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BASE SET STRATEGY FOR SOLVING
LINEARLY CONSTRAINED CONVEX PROGRAMS

by

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1. Introduction

We discuss the active set algorithm for solving linearly inequality constrained programs, via pivotings in special kind of active sets, namely, base sets, which will be defined later.

There is a large literature about the active set algorithm for solving linearly inequality constrained programs. See [1], [2], [3], [5] and their references. The active set is defined as the set of indices of constraints for which equality holds at the current iteration. The method consists of

- Test whether x^k is optimal or not. If it is, stop.
- Decide whether to add constraints to the active set or to delete constraints from the active set.
- Optimize in a subspace corresponding to the active set.
- Add constraints to the active set.
- Delete constraints from the active set.

If we only solve unconstrained problems in the subspace, it is very possible that we may return to the same active set again. In an extreme situation, the well-known phenomenon of <u>zigzagging</u> occurs when a constraint is repeatedly dropped from the active set at one iteration, only to be added again at a subsequent iteration. Zigzagging can cause slow progress, or even convergence to a non-optimal point [3].

In section 2, we distinguish a class of base sets among the sets of indices. It is proved that the correct active set [3] is a base set. By means of this concept, we give an algorithm in section 3 such that it converges in a finite number of steps. In section 4, we discuss the technique to find a base set with lower optimal value from a feasible point.

Convex programs with linear constraints arise naturally in applications, e.g. stochastic programs [4] [6]. The deterministic problems equivalent to stochastic programs with fixed recourse have separable convex objective functions. For such problems, unconstrained optimization in a subspace is not difficult. Therefore, the base set strategy may have use in these cases.

In the theorems, we assume that the objective function is uniformily quasi-convex and differentiable. A function $f: \mathbb{R}^n \to \mathbb{R}^n$ is uniformily quasi-convex if $\forall x, y \in \mathbb{R}^n, t \in (0,1)$, we have

$$f(tx+(1-t)y) \leqq \max \{f(x), f(y)\} - ct(1-t)\|x-y\|_2^2 ,$$

where c is a positive constant.

2. Base Sets

Consider

$$\begin{aligned} &\inf \quad f(x) \\ &\text{s.t.} \quad a_i x \le b_i, \quad i=1,\dots,m \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $i=1,\dots,m$, $f: \mathbb{R}^n \to \mathbb{R}$ is uniformily quasi-convex and differentiable. Suppose (2.1) is solvable i.e., it has an optimal solution x^*. Since f is uniformily quasi-convex, this optimal solution is unique. Let $M = \{1,\dots,m\}$.

Definition 2.1. The active set of a point $x \in \mathbb{R}^n$ (x is not necessarily feasible for (2.1)) is

$$S(x) := \{i \in M \mid a_i x = b_i\} . \tag{2.2}$$

Write $S^* = S(x^*)$. In [3], S^* is called the correct active set. If we can find S^*, then the problem becomes an equality constrained problem:

$$\begin{aligned} &\min \quad f(x) \\ &\text{s.t.} \quad a_i x = b_i, \quad i \in S^* . \end{aligned} \tag{2.3}$$

In general, S^* is unknown. However, we can show that S^* belongs to a subclass of subsets of M, the base set class. The key idea is to perform descent pivoting in this base set class.

Suppose $S \subset M$. Consider

$$\begin{aligned} &\inf \quad f(x) \\ &\text{s.t.} \quad a_i x = b_i, \quad i \in S . \end{aligned} \tag{2.4}$$

Denote its optimal value by $F(S)$.

Definition 2.2. If (2.4) is solvable for some $S \subset M$ and the optimal solution $x(S)$ is a feasible solution of (2.1), then S is called a base set of (2.1).

Theorem 2.1. S^* is a base set of (2.1). ∎

Proof. It suffices to prove that x^* is the optimal solution of (2.3). Suppose not. Then there exists a point $\bar{x}$, which is feasible for (2.3) and satisfies

$$f(\bar{x}) < f(x^*) .$$

By the definition of x^*, $\bar{x}$ is infeasible to (2.1) i.e.,

$$D := \{i \in M | a_i \bar{x} > b_i\} \neq \phi .$$

Let

$$\lambda = \min_{i \in D} \left\{ \frac{b_i - a_i x^*}{a_i \bar{x} - a_i x^*} \right\} .$$

Then $0 < \lambda < 1$. Let $y = \lambda \bar{x} + (1-\lambda)x^*$. Then y is feasible to (2.1) and $f(y) < f(x^*)$. That is impossible. Therefore x^* optimizes (2.3). ∎

Corollary 2.1. If S is a base set of (2.1) and $S \neq S^*$, then

$$F(S^*) < F(S) .$$

3. Algorithms

A1. Start from a feasible point x^0 of (2.1).

A2. Having a point x^k feasible for (2.1) determine a base set S^k of (2.1) such that

$$F(S^k) \leq f(x^k) . \tag{3.1}$$

A3. Having S^k, find $x(S^k)$. If it is optimal for (2.1), stop, else determine a feasible point x^{k+1} for (2.1), such that

$$f(x^{k+1}) < F(S^k) , \tag{3.2}$$

and go to A2.

Theorem 3.1. This algorithm produces a finite sequence $\{S^k, k=0,1,\dots,k_0\}$ of base sets of (2.1), such that

$$F(S^k) < F(S^{k-1}) , \quad k=1,\dots,k_0 \tag{3.3}$$

and $S^{k_0} = S^*$.

Proof. (3.3) is according to (3.1) and (3.2). Therefore, we get a base set sequence $\{S^k\}$, $S^k \neq S^\ell$, $\forall k \neq \ell$. Since the number of base sets of (2.1) is finite, $\{S^k\}$ must have only finitely many elements. By A3, the last element of $\{S^k\}$ must be S^*. ∎

There are many methods for finding a starting feasible solution. There are also many methods for finding a better feasible solution from a feasible solution $x(S^k)$, such as using negative Lagrange multiplier or gradient projection methods [3]. Therefore, we will not discuss the techniques of A1 and A3. In the following, we discuss the technique of A2.

4. Technique of A2

We omit the subscript k, use z instead of x^k, T instead of S^k, and rewrite A2:

A2. Having a feasible point z of (2.1), determine a base set T of (2.1) such that

$$F(T) \leq f(z) . \tag{4.1}$$

If $S(z)$ is a base set, then we are finished. In general, we have

Theorem 4.1. For any feasible point z of (2.1), there is a base set of (2.1) such that

$$T \supseteq S(z) \tag{4.2}$$

and (4.1) holds. ∎

We first prove the following lemma:

Lemma 4.1. Suppose z is a feasible point of (2.1) and $S(z)$ is not a base set of (2.1). Then there exists another feasible point $\bar{z}$ such that

$$f(\bar{z}) < f(z) , \quad S(\bar{z}) \supsetneqq S(z) . \quad ∎ \tag{4.3}$$

Proof. Consider

$$\begin{aligned} &\inf \quad f(x) \\ &\text{s.t.} \quad a_i x = b_i , \quad i \in S(z) . \end{aligned} \tag{4.4}$$

First, the constraints of (4.4) are consistent (z satisfies them).

Since $S(z)$ is not a base set of (2.1), there exists a point z^* which is feasible for (4.4), but infeasible for (2.1), and which satisfies

$$f(z^*) < f(z) \tag{4.5}$$

Therefore

$$D := \{i \in M | a_i z^* > b_i\} \neq \phi \tag{4.6}$$

and

$$D \cap S(z) = \phi\ . \tag{4.7}$$

Let

$$\lambda = \min_{i \in D} \left\{ \frac{b_i - a_i z}{a_i z^* - a_i z} \right\} . \tag{4.8}$$

Then $0 < \lambda < 1$. Let $\bar{z} = (1-\lambda)z + \lambda z^*$. Then $\bar{z}$ is a feasible point of (2.1) and

$$S(z) \supseteq (S(z) \cap S(z^*)) = S(z)\ . \tag{4.9}$$

From (4.5), we know

$$f(\bar{z}) < f(z)\ . \tag{4.10}$$

By (4.8), we know

$$S(\bar{z}) \cap D \neq \phi\ . \tag{4.11}$$

From (4.7), (4.9) and (4.11), we know

$$S(\bar{z}) \supsetneq S(z)\ .$$

This proves the lemma. ∎

Proof of Theorem 4.1. If $S(z)$ is a base set, we are done. Suppose it is not. Write $z^0 = z$. By Lemma 4, we have a feasible point z^1 of (2.1) such that

$$f(z^1) \leq f(z^0),\ S(z^1) \supsetneq S(z)\,.$$

If $S(z^1)$ is not a base set, we can find z^2, and so on. We thus have a descending sequence $\{z^r\}$ of feasible solutions such that for $r=0,1,\ldots,$

$$f(z^{r+1}) \leq f(z^r) \tag{4.12}$$

and

$$S(z^{r+1}) \supsetneq S(z^r)\,. \tag{4.13}$$

Since $S(z^r)$ has at most m members $\{z^r\}$ has at most $m+1$ members ($S(z^0)$ may be ϕ). Thus we have $z^0,\ldots,z^t$, where $0 \leq t \leq m$, and $S(z^t)$ is a base set of (2.1). Let $T := S(z^*)$. From (4.13), $T \supseteq S(z)$. From (4.12),

$$F(T) \leq f(z^t) \leq f(z)\,.$$

This proves the theorem. □

Lemma 4.1 and (4.8) give us a constructive way to find T. If we combine this method with the algorithm in Section 3, some active sets may be repeatedly met in distinct steps (for different k). However, the base sets will never be repeated.

One might argue that the concept of base set is not necessary. One might say that if one insisted on minimizing the function for the given active set before considering removal of any constraints, then there would be no points with lower function value for which the active set was the same. However, if an active set is not a base set, this implies minimizing the function under the equality constraints corresponding to the active set and the inequality constraints corresponding to remain indices. This is no simpler than the original problem except insofar as it reduces the number of constraints.

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