{\text{maximum}} \frac{|v_k|}{d_k}$$

where $\delta > 0$ since $v \neq 0$. By negating v if necessary we can assume $v_i>0$. Expanding the $i-th$ component of (V.3), we have

$$0=\sum_{j\neq i}A_{ij}d_j\frac{v_j}{d_j}+(A_{ii}-\lambda)d_i\frac{v_i}{d_i}$$

$$\geq\sum_{j\neq i}-|A_{ij}|d_j\delta+(A_{ii}-\lambda)d_i\delta$$

$$\geq(\sum_{j\neq i}-|A_{ij}|d_j+A_{ii}d_i)\delta-\lambda d_i\delta\geq-\lambda d_i\delta\quad(by\ V.1)\qquad\text{(V.4)}$$

Hence $\lambda\geq 0$. If $A\in\mathrm{QD}_+$, then the last inequality in (V.4) is strict and we have $\lambda>0$.

(iii) Since $C(M)$ is a Z-matrix, $C(M)$ has a nonnegative inverse iff $C(M)\in\mathrm{QD}_+$ [Berman and Plemmons 79, Chap. 6, Thm. (2.3)] and this is true iff $M\in\mathrm{QD}_+$ by (i). ■

By (ii), if $M\in\mathrm{QD}_+$, then M is a P-matrix, so (M,q) has a unique solution for every q [Samelson, Thrall and Wesler 58]. However, if $M\in\mathrm{QD}_o$, although we know M is a P_o-matrix, neither existence nor uniqueness of solutions to (M,q) exists for a general P_o-matrix M as can be shown by simple examples [see (5.2.9)]. By (iii), $M\in\mathrm{QD}_+$ iff M is an H-matrix with positive diagonals [Pang 79].

(5.2.3) Lemma

(i) $M \in \mathrm{QD}_+$ iff $M^T \in \mathrm{QD}_+$.

(ii) $M \in \mathrm{QD}_o$ and $M^T \in \mathrm{QD}_o$ do not imply each other.

Proof.

(i) By (5.2.2) (iii), $M \in \mathrm{QD}_+$ iff $C(M)^{-1} \geq 0$ iff $C(M^T)^{-1} \geq 0$ iff $M^T \in \mathrm{QD}_+$.

(ii) Let

$$M := \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

then $M \in \mathrm{QD}_{o+}$ but $M^T \notin \mathrm{QD}_o$. By setting the above matrix equal to M^T, it follows that $M^T \in \mathrm{QD}_{o+}$ but $M \notin \mathrm{QD}_o$. ∎

So we see that the property $M \in \mathrm{QD}_+$ is preserved under matrix transposition while $M \in \mathrm{QD}_o$ is not. Note that the example in (ii) is a *reducible* matrix. We will see that for irreducible matrices, the asymmetry disappears.

(5.2.4) Definition

M is *reducible* iff there is a *proper* subset K of $\{1,2,\ldots,n\}$ such that

$$M_{ij} = 0, \text{ for all } i \in K,\ j \notin K.$$

In other words, after reordering of the rows and columns correspondingly (so that the indices in K follow the indices of the complement of K) M becomes

$$P^T MP = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A and C are square matrices and P is the permutation matrix corresponding to the reordering. M is ***irreducible*** if it is not reducible.

(5.2.5) Remarks

(i) If we reorder the matrix M so that K precedes its complement, we will have the zero submatrix in the upper right corner instead of the lower left.

(ii) It is clear by the definition that M is irreducible iff M^T is.

(iii) Since z is a solution of $(P^T MP, P^T q)$ iff $x := Pz$ is a solution of (M,q), reordering of the indices does not affect the geometric and algebraic properties of the solution set of (M,q) if we permute the components of q accordingly.

(iv) Given any M, if it is reducible we can reorder the indices as in Definition (5.2.4) and get the square matrices A and C. If A or C or both are reducible we can repeat the process on A or C or both. So at last, after some reordering, M becomes

$$P^T MP = \begin{bmatrix} N_1 & * & * & * \\ 0 & N_2 & * & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_k \end{bmatrix} \tag{V.5}$$

i.e., a block upper triangular matrix with diagonal blocks N_1,

$N_2,\ldots,N_k$ all irreducible. If M itself is irreducible then $k=1$, otherwise $k \geq 2$. In fact, if we defined a directed graph G(M) with n nodes numbered from 1 to n such that G(M) contains an edge from node i to node j iff $M_{ij} \neq 0$, then irreducibility of M is equivalent to strongly connectedness of G(M) [see Varga 62, 16-19]. A directed graph is strongly connected if for any two nodes, there is a path that goes from one node to another following the (directed) edges. Hence the irreducible components N_i correspond to the strongly connected components of G(M) which can be found in time proportional to the number of nonzero entries of M [Tarjan 72]. For more details see [Aho, Hopcroft and Ullman 74, Ch. 5], in which there is a linear-time algorithm (Algorithm 5.4) that can be used to find N_i $i=1,\ldots,k$, in reverse order.

(v) If x is a solution of (M,q), then x_k is a solution of (N_k, q_k), where x_k and q_k are the appropriate *sub-vector* of x and q respectively. Hence we may want to solve (N_k, q_k) first. If the solution is unique we can proceed to solve x_{k-1} by solving $(N_{k-1}, q_{k-1}-Bx_k)$ where B is the block to the right of N_{k-1} in (V.5). The process can be kept going as long as uniqueness is guaranteed for each block. At last, we either reduce the size of the problem or solve it completely. Note that at each step the LCP has an ***irreducible*** matrix.

So let us start with irreducibe quasi-diagonally dominant M.

(5.2.6) Theorem Let M be irreducible and $M \in \mathrm{QD}_o$. Then (i) and (ii) defined below are equivalent,

(i) (M,q) has more than one solution

(ii) (a) $M \notin \mathrm{QD}_{o+}$, i.e. for any $d>0$ satisfying (V.1), equality holds in (V.1) for all i, and

(b) for some permutation matrix P,

$$P^T MP = \begin{bmatrix} Z_1 & P_2 \\ P_1 & Z_2 \end{bmatrix} \tag{V.6}$$

where Z_1 and Z_2 are Z-matrices, $P_1 \geq 0$ and $P_2 \geq 0$, Z_2 may be vacuous (so M is a Z-matrix), and

(c) $Mx+q=0$ for some $x>0$.

Moreover, either (i) or (ii) implies that

(c') $Mx+q=0$ for any solution x.

In particular, if $M \in \mathrm{QD}_{o+}$ then (M,q) has at most one solution for any q.

Proof.

(i) => (ii) Assume (M,q) has more than one solution.

Let x, y be two distinct solutions, let

$$v := Mx+q\,,\; w := My+q\,,\; \text{so } v-w=M(x-y),\; xv=yw=0. \quad \text{(V.7)}$$

Let $d>0$ satisfying (V.1). Define

$$K := \{k \mid 1\le k\le n,\; \frac{|x_k-y_k|}{d_k} = \max_{1\le j\le n} \frac{|x_j-y_j|}{d_j} =: \delta\}, \quad \text{(V.8)}$$

then $\delta>0$ and $K\neq\phi$ since $x\neq y$. Define

$$K^+ := \{k\in K \mid x_k-y_k>0\},\;\; K^- := \{k\in K \mid x_k-y_k<0\}. \quad \text{(V.9)}$$

For $k\in K^+$, $x_k>y_k\ge 0$, so by complementarity,

$$v_k=0, \text{ and} \quad \text{(V.10)}$$

$$0 \ge -w_k = v_k-w_k = (M(x-y))_k \quad \text{(V.11)}$$

$$= M_{kk}\,d_k\,\frac{x_k-y_k}{d_k} + \sum_{j\neq k} M_{kj}\,d_j\,\frac{x_j-y_j}{d_j} \quad \text{(V.12)}$$

$$\ge M_{kk}\,d_k\,\delta - \sum_{j\neq k} |M_{kj}|\,d_j\,\delta \ge 0 \qquad (\,by\;\; V.1\,) \quad \text{(V.13)}$$

Hence equalities hold in (V.11) and (V.13). Together with (V.10) the first (implied) equality in (V.11) implies that

(1) $v_k=w_k=0$.

The last (implied) equality in (V.13) implies that

(2) equality in (V.1) holds for $i=k$,

while the first equality implies that

$$M_{kj}\, d_j \frac{x_j - y_j}{d_j} = -|M_{kj}|\, d_j \delta \quad \text{for all } j \neq k \tag{V.14}$$

which in turn implies that

(3) If $M_{kj} \neq 0$ then $\dfrac{|x_j - y_j|}{d_j} = \delta$. So $x_j \neq y_j$.

(4) If $j \in K^+$, $j \neq k$, then $M_{kj} = -|M_{kj}| \leq 0$,

(5) If $j \in K^-$, then $M_{kj} = |M_{kj}| \geq 0$, or

(6) If $j \notin k$, then $M_{kj} = 0$.

In summary, we have proved (1-6) if $k \in K^+$. Similarly for $k \in K^-$, $y_k > 0$ and therefore $w_k = 0$, the same argument follows by interchanging v and w, and x and y in (V.10-14), and we have, under the assumption that $k \in K^-$, that (1-3) hold and

(4') If $j \in K^-$, $j \neq k$, then $M_{kj} = -|M_{kj}| \leq 0$,

(5') If $j \in K^+$, then $M_{kj} = |M_{kj}| \geq 0$, or

(6') If $j \notin k$, then $M_{kj} = 0$.

If $k \in K$, $j \notin K$ then either (6) or (6') applies and we have $M_{kj} = 0$, hence $K = \{1,2,\ldots,n\}$ since M is irreducible. Therefore (a) (V.1) holds for all i by (2), (c') $v = w = 0$ by (1), and (b) by reordering the indices so that K^+ procedes K^-, M becomes the form in (V.6) by

(4,5,4',5'). When either K^+ or K^- is empty, M is a Z-matrix. Moreover, for any j $x_j+y_j>0$. Otherwise, $x_j=y_j=0$, therefore $M_{kj}=0$ for all $k\neq j$ by (3) which contradicts the irreducibility of M. Hence $z:=x+y>0$ satisfies (c).

(ii) => (i) Assume (ii).

By (b) we can assume, without loss of generality, that

$$M = \begin{bmatrix} Z_1 & P_2 \\ P_1 & Z_2 \end{bmatrix} \tag{V.14}$$

where Z_2 may be vacuous. Let k_1, k_2 be the dimensions of Z_1, Z_2 respectively. Let $d>0$ satisfy (V.1). Define the vector $\bar{d}$ by

$$\bar{d}_k := \text{if } k\leq k_1 \; d_k \; else \; -d_k \tag{V.15}$$

Then $M\bar{d}=0$ by (a) and (V.14). By (c) there is a vector $\bar{x}>0$ such that $M\bar{x}+q=0$. Define

$$\Lambda := \{\lambda\in R \mid \bar{x}+\lambda\bar{d}\geq 0\} \tag{V.16}$$

For $\lambda\in\Lambda$, $\bar{x}+\lambda\bar{d}$ is a solution of (M,q) since, by the definition of $\bar{d}$,

$$M(\bar{x}+\lambda\bar{d})+q \;=\; M\bar{x}+q+\lambda M\bar{d} \;=\; 0+0 = 0 \tag{V.17}$$

Since $\bar{x}>0$, Λ contains an open neighborhood of the origin. In particular, Λ contains a nonzero element, say $\bar{\lambda}$, so that $\bar{x}$ and $\bar{x}+\bar{\lambda}\bar{d}$ are two distinct solutions of (M,q). ■

The following corollary characterizes the solution set when it is not unique.

(5.2.7) Corollary

Let M be irreducible and $M \in \mathrm{QD}_0$. If (M,q) has more than one solution, then the solution set of (M,q) is

$$S := \{x \mid x \geq 0,\ Mx+q=0\}$$

Let

$$\bar{x} := \arg\min x_1 \ \ s.t.\ \ Mx+q=0,\ x\geq 0 \tag{V.18}$$

then S is the intersection of the feasible region of (M,q) and a half-line starting at $\bar{x}$ with some direction $\bar{d}$ satisfying

$$M\bar{d} = 0,\ \bar{d}_1 > 0,\ |\bar{d}_i| > 0 \text{ for all } i. \tag{V.19}$$

i.e. $S = \{\bar{x}+\lambda\bar{d} \mid \lambda\geq 0,\ \bar{x}+\lambda\bar{d}\geq 0\}$. Moreover, S is unbounded (iff $\bar{d}>0$) iff M is a Z-matrix.

Proof. Clearly every vector in S is a solution of (M,q). On the other hand, all solutions are in S by Theorem (5.2.6) (ii) (c'). So S is the solution set of (M,q). $\bar{x}$ is well-defined by (V.18) since it is a solution of an LP which is feasible and bounded below. Let $d>0$ satisfy (V.1). By (ii) (b) of Theorem (5.2.6) there is a partition $\{L^+, L^-\}$ of the indices such that $1 \in L^+$ and

$$M_{ij} \geq 0 \ \text{ for } \ (i,j) \in (L^+ \times L^-) \cup (L^- \times L^+) \tag{V.20}$$

$$M_{ij} \leq 0 \text{ for } i \neq j, \ (i,j) \in (L^+ \times L^+) \cup (L^- \times L^-) \tag{V.21}$$

Note that L^- is empty iff Z_2 in (5.2.6) (ii) (b) is vacuous. Let x be any solution other than $\bar{x}$, then $M(x-\bar{x})=0$ since $x \in S$. By the same argument as in the proof of (5.2.6), defining

$$K^+ := \{j \mid \frac{x_j - \bar{x}_j}{d_j} = \delta\}, \ K^- := \{j \mid \frac{x_j - \bar{x}_j}{d_j} = -\delta\},$$

where $\delta := \max\limits_{1 \leq k \leq n} \frac{|x_k - \bar{x}_k|}{d_k} > 0$, we have that $K^+ \cup K^-$ is the whole index set, and

$$M_{kj} d_j \frac{x_j - \bar{x}_j}{d_j} = -|M_{kj}| d_j \delta \text{ for } j \neq k \in K^+ \tag{V.14'}$$

$$M_{kj} d_j \frac{\bar{x}_j - x_j}{d_j} = -|M_{kj}| d_j \delta \text{ for } j \neq k \in K^- \tag{V.14''}$$

Claim: $K^+ = L^+$, $K^- = L^-$.

Indeed, let $J := (K^+ - L^+) \cup (K^- - L^-)$, then $M_{kj} = 0$ for all $k \in J$, $j \notin J$. For

(1) Case $k \in K^+$ (so $k \notin L^+$ since $k \in J$), $j \in K^+$ (so $j \in L^+$ since $j \notin J$). Then by (V.14') $M_{kj} = -|M_{kj}| \leq 0$. On the other hand $M_{kj} \geq 0$ by (V.20) and $(k,j) \in L^- \times L^+$ [see previous two parentheses].

(2) Case $k \in K^+$ (so $k \notin L^+$ since $k \in J$), $j \in K^-$ (so $j \in L^-$ since $j \notin J$). Then by (V.14') $M_{kj} = |M_{kj}| \geq 0$. On the other hand $M_{kj} \leq 0$ by

(V.21) and $(k,j) \in L^- \times L^-$ [see previous two parentheses].

(3) Case $k \in K^-$ (so $k \notin L^-$ since $k \in J$), $j \in K^+$ (so $j \in L^+$ since $j \notin J$). Then by (V.14'') $M_{kj} = |M_{kj}| \geq 0$. On the other hand $M_{kj} \leq 0$ by (V.21) and $(k,j) \in L^+ \times L^+$ [see previous two parentheses].

(4) Case $k \in K^-$ (so $k \notin L^-$ since $k \in J$), $j \in K^-$ (so $j \in L^-$ since $j \notin J$). Then by (V.14'') $M_{kj} = -|M_{kj}| \leq 0$. On the other hand $M_{kj} \geq 0$ by (V.20) and $(k,j) \in L^+ \times L^-$ [see previous two parentheses].

Hence J is either the whole index set or empty by irreducibility of M. Since $1 \notin J$, J is empty. So $K^+ \subseteq L^+$ and $K^- \subseteq L^-$, which proves the claim since both $\{K^+, K^-\}$ and $\{L^+, L^-\}$ are partitions of the indices.

Therefore,

$$\frac{x_j - \bar{x}_j}{d_j} = \delta \quad \text{for } all \ j \in L^+$$

$$\frac{x_j - \bar{x}_j}{d_j} = -\delta \quad \text{for } all \ j \in L^-$$

Hence $x = \bar{x} + \delta \bar{d}$ where

$$\bar{d}_j := \text{if } j \in L^+ \ d_j \ \ else \ -d_j \tag{V.22}$$

We have proven that for any solution x, $x \in S'$ where

$$S' := \{\bar{x} + \lambda \bar{d} \mid \lambda \geq 0, \ \bar{x} + \lambda \bar{d} \geq 0\}.$$

So $S=S'$ since obviously $S'\subseteq S$. Moreover, by (V.22), $\bar{d}>0$ iff L^- is empty iff M is a Z-matrix (the last "if" part follows in part by the irreducibility of M). It is obvious that S' is unbounded iff $\bar{d}\geq 0$ iff $\bar{d}>0$ since $|\bar{d}_i|>0$ for all i. So the proof is complete. ■

(5.2.8) Remarks

(i) Under the assumption of (5.2.7), the vector d satisfying (V.1) is unique up to a positive scaling since every d uniquely defines a direction $\bar{d}$ by (V.22) and by Corollary the direction should be unique.

(ii) (V.19) uniquely determines $\bar{d}$ (up to positive scaling), i.e. instead of finding $d>0$ satisfying (V.1) and computing $\bar{d}$ by (V.22) we can simply solve (V.19) for a (unique) $\bar{d}$ which will give the direction of the solution set. For if $\bar{d}'$ is another direction satisfying (V.19), then since there exists an $x>0$, $Mx+q=0$ by Theorem (5.2.6)(ii)(c), at x the solution set has two directions, i.e. both $x+\lambda\bar{d}$ and $x+\lambda\bar{d}'$ are solutions for all lambda sufficiently small, contradicting the fact that S is one dimensional.

(iii) Hence for M irreducible, $M\in\mathrm{QD}_0$, we have the following computational procedure:

Step (1). Solve $\bar{x}$ by (V.18), exit if the LP is infeasible, in which case (M,q) has at most one solution which can be solved by some other algorithm, e.g. (3.4.1) [Example (5.2.9) 1,2].

Step (2). Solve $\bar{d}$ by (V.19), exit if no solution found, in which case (M,q) has the unique solution $\bar{x}$ [(5.2.9) 3,4].

Step (3). $S \leftarrow \{\bar{x}+\lambda\bar{d} \mid \bar{x}+\lambda\bar{d}\geq 0\}$, stop. In this case S may still be a singleton if the only feasible λ is zero [see Example (5.2.9) 5,6,7].

(5.2.9) Examples

(1) Exit at Step (1) of (5.2.8) (iii), (M,q) has a unique solution $x=0$.

$$M := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad q := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Note that (ii)(a) and (b) of Theorem (5.2.6) hold, but not (ii)(c).

(2) Exit at Step (1), (M,q) has no solution.

$$M := \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad q := \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Note that (M,q) is infeasible. Again only (c) is violated in (5.2.6) (ii).

(3) Exit at Step (2), (M,q) has the unique solution $x=[1\ 1]^T$.

$$M := \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad q := \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Note that (ii) (a) (c) hold in (5.2.6), but not (ii) (b).

(4) Exit at Step (2), (M,q) has the unique solution $x=[1\ 1]^T$.

$$M := \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad q := \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Note that (ii) (b) (c) hold in (5.2.6), but not (ii) (a).

(5) Exit at Step (3), (M,q) has a unique solution $x=[1\ 0\ 0]^T$.

$$M := \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad q := \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

Note that $\bar{d}=[1\ 1\ -1]^T$, and that only (ii)(c) does not hold in (5.2.6)(ii).

(6) Exit at Step (3), the solutions set is a line segment.

$$M := \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad q := \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix}$$

Then $\bar{x}=[1\ 0\ 1]^T$, $\bar{d}=[1\ 1\ -1]^T$, $K^{+}=\{1,2\}$, and

$$S = \{\bar{x}+\lambda\bar{d} \mid 0 \le \lambda \le 1\}$$

(7) Exit at Step (3), the solutions set is a half line.

$$M := \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad q := \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Then $\bar{x}=[1\ 0]^T$, $\bar{d}=[1\ 1]^T$, $K^{+}=\{1,2\}$, and

$$S = \{\bar{x}+\lambda\bar{d} \mid 0 \le \lambda < +\infty\}$$

Note that M is a Z-matrix, which is the only case S is unbounded.

5.3 EXISTENCE FOR DIAGONALLY DOMINANT–IRREDUCIBLE MATRICES

We assumed $M \in \mathrm{QD}_o$ in the previous section, and we got a complete characterization of the *uniqueness* of the solution set of (M,q) when M is irreducible. In this section we shall assume that $M^T \in \mathrm{QD}_o$, i.e. M is quasi-diagonally column dominant. It is interesting to note that what we are going to get are characterizations of *existence* instead of uniqueness.

(5.3.1) Theorem

Let M be irreducible and $M^T \in \mathrm{QD}_o$.

(i) Then every KKT point of (QP) is a solution of (M,q) and hence $M \in \mathrm{Q}_o$, i.e. (M,q) has a solution for all q for which (M,q) is feasible.

(ii) In fact, if (x, y, v) satisfies the KKT conditions of (M,q), i.e.

$$(Mx+q+M^T x)-M^T y-v = 0 \tag{V.23}$$

$$y_i(Mx+q)_i = v_i x_i = 0, \text{ for all } i \tag{V.24}$$

$$y,\, v,\, Mx+q,\, x \geq 0 \tag{V.25}$$

then (1) $M^T(x-y)=0$, (2) $v=Mx+q$, (3) $x^T(Mx+q)=0$ so x is a solution of (M,q).

(iii) There exists (x,y,v), $x \neq y$, satisfies (V.23-25) iff

(a) $M^T \notin \mathrm{QD}_{o+}$, (b) the statement (ii)(b) of (5.2.6) hold, and (c) there exists $\bar{y}>0$ such that $(x,\bar{y},v)$ satisfies (V.23-25), hence (c') $Mx+q=0$ by complementarity.

In particular, if $M^T \in \mathrm{QD}_{o+}$ then $x=y$

Proof.

(i) If (M,q) is feasible, then (QP) is bounded below by zero and hence has a global minimum [Frank and Wolfe] which is a KKT point. Hence it suffices to prove (ii) for (ii) implies that every KKT point is a solution of (M,q).

(ii) Assume (x,y,v) satisfies (V.23-25). If $x=y$ then (1)-(3) follow immediately by (V.23-25).

Assume $x \neq y$, we will show (ii) and the "only if" part of (iii) together. Let $d>0$ be as in Definition (5.2.1), i.e.

$$M_{ii} d_i \geq \sum_{j \neq i} |M_{ji}| d_j, \quad i=1,\dots,n. \tag{V.26}$$

Similar to (5.2.6), define

$$\delta := \max_{1 \leq j \leq n} \frac{|x_j - y_j|}{d_j} > 0,$$

$$K^+ := \{k \mid \frac{x_k - y_k}{d_k} = +\delta\}, \;\; K^- := \{k \mid \frac{x_k - y_k}{d_k} = -\delta\}$$

Then for $k \in K^+$, $x_k > y_k \geq 0$. So $v_k = 0$ by (V.24). Hence, by the k-th component of (V.23), we have

$$0 = (Mx + q)_k + (M^T(x - y))_k$$

$$\geq (M^T(x - y))_k \quad (Mx + q \geq 0 \; by \; V.25) \tag{V.27}$$

$$= M_{kk} d_k \frac{x_k - y_k}{d_k} + \sum_{j \neq k} M_{jk} d_j \frac{x_j - y_j}{d_j}$$

$$\geq M_{kk} d_k \delta - \sum_{j \neq k} |M_{jk}| d_j \delta \geq 0 \quad (by \; V.26) \tag{V.28}$$

Hence equalities hold in (V.27) and (V.28). Hence we have, by the (implied) equality of (V.27) and the one preceding it,

$$(Mx + q)_k = 0, \; (M^T(x - y))_k = 0, \; \text{ for } k \in K^+ \tag{V.29}$$

and, by the last (implied) equality of (V.28),

$$M_{kk} d_k = \sum_{j \neq k} |M_{jk}| d_j, \; \text{ for } k \in K^+ \tag{V.30}$$

and, by the first equality of (V.28)

$$M_{jk} d_j \frac{x_j - y_j}{d_j} = -|M_{jk}| d_j \delta \;\; \text{for all } j \neq k \in K^+ \tag{V.31}$$

Similarly for $k \in K^-$, $y_k > 0$ and therefore by (V.24),

$$(Mx+q)_k = 0, \text{ (for } k\in K^-) \tag{V.32}$$

Hence, by changing sign of the k-th component of (V.23), we have

$$0 = -(M^T(x-y))_k + v_k$$

$$\geq (M^T(y-x))_k \quad (*\ v\geq 0\ bx\ V.25\ *) \tag{V.27'}$$

$$= M_{kk}\, d_k \frac{y_k - x_k}{d_k} + \sum_{j\neq k} M_{jk}\, d_j \frac{y_j - x_j}{d_j}$$

$$\geq M_{kk}\, d_k\, \delta - \sum_{j\neq k} |M_{jk}|\, d_j \delta \geq 0 \quad (*\ by\ V.26\ *) \tag{V.28'}$$

Hence equalities hold in (V.27') and (V.28'), and by similar argument we have

$$v_k = 0,\ (M^T(y-x))_k = 0, \text{ for } k\in K^- \tag{V.29'}$$

$$M_{kk}\, d_k = \sum_{j\neq k} |M_{jk}|\, d_j, \text{ for } k\in K^- \tag{V.30'}$$

$$M_{jk}\, d_j \frac{y_j - x_j}{d_j} = -|M_{jk}|\, d_j \delta \text{ for all } j\neq k\in K^- \tag{V.31'}$$

By similar argument as in (5.2.6), (V.30-31,30'-31') implies that $K^+ \cup K^-$ is the whole index set and (iii)(a) and (iii)(b) hold. Therefore it follows that (1) $M^T(x-y)=0$ by (V.29,29'). (2) and (3) follow immediately by (1) and (V.23-25). Also (iii)(c')

$Mx+q=0$ by (V.29,32), and $\bar{y}:=\frac{1}{2}(x+y)>0$ by the definitions of K^+ and K^-. Finally, (iii)(c) follows since $(x,\bar{y},v)$ is a KKT point by (V.23-25), (1) and (iii)(c').

(iii) "only if" part has been proven in (ii). To show "if" part. Assume (a)(b)(c).

By the same arguments as in (5.2.7) [with M replaced by M^T], (a) (b) implies the existence of a $\bar{d}$ satifying

$$M^T\bar{d} = 0,\ \bar{d}_1 > 0,\ |\bar{d}_i| > 0 \text{ for all } i. \tag{V.33}$$

By (c), there exists $(x,\bar{y},v)$, $\bar{y}>0$, satisfying (V.23-25). It follows that $\bar{y}+\lambda\bar{d}>0$ for all sufficiently small λ. Besides, (1)-(3) hold by (ii). It is easy to see that $(x,\bar{y}+\lambda\bar{d},v)$ is a KKT point for all sufficiently small λ by (V.23-25) and (1)-(3). The proof is complete since $x\neq\bar{y}+\lambda\bar{d}$ for all but at most one λ. ▪

We have proven that (M,q) is solvable iff it is feasible. In the next theorem, we characterize the feasibility of (M,q).

(5.3.2) Theorem

Let M be irreducible and $M^T \in \mathrm{QD}_o$. Then

(M,q) is infeasible iff (a) $M^T \notin \mathrm{QD}_{o+}$, (b) M is a Z-matrix, and (c) $qd<0$ for some $d>0$ satisfying (V.26) [d is unique up to a positive scaling by (5.2.8)(i)].

In particular, if $M^T \in \mathrm{QD}_{o+}$, then (M,q) is feasible [hence solvable by (5.3.1)] for any q.

Proof. By a theorem of alternative [Mangasarian 79, 2.4.12 or Gale 60],

$$\left[I. \begin{array}{l} Mx+q \geq 0 \\ x \geq 0 \end{array} \text{ has no solution} \right] \text{ iff } \left[II. \begin{array}{l} M^T y \leq 0 \\ y \geq 0,\ qy < 0 \end{array} \text{ has a solution} \right]$$

So it suffices to show that II has a soluion iff (a), (b), (c) hold.

(i) Assume y is a solution of II, then $y \neq 0$ since $qy < 0$. So

$$\delta := \max_{1 \leq k \leq n} \frac{y_j}{d_j} > 0, \quad K^+ := \{k \mid 1 \leq k \leq n,\ \frac{y_k}{d_k} = \delta\} \neq \phi \tag{V.34}$$

where $d > 0$ be any positive vector satisfying (V.26). For $k \in K^+$,

$$0 \geq (M^T y)_k = M_{kk} d_k \frac{y_k}{d_k} + \sum_{j \neq k} M_{jk} d_j \frac{y_j}{d_j}$$

$$\geq M_{kk} d_k \delta - \sum_{j \neq k} |M_{jk}| d_j \delta \geq 0 \quad (*\ by\ V.26\ *)$$

Again, by similar arguments as before [note that K^- is empty], we have $K^+ = \{1,2,\ldots,n\}$, (a) and (b) hold, $M^T y = 0$, and $y = \delta d$ by (V.34). Hence $qd = \frac{1}{\lambda} qy < 0$ proving (c).

(ii) Assume (a), (b), (c) hold. By (c) there exists $d > 0$, $qd < 0$ satisfying (V.26). By (a), equality holds in (V.26) for all i. By (b), (V.26) [with equality] is equivalent to $M^T d = 0$. Hence d is a solution of

II. ■

5.4 EQUIVALENCE OF DIAGONAL ROW/COLUMN DOMINANCE FOR IRREDUCIBLE MATRICES

In this section, we shall prove that quasi-diagonally row dominance is equivalent to quasi-diagonally column dominance for irreducible matrices. Hence we can combine the results of previous two sections. Note that in (5.2.3) we have shown that without the irreducibility assumption, row dominance and column dominance are different unless the dominance is strict.

(5.4.1) Theorem

Let M be irreducible. Then

(i) $M \in \mathrm{QD}_o$ iff $M^T \in \mathrm{QD}_o$.

(ii) $M \in \mathrm{QD}_{o+}$ iff $M^T \in \mathrm{QD}_{o+}$.

Proof.

(i) "only if" part. Let $M \in \mathrm{QD}_o$.

Since (V.26) is equivalent to $C(M)^T d \geq 0$, where $C(M)$ is the comparison matrix of M defined by (V.2), it follows that

$$M^T \in \mathrm{QD}_o$$
$$\text{iff}$$
$$C(M)^T d \geq 0,\ d > 0 \text{ has a solution}$$

iff

$$C(M)x+z=0,\ x,z\geq 0,\ z\neq 0 \text{ has no solution} \tag{V.35}$$

where the last equivalence comes from a theorem of alternative [Mangasarian 79, Ch. 2]. So it suffices to show that (V.35) holds, which is equivalent to:

$$x\geq 0,\ C(M)x=-z\leq 0 \text{ implies } z=0. \tag{V.35'}$$

Let $x\geq 0$, $C(M)x=-z\leq 0$. Define

$$K^+ := \{k \mid 1\leq k\leq n,\ \frac{x_k}{d_k} = \delta := \max_{1\leq j\leq n} \frac{x_j}{d_j}\}$$

where $d>0$ satisfies (V.1). Then for $k\in K^+$,

$$0\geq -z_k = M_{kk} d_k \frac{x_k}{d_k} + \sum_{j\neq k} |M_{kj}| d_j \frac{x_j}{d_j}$$

$$\geq M_{kk} d_k \delta - \sum_{j\neq k} |M_{kj}| d_j \delta \geq 0 \quad (* \text{ by } V.1 *)$$

Hence $z_k=0$ for all $k\in K^+$. But, as before, K^+ is the whole index set, so $z=0$ proving (V.35'). Hence $M^T\in \mathrm{QD}_{o+}$.

(i) "if" part. Let $M^T\in \mathrm{QD}_o$.

Then by the "only if" part $(M^T)^T\in \mathrm{QD}_o$, i.e. $M\in \mathrm{QD}_o$.

(ii) "only if" part. Let $M\in \mathrm{QD}_{o+}$.

Then for some $d>0$, (V.1) holds and for some index, say k, the strict inequality holds. Then by decreasing the M_{kk} a little, (V.1) still holds, i.e. for some small positive number α, $M-\alpha E_{kk} \in \mathrm{QD}_o$ where E_{kk} is the matrix with all entries equal to zero except the kk-th entry which is 1. Hence by (i),

$$M^T-\alpha E_{kk} \equiv (M-\alpha E_{kk})^T \in \mathrm{QD}_o$$

Hence $M^T \in \mathrm{QD}_{o+}$ since any $d>0$ satisfying (V.1) for the matrix $M^T-\alpha E_{kk}$ will satisfies (V.1) for M^T with strict inequality hold for index k.

(ii) "if" part follows by "only if" part as in (i). ∎

Hence the results of the previous two sections hold under a single assumption $M \in \mathrm{QD}_o$ (or equivalently $M^T \in \mathrm{QD}_o$). Here we summarize some of them.

(5.4.2) Corollary [Aganagic 81, Thm 2.2]

Let M be irreducible, and $M \in \mathrm{QD}_o$ or equivalently under the irreducibility assumption $M^T \in \mathrm{QD}_o$. Then

(i) If $M \in \mathrm{QD}_{o+}$ or equivalently $M^T \in \mathrm{QD}_{o+}$ then (M,q) has a unique solution for any q, i.e. M is a P-matrix.

(ii) Suppose $M \notin \mathrm{QD}_{o+}$. Then (M,q) has a soution for any q, i.e. $M \in Q$. iff M is not a Z-matrix. Moreover, if M is a Z-matrix then (M,q) has a solution iff $q^T d \geq 0$ where d is the unique (up to a positive scaling) positive vector satisfying $M^T d=0$.

Proof. Uniqueness follows by (5.2.6) while the existence follows by (5.3.2). ∎

5.5 DIAGONAL ROW/COLUMN DOMINANCE WITHOUT IRREDUCIBILITY

Now we are ready to give characterizations for general QD_o-matrix without the irreducibility assumption. Recall that row dominance and column dominance are different for reducible matrices.

(5.5.1) Existence Theorem For QD_o-matrix

Let $M \in QD_o$. Then any KKT point of (QP) is a solution of (M,q). Hence (M,q) has a solution for all q for which (M,q) is feasible, i.e. $M \in Q_o$.

Proof. Let (M,q) be feasible and (x, y, v) be a KKT point of (QP). To show X is a solution of (M,q) we use induction on n. If M is irreducible we are done by (5.4.1) and (5.3.1). If M is reducible, then by reordering the indices we can assume that

$$M = \begin{bmatrix} M_1 & C \\ 0 & M_2 \end{bmatrix}$$

where M_1, M_2 are nonvacuous sqare matrices and M_1 is **irreducible**, i.e. M_1 is the matrix N_1 defined in (V.5) [M_2 is reducible unless it happens to equal to N_2 defined in (V.5)].

(i) Case $C=0$.

Then (M,q) can be decomposed to (M_1,q_1) and (M_2,q_2) where $[q_1,q_2]=q$. Since (M,q) is feasible, so are the two sub-problems. Therefore by the induction hypothesis, any KKT point of (QP_i) is a solution of (M_i,q_i) for $i=1,2$ where (QP_i) is defined by

$$\text{minimize } z_i(M_i z_i+q_i) \quad s.t. \;\; z_i\geq 0,\; M_i z_i+q_i\geq 0$$

On the other hand, it is straightforward to check that $z\equiv[z_1,z_2]$ is a KKT point of (QP) iff z_i is a KKT point of (QP_i) for $i=1,2$. So we are done.

(ii) Case $C\neq 0$.

Then $M_1\in\mathrm{QD}_{o+}$ [strict inequality holds in (V.1) for some indices after deleting from the right hand side some terms corresponding to the nonzero terms of C]. So $M_1^T\in\mathrm{QD}_{o+}$ by (5.4.1).

Let x_i,y_i,v_i $i=1,2$ be the appropriate components of the KKT point (x,y,v). By (V.23)

$$v_1 = M_1x_1+Cx_2+q_1+M_1^T(x_1-y_1) \tag{V.36}$$

$$v_2 = M_2x_2+q_2+C^T(x_1-y_1)M_2^T(x_2-y_2) \tag{V.37}$$

It follows by (V.24-25) and (V.36) that (x_1,y_1,v_1) satisfies the KKT conditions of the QP corresponding to (M_1,Cx_2+q_1). So

$x_1=y_1$ by (5.3.1) (iii), and therefore the term $C^T(x_1-y_1)$ in (V.37) can be dropped. Hence (x_2,y_2,v_2) is a KKT point of the QP corresponding to (M_2,q_2). Let w_1,w_2 be the appropriate components of $w:=Mx+q$, then $x_2w_2=0$ by the induction hypothesis while $x_1w_1=0$ by (5.3.1) since M_1 is irreducible. So x is a solution of (M,q), and the induction is complete.

Now for the same reason as in (5.3.1) (QP) has a KKT point provided that (M,q) is feasible, hence $M \in \mathrm{Q}_o$. ■

The result $M \in \mathrm{Q}_o$ in Theorem (5.5.1) has also been obtained by Moré [Moré 74] and Aganagic [Agnagic 81] by utilizing Lemke's method instead of the KKT conditions of (QP).

(5.5.2) Characterization of QD $_o \cap$Q

Let $M \in \mathrm{QD}_o$. Let N_i, $i=1,\ldots,k$ be defined as in (V.5), i.e. after reordering the indices so that M becomes upper block triangular with irreducible diagonal blocks, N_i, $i=1,\ldots,k$ are the diagonal blocks. Then

$M \in \mathrm{Q}$ iff for $i=1,\ldots,k$, either $N_i \in \mathrm{QD}_{o+}$ or $N_i \not\in \mathrm{Z}$.

Proof. Induction on k. If $k=1$, i.e. M is irreducible, we are done by (5.4.1) and (5.3.2). Assume $k \geq 2$, then by reordering the indices we can assume

$$M = \begin{bmatrix} M_1 & C \\ 0 & M_2 \end{bmatrix}$$ M_1, M_2 be nonvacuous sqare matrices,

and M_1 is **irreducible** , i.e. $M_1 := N_1$ [M_2 is reducible unless $k=2$].

(i) "if" part, assume for $i=1,\ldots,k$, $N_i \in \mathrm{QD}_{o+}$ or $N_i \notin \mathrm{Z}$.

Given any vector $q \equiv [q_1\ q_2]$, (M_2, q_2) has a solution by induction hypothesis. So it has a feasible point, say x_2. Since $M_1 \equiv N_1$ is irreducible, $(M_1, Cx_2 + q_1)$ is feasible by (5.4.1) and (5.3.2). Let x_1 be any feasible point, then $x := [x_1\ x_2]$ is a feasible point of (M,q). So (M,q) has a solution by (5.5.1).

(ii) "only if" part, assume $M \in \mathrm{Q}$.

Given any vector q_2 and any q_1, let $q := [q_1\ q_2]$. Since $M \in \mathrm{Q}$ (M,q) has a solution $[x_1\ x_2]$. It follows easily that x_2 is a solution of (M_2, q_2). Hence $M_2 \in \mathrm{Q}$. Hence by induction hypothesis, for $i=2,\ldots,k$ N_i satisfies the asserted property. For N_1, assume it is not in QD_{o+}, hence $C=0$ and x_1 is a solution of (M_1, q_1), we only need to show that it is not a Z-matrix. Since q_1 was arbitrary, $M_1 \in \mathrm{Q}$, hence by the induction hypothesis on M_1 which is N_1, N_1 is not a Z-matrix. ∎

A weaker result than Theorem (5.5.2) has also been obtained [Aganagic 81] which needs to check all of the $2^n - 1$ pricipal submatrices of M to determine whether $M \in \mathrm{Q}$. By Theorem (5.5.2) only k

of them needs be checked and these k submatrices can easily be found by a linear-time graph algorithm [see Remarks (5.2.5) (iv)].

(5.5.3) Theorem

Let $M^T \in \mathrm{QD}_o$. By reordering the indices we can assume M equals the right hand side of (V.5). Let n_i be the dimension of the irreducible diagonal block N_i for $i=1,\ldots,k$. Let S be the solution set of (M,q). Then

$$S = S_1 \times S_2 \times \cdots \times S_k \tag{V.38}$$

where S_i is the solution set of (N_i,p_i) for some vector $p_i \in R^{n_i}$. So S is a singleton iff every S_i is a singleton.

Moreover if S_i contains more than one point, then (5.2.6) and (5.2.7) apply to (N_i,p_i), e.g. S_i is a half line [if N_i is a Z-matrix] or a line segment.

Proof. Induction on k. If $k=1$ then M is irreducible and we are done by (5.4.1) and (5.2.6-7). For $k \geq 2$, i.e. M is reducible, then

$$M =: \begin{bmatrix} M_1 & C \\ 0 & M_2 \end{bmatrix} \quad M_1,\ M_2 \text{ be nonvacuous sqare matrices,}$$

where $M_2 := N_k$ is **irreducible** , [M_1 is reducible unless $k=2$]

(i) Case $C=0$.

Then (M,q) can be decomposed to (M_1,q_1) and (M_2,q_2) where

$[q_1, q_2] = q$ such that S is the product of the soution sets of the two sub-problems. So we are done by applying induction hypothesis on those two sub-problems.

(ii) Case $C \neq 0$.

Then $M_2^T \in \mathrm{QD}_{o+}$ [see (5.5.1)] So $M_2 \in \mathrm{QD}_{o+}$ by (5.4.1). Hence (M_2, q_2) has a unique solution, say x_2 (5.2.6). Let $S_k := \{x_2\}$, then it is straightforward to check that $S = T \times S_k$ where T is the solution set of $(M_1, Cx_2 + q_1)$. The induction is complete by applying the induction hypothesis to T. ∎

(5.5.4) Remarks

(i) Let us go back to the procedure defined in (5.2.8) (iii) for solving the set of all solutions when M is irreducible. In (5.2.8) (iii) step 1, we did not specify what to do if the LP is infeasible. Thanks to (5.3.2) [and (5.4.1), of course, which enables us to apply (5.3.2)] we can tell whether the LCP is feasible without solving another LP. And if it is feasible, then Algorithm (3.4.1) will find the unique solution. Hence we have a complete algorithm.

(ii) Recall the **reduction procedure** defined in (5.2.5) (iv) where we solve S_i for i from k down to 1, and we claimed it worked when each S_i is a singleton. For $M^T \in \mathrm{QD}_o$, as we see in the previous lemma the reduction procedure still works even when some S_i contains more than one point, since the problem is then completely decoupled [see case $C=0$]. In fact, we can solve for the whole set S by applying (5.2.8) (iii) for each S_i. ■

In Theorem (5.5.2) we characterize the class of $M \in \mathrm{QD}_o$ for which (M,q) has **at least one solution for all** q. In the next theorem we characterize $M^T \in \mathrm{QD}_o$ for which (M,q) has **at most one solution for all** q.

(5.5.5) Theorem

Let $M^T \in \mathrm{QD}_o$. By reordering the indices we can assume M equals the right hand side of (V.5). Let n_i be the dimension of the irreducible diagonal block N_i for $i=1,\dots,k$. Then the following are equivalent,

(i) (M,q) has more than one solution for some q

(ii) for some $1 \le i \le k$, $N_i \notin \mathrm{QD}_{o+}$ and $P^T N_i P$ equals the right hand side of (V.6) for some permutation matrix P.

Proof.

Suppose (i), let S be the solution set of (M,q) for some q for which (M,q) has more than one solution. Then by (5.5.3), for some i, S_i, defined in (5.5.3), contains more than one point. By (5.5.3) and (5.2.6) S_i satisfies (ii).

Suppose (ii), let N_i have the properties in (ii). Pick any $x_i>0$ and $x_j=0$ for $j\neq i$ and let $x=[x_1\, x_2 \cdots x_k]$. Let $q:=-Mx$ then x is a solution of (M,q). Define S, S_i as in (5.5.3), then $x_i\in S_i$. By (5.5.3) and (5.2.6), S_i contains more than one point. It follows immediately that S contains more than one point, proving (i). ▪

(5.5.6) Corollary Let $M^T\in\mathrm{QD}_o$, $x\equiv[x_1\, x_2 \cdots x_k]$ be a solution of (M,q). Let $w:=Mx+q$. If for all i having property in (5.5.5)(ii), $w_i\neq 0$ then the solution is unique.

Proof. We use the same notation as in (5.5.5). Since $x\in S$, then $x_i\in S_i$ by (5.5.3). If the solution is not unique then S_i is not a singleton for some i. By (5.5.3) and (5.2.6), N_i has the property in (5.5.5)(ii) and $w_i=0$. ▪

5.6 SUMMARY

We considered the thoretic and computational aspects of the linear complementarity problem (M,q) when M is quasi-diagonally row, or column, dominant. Row dominance and column dominance do not imply each other in general. **They are equivalent if the dominance is strict or the matrix is irreducible.**

For quasi-diagonally row dominant matrix M, (M,q) is solvable for q for which (M,q) is feasible, and in which case Algorithm (3.4.1) will find a vertex solution. Moreover, a characterization is given for M for which (M,q) is feasible [hence solvable] for all q, i.e. for $M \in$Q.

For quasi-diagonally column dominant matrix M, the solution set of (M,q) is a product of k sets where k is the number of irreducible components of M and each set is either a closed half line, a closed line segment, a single point, or is empty, and all the k sets can easily be computed in certain order by a reduction procedure. Given a solution we can easily check whether it is unique. Finally, we give a characterization for M for which (M,q) has at most one solution for every q.

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