ANALYSIS OF A DYNAMIC, DECENTRALIZED ECONOMIC MODEL

by

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Table of contents

Introduction

We present a simple *Dynamic Economic Model* , an exchange economy in which nothing is produced (the set of available commodities is fixed). We show that the market, at each moment, can adjust prices in such a way as to lead the consumers to make choices satisfying the scarcity constraints . We say that the model is decentralized because each consumer makes decisions independent of the rest of the consumers; the only information the consumer receives from the market is the price vector.

This result is well known in the framework of a *Static Equilibrium*. The model there is described by a set of available commodities $M \subset \mathbf{R}^m$ and by n consumers i=1,..,n. The set of normalized positive prices is $S = \left\{ p \in \mathbf{R}^m_+ \mid \sum_{j=1}^{m} p_j = 1 \right\}$. For each price vector the total revenue, defined as $r(p) := \sup_{w \in M} \langle p,w \rangle$, is allotted to the consumers; consumer i receives an income $r_i(p)$ where $r(p) = \sum_{i=1}^{n} r_i(p)$. Each consumer is represented by his demand function $d_i(p,r)$, that to each price vector and to each income associates a basket of goods (the demand). This demand function satisfies the Walras law: for each price vector and for each income the value of the demand does not exceed the income, i.e. $\langle p,d_i(p,r) \rangle \leq r$. The problem is then to find a price vector $\overline{p} \in S$, called a Walrasian equilibrium, such that the aggregate demand satisfies the scarcity constraints of the market, i.e. $\sum_{i=1}^{n} d_i(\bar{p}, r_i(\bar{p})) \in M$. It is possible to show that if M is closed convex with $M = M - R_+^m$, and the functions $p \to d_i(p, r_i(p))$ are continuous on S, then such a Walrasian equilibrium exists (see for example Aubin [4] p. 378).

This type of model, interesting insofar as it allows us to describe and eventually to predict the behavior of a market, has a substantial drawback due to its static character. Recently there has been an increasing interest in endowing economics with dynamic models that better reflect the changing behavior of the market. The Walras tatonnement is the best known. In the framework of a pure exchange economy, the evolution of prices is explained by the following differential equation:

$$p'_j(t) = \begin{cases} 0 & \text{if } p_j \leq 0 \text{ and } z_j(p) < 0 \\ G_j(z_j(p)) & \text{otherwise} \end{cases}$$

with initial condition $p(0) = p_0$ ($p_0 \in R_+^m$ given). $z_j(p)$ denotes the excess demand function for commodity j (demand less offer), and G_j is a differentiable sign preserving function from **R** into **R** with $G'_j(z) > 0$ for each $z \in \mathbf{R}$ (a description of this model can be seen in Arrow and Hahn [1]). However the method is not operational, the transactions cannot be carried out unless the current prices represent a Walrasian equilibrium. If the excess demand function for a commodity is strictly positive, then the demand is greater than the offer and we cannot allow any transaction. It is then necessary to imagine "the existence of a super-auctioneer who calls a given set of prices p and receives transaction offers from the agents in the economy. If these do not match he calls another set of prices..." [1].

Smale [23] describes a second class of evolution model that is not decentralized.

The model of Aubin [7] has been the inspiring source for this work. There a dynamic decentralized model is described. However he makes the rather strong assumption that the instantaneous demand functions are linear on the endowment. His economic model can be summarized by the following evolution inclusion:

$$\begin{cases} \frac{dx}{dt}(t) \in D(x(t)) \quad \text{a.e. } (0,T) \\ x(t) \in K \quad \forall t \in [0,t] \\ x(0) = x^0 \quad (x^0 \in K \text{ given}) \end{cases}$$

In order to show the existence of a solution to the above evolution inclusion, Aubin, Cellina and Nohel [2] require K to be a convex and compact subset of $\mathbb{R}^m$, D to be an upper hemicontinuous multifunction from K into $\mathbb{R}^m$ with nonempty closed convex images and $D(x) \cap T_K(x) \neq \phi$ for each $x \in K$, where $T_K(x)$ denotes the tangent cone to K at x. Aubin [6] discusses other possibilities. Essentially, in order to relax the convexity assumption on the images of D, he requires that D be continuous (upper and lower semicontinuous) and that $D(x) \subset T_K(x)$ for all $x \in K$. This last assumption is equivalent to the global constraint imposed in the model by Smale [23] and it destroys the decentralization. It amounts to requiring that the aggregate instantaneous demand never point in an inadmissible direction for the set of available commodities. This is the same as asking that each consumer know the behavior of the other consumers. By definition of the multifunction D, the requirement that the instantaneous demand functions be linear in the endowment is the only way to guarantee that D(x) is convex for each x in K.

In this setting, we show the existence of a solution to the above evolution equation without asking D to have convex images but also without destroying the decentralization of the model.

The model has important theoretical and practical implications. An increasing number of large-scale economic models is being generated to study, for example, whole sectors of the economy of a country (see for instance Hogan and Weyant [12]). Usually their goal is to get an optimal point (equilibrium point) of a mathematical programming model. Thus the economy is studied as a sequence of equilibrium points.

The size of these models is a major problem. Often they are made up of pieces generated by different researchers, using different techniques but sharing the same information. Each one of these components represents a group of economic agents. The sizes of these problems require that decomposition techniques be used. The decomposition is achieved by relaxing some of the constraints and by the use of iterative methods. Sometimes the convergence of these iterative methods is not proved.

The decentralized model we propose is a contribution toward a different and, we hope, promising way of approaching such problems. In our approach, the components can be modeled individually; the only information they share is the price vector. The evolution of the system is built into the model in a natural way, and we do not have to assume that the economy moves from one equilibrium point to another. We therefore suggest this decentralized approach as a possible useful technique for analyzing and solving the kinds of large-scale economic models just mentioned.

In chapter 1 we will review some basic properties of multifunctions, we will define the tangent and normal cones to a convex set and state some of their properties. We will also define some notation we will be using in the remainder of the thesis.

We present the economic model and show the existence of a solution for the model in chapter 2. In section 2.1 we state most of the assumptions our model requires. These assumptions are essentially the same that Aubin makes in [7]. Some of Aubin's assumptions have been strengthened and the linearity condition on the instantaneous demand functions has been removed. In place of the linearity condition we propose three new conditions. In section 2.2 we prove the existence theorem, and at the same time we study the three new assumptions. Finally, in section 2.3 we compare Aubin's model with ours. We show that the linearity condition, that he requires for the instantaneous demand functions, imply two of our three assumptions. The third is not found in Aubin's model. However, as the number of consumers increases, we expect this assumption to be satisfied.

In chapter 3 we study a family of instantaneous demand functions. They are obtained by solving a "natural" maximization problem and satisfy most of the assumptions made in chapter 2. For economic models with all consumers having instantaneous demand functions in this family, we study the behavior of the solutions as time goes to infinity in section 3.4.

Finally, in chapter 4, we propose two algorithms to solve numerically the economic model and we study their implementation. We do not prove their correctness but we use them to get approximate solutions for a few small examples.

1. Multifunctions

1.1 Continuity Notions

We define here the most used continuity notions for multifunctions and their relationships. Most of the results in this section are well known (see for example Berge [8]). Our definition of upper semicontinuity does not require the images of the multifunction to be compact; we prefer to explicitly require compactness when needed.

Theorems 1.1.5 and 1.1.12 will be used in the proof of the existence theorem of chapter 2.

Let X be a topological space and A a subset of X. We will denote by $\operatorname{int} A$ the interior of A, by $\bar{A}$ the closure of A and by A^c the complement of A in X.

If (X,d) is a metric space we will denote by B(x,r) the open ball of radius r with center at x, i.e.

$$B(x,r) := \left\{ y \in X \mid d(y,x) < r \right\}.$$

For a normed space $(X, \| \ \|)$ we will denote by **B** the open ball of radius 1 centered at the origin. Then, the ball of radius r and center x can be written $x + r\mathbf{B}$.

$\mathbf{R}^m_+$ will denote the positive orthant in $\mathbf{R}^m$, i.e.

$$\mathbf{R}^m_+ := \left\{ x \in \mathbf{R}^m \mid x_j \geq 0 \;\; \forall j=1,..,m \right\}$$

and for x and y in $\mathbf{R}^m$, $<x,y>$ is the usual inner product of x and y, i.e.

$$<x,y> := \sum_{j=1}^{m} x_j y_j$$

Finally, **N** will denote the set of natural numbers.

1.1.1 Definition

A multifunction F from a set X into a set Y is a function from X into P(Y), the set of all subsets of Y (sometimes also denoted by 2^Y).

The graph of F is $\text{graph}(F) := \left\{ (x,y) \in X{\times}Y \mid y \in F(x) \right\}$. Sometimes we will write $(x,y) \in F$ to mean $(x,y) \in \text{graph}(F)$.

The domain of F is $\text{dom}(F) := \left\{ x{\in}X \mid F(x){\neq}\phi \right\}$ and the range of F is $R(F) := \bigcup_{x \in X} F(x)$. The inverse of F is the multifunction $F^{-1}(y) := \left\{ x \mid y \in F(x) \right\}$.

It is easy to see that the graph of F^{-1} is just a reorientation of that of F: $\text{graph}(F^{-1}) = \left\{ (y,x) \mid (x,y) \in \text{graph}(F) \right\}$

If Y is a topological space we can define the multifunction $\overline{F}$ by $\overline{F}(x) := \overline{F(x)}$ for each $x \in X$.

1.1.2 Definition

Let X and Y be topological spaces. We will say that a multifunction $F : X \to Y$ is upper semicontinuous (u.s.c.) at x if for any open set W containing F(x) there is a neighborhood N of x such that $F(y) \subset W$ for all $y \in N$ (or, equivalently $F(N) \subset W$). The multifunction F is u.s.c. (on X) if it is u.s.c. at every x in X.

Note that the multifunction $F : \mathbf{R} \to \mathbf{R}$ defined by $F(t) := (-|t|, +|t|)$ is not u.s.c. However, the multifunction $\overline{F}(t) = [-|t|, +|t|]$ is u.s.c.

1.1.3 Lemma

Assume (Y,d) is a metric space and let $F : X \to Y$ be a multifunction. If $\overline{F}$ is u.s.c. at x then for each $\varepsilon > 0$ there is a neighborhood N of x such that $\overline{F}(y) \subset B(F(x), \varepsilon)$ for every $y \in N$, where:

$$B(A,\varepsilon) := \left\{ z \in Y \mid d(z,A) < \varepsilon \right\} \quad \text{and}$$

$$d(z,A) := \inf \left\{ d(z,a) \mid a \in A \right\} \quad \text{for any } A \subset Y.$$

Conversely, if F(x) is relatively compact and for each $\varepsilon > 0$ there is a neighborhood N of x such that $F(y) \subset B(F(x), \varepsilon)$ for all $y \in N$, then $\overline{F}$ is u.s.c. at x.

Proof:

If $\overline{F}$ is u.s.c., for any $\varepsilon > 0$ take $W = B(F(x),\varepsilon)$. Then $\overline{F(x)} \subset W$ and there is a neighborhood N of x such that $\overline{F}(N) \subset B(F(x),\varepsilon)$.

Conversely, assume F(x) is relatively compact. If W is any open set containing $\overline{F(x)}$, then $r := \inf \left\{ d(z,W^c) \mid z \in F(x) \right\} = \min \left\{ d(z,W^c) \mid z \in \overline{F(x)} \right\}$ is positive since $d(.,W^c)$ is (Lipschitz) continuous.

Let N be a neighborhood of x such that $F(y) \subset B(F(x),r/2)$ for all $y \in N$; then $\overline{F(y)} \subset W$ for all $y \in N$.

∎

1.1.4 Definition

The multifunction $F : X \to Y$ is closed if its graph is a closed set in $X \times Y$ with the product topology. This is equivalent to saying that if $y \notin F(x)$ then there is a neighborhood N of x and a neighborhood W of y such that $(N \times W) \cap \text{graph}(F) = \phi$ or $v \notin F(u)$ for each $u \in N$ and each $v \in W$.

1.1.5 Theorem

Let X and Y be topological spaces. If $F_1 : X \to Y$ is closed and $F_2 : X \to Y$ is u.s.c. at x with $F_2(x)$ compact, then $F := F_1 \cap F_2$ is u.s.c. at x.

Proof:

See theorem VI.1.7 on page 112 in Berge [8]

■

1.1.6 Definition

The multifunction $F: X \to Y$ is lower semicontinuous (l.s.c.) at x if for every open set W such that $W \cap F(x) \neq \phi$, there is a neighborhood N of x such that $F(y) \cap W \neq \phi$ for all $y \in N$. We say F is l.s.c. on X if it is l.s.c. at every x in X.

1.1.7 Lemma

$F: X \to Y$ is l.s.c. at x if and only if $\overline{F}$ is l.s.c. at x.

Proof:

Let us note that if W is any open set in Y and A is any subset of Y then $W \cap \overline{A} \neq \phi$ if and only if $W \cap A \neq \phi$.

Assume that F is l.s.c. at x. If W is any open set in Y such that $W \cap \overline{F}(x) \neq \phi$ then $W \cap F(x) \neq \phi$. Therefore, there exists a neighborhood N of x such that $W \cap F(y) \neq \phi$ for all $y \in N$. But then $W \cap \overline{F}(y) \neq \phi$ for all $y \in N$ and $\overline{F}$ is l.s.c. at x.

The proof for the converse is similar.

■

1.1.8 Definition

Let (X,d) be a metric space and A and B two subsets of X. The Hausdorff distance between A and B, denoted by $\delta(A,B)$, is defined by

$$\delta(A,B) := \max\left\{\sup_{x\in A} d(x,B)\ ,\ \sup_{y\in B} d(y,A)\right\}$$

1.1.9 Definition

Let X be a topological space and (Y,d) a metric space. We say that the multifunction $F : X \to Y$ is continuous at $x \in X$ in the sense of Hausdorff if for each $\varepsilon > 0$ there exists a neighborhood N of x such that $\delta(F(y),F(x)) < \varepsilon$ for all $y \in N$.

It is easy to check that $\delta(\overline{A},\overline{B}) = \delta(A,B)$ for any two subsets A and B of the metric space Y. Therefore we have the following lemma:

1.1.10 Lemma

$F : X \to Y$ is continuous at $x \in X$ in the sense of Hausdorff if and only if the multifunction $\overline{F}$ is continuous at x in the sense of Hausdorff.

If (X,Δ) is also metric, the same argument used for functions shows that if the multifunction $F : X \to Y$ is continuous in the sense of Hausdorff and X is compact then F is uniformly continuous, i.e.

$$\forall\varepsilon > 0\ \exists\gamma > 0 \text{ such that } \delta(F(y),F(x)) < \varepsilon \quad \forall x,y \in X \text{ with } \Delta(x,y) < \gamma$$

1.1.11 Definition

The multifunction $F : X \to Y$ is continuous at x if it is u.s.c. and l.s.c. at x.

1.1.12 Theorem

Let X be a topological space, (Y,d) be a metric space and $F : X \to Y$ be a multifunction such that $F(x)$ is precompact. Then F is continuous at x in the sense of Hausdorff if and only if $\overline{F}$ is continuous at x.

Proof:

Assume first that F is u.s.c. and l.s.c. at $x \in X$. For $\varepsilon > 0$, let $U_1(x)$ be a neighborhood of x such that $F(y) \subset B(F(x),\varepsilon)$ for all $y \in U_1(x)$. In particular this means that $d(z,F(x)) < \varepsilon$ for every $z \in F(y)$ and every $y \in U_1(x)$.

For each $z \in F(x)$ let $U_z(x)$ be a neighborhood of x such that $F(y) \cap B(z,\varepsilon/2) \neq \phi$ for all $y \in U_z(x)$.

Since F(x) is precompact, there exist $z_1,..,z_q \in F(x)$ such that $F(x) \subset \bigcup_{i=1}^{q} B(z_i,\varepsilon/2)$.

Define $U_2(x) := \bigcap_{i=1}^{q} U_{z_i}(x)$; then $F(y) \cap B(z_i,\varepsilon/2) \neq \phi$ for all i=1, .. ,q and all $y \in U_2(x)$.

For each $z \in F(x)$ and each $y \in U_2(x)$ there exists i such that $d(z,z_i) < \varepsilon/2$. For this i, $d(z,F(y)) \leq d(z,z_i) + d(z_i,F(y))$; therefore $d(z,F(y)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Let $U(x) = U_1(x) \cap U_2(x)$; then $\delta(F(x),F(y)) < \varepsilon$ for all $y \in U(x)$ and F is continuous at x in the sense of Hausdorff.

Conversely, if F is continuous at x in the sense of Hausdorff, for each $\varepsilon > 0$ there exists a neighborhood $N(x)$ of x such that $\delta(F(x),F(y)) < \varepsilon$ for all $y \in N(x)$. Then

$$\begin{cases} d(z,F(x)) < \varepsilon & \forall z \in F(y) \quad \forall y \in N(x) \\ d(z,F(y)) < \varepsilon & \forall z \in F(x) \quad \forall y \in N(x). \end{cases}$$

The first inequality implies that $F(y) \subset B(F(x),\varepsilon)$ for all $y \in N(x)$. By lemma 1.1.3 $\overline{F}$ is u.s.c. at x.

From the second inequality we have that $B(z,\varepsilon) \cap F(y) \neq \phi$ for every $z \in F(x)$ and $y \in N(x)$. Let W be any open set such that $W \cap \overline{F}(x) \neq \phi$. Take any $z_0 \in W \cap F(x)$ and let r>0 such that $B(z_0,r) \subset W$. Taking $0 < \varepsilon \leq r$ we have that for some neighborhood $N(x)$ of x, $\phi \neq B(z_0,\varepsilon) \cap F(y) \subset W \cap F(y)$ for all $y \in N(x)$ and F is l.s.c. at x.

■

1.2 Tangent and normal cones

This section is mainly a collection of some basic definitions in convex analysis. Almost all the results are standard and can be found in, for instance, Rockafellar [20].

Lemma 1.2.11 will be helpfull in the proof of lemma 2.2.1, which is part of the proof of the existence theorem. It will also allows us to simplify assumption I in chapter 2.

1.2.1 Definition

A subset C of $\mathbf{R}^m$ is convex if $\lambda x + (1-\lambda)y \in C$ for every $\lambda \in [0,1]$ and every $x,y \in C$.

Clearly, the image of a convex set under a linear function is a convex set.

1.2.2 Definition

Let $C \subset \mathbf{R}^m$. The convex hull of C, denoted conv C, is the smallest convex set containing C, i.e. is the intersection of all convex subsets of $\mathbf{R}^m$ containing C. One can show that conv C is the set of all the convex combinations of points from C, i.e.

$$\operatorname{conv} C = \left\{ \sum_{i=1}^{k} \lambda_i x_i \mid k \in \mathbf{N},\ \lambda_i \geq 0 \ \forall i=1,..,k \text{ and } \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

1.2.3 Definition

The set $M \subset \mathbf{R}^m$ is affine if $\lambda x + (1-\lambda)y \in M$ for every $\lambda \in \mathbf{R}$ and every $x,y \in M$.

It is easy to check that M is affine if and only if there exists a subspace H of $\mathbf{R}^m$ and a point $\overline{m}$ of M such that $M = \overline{m} + H$. Also M is affine if and only if there exists a $k \times m$ matrix A and a point $a \in \mathbf{R}^k$ such that $M = \left\{ x \in \mathbf{R}^m \mid Ax=a \right\}$.

1.2.4 Definition

Let C be a subset of $\mathbf{R}^m$. The affine hull of C, denoted aff C, is the smallest affine set containing C.

It is possible to show that aff C is the set of all affine combinations of points in C, i.e.

$$\operatorname{aff} C = \left\{ \sum_{i=1}^{k} \lambda_i x_i \mid k \in \mathbf{N},\ \lambda_i \in \mathbf{R}\ \forall i=1,..,k \text{ and } \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

1.2.5 Definition

The relative interior of a subset C of $\mathbf{R}^m$ is

$$\operatorname{ri} C := \left\{ x \in \operatorname{aff} C \mid (x + \varepsilon \mathbf{B}) \cap \operatorname{aff} C \subset C \right\}.$$

C is relatively open if C = ri C; the relative boundary of C is $\overline{C} \setminus \operatorname{ri} C$. If C is convex and $\operatorname{int} C \neq \phi$ then $\operatorname{int} C = \operatorname{ri} C$.

If C is convex, the relative boundary of C will be denoted by ∂C, otherwise ∂C will denote the boundary of C.

1.2.6 Definition

A subset C of $\mathbf{R}^m$ is strictly convex if $\lambda x + (1-\lambda) y \in \operatorname{ri} C$ for every $\lambda \in (0,1)$ and every $x, y \in C$.

Two of the important multifunctions associated with a convex set are the normal and tangent cones.

1.2.7 Definition

A set $K \subset \mathbf{R}^m$ is a cone if $\lambda x \in K$ for all $\lambda > 0$ and $x \in K$.

Let S be a subset of $\mathbf{R}^m$. The cone generated by S is $\operatorname{cone} S := \bigcup_{\lambda > 0} \lambda S$. It is interesting to note that in general $\overline{\operatorname{cone} S} = \overline{\bigcup_{\lambda > 0} \lambda S} \neq \bigcup_{\lambda \geq 0} \lambda \bar{S}$ (the inclusion $\overline{\operatorname{cone} S} \supset \bigcup_{\lambda \geq 0} \lambda \bar{S}$ is always satisfied).

Take for example $S = \left\{ (x,y) \in \mathbf{R}^2 \mid y=x^2 \quad x \geq 0 \right\}$, then $\operatorname{cone} S = \operatorname{int} \mathbf{R}^2_+ \cup \{0\} = \bigcup_{\lambda \geq 0} \lambda \bar{S}$ but $\overline{\operatorname{cone} S} = \mathbf{R}^2_+$.

On the other hand, if S is relatively compact and bounded away from 0, then $\overline{\operatorname{cone} S} = \bigcup_{\lambda \geq 0} \lambda \bar{S}$.

1.2.8 Definition

Let $C \subset \mathbf{R}^m$ be convex and let $\bar{x} \in \bar{C}$. A point $p \in \mathbf{R}^m$ is normal to C at $\bar{x}$ if $\langle p, x - \bar{x} \rangle \leq 0$ for every $x \in C$. The normal cone to C at $\bar{x}$, denoted by $N_C(\bar{x})$, is the set of all points normal to C at $\bar{x}$, i.e.

$$N_C(\bar{x}) := \left\{ p \in \mathbf{R}^m \mid \langle p, x - \bar{x} \rangle \leq 0 \quad \forall x \in C \right\}$$

It is immediate that the normal cone is indeed a cone, easily shown to be closed and convex.

1.2.9 Definition

The (negative) polar cone of a nonempty cone $K \subset \mathbf{R}^m$ is the set

$$K^- := \left\{ p \in \mathbf{R}^m \mid \langle p,x \rangle \leq 0 \quad \forall x \in K \right\}.$$

It can be shown that K^- is a nonempty closed convex cone and that $K^{--} = \overline{\operatorname{conv} K}$.

1.2.10 Definition

Let $C \subset \mathbf{R}^m$ be convex and let $\bar{x} \in \bar{C}$. The tangent cone to C at $\bar{x}$, denoted by $T_C(\bar{x})$, is the polar of the normal cone to C at $\bar{x}$.

One can show that:

$$T_C(\bar{x}) = \overline{\bigcup_{\lambda \geq 0} \lambda (C - \bar{x})} \quad \text{and also that}$$

$$T_C(\bar{x}) = \left\{ h \in \mathbf{R}^m \mid d(\bar{x} + \lambda h, C) = o(\lambda),\ \lambda > 0 \right\}.$$

In particular we have that $T_C(\bar{x}) \subset \operatorname{aff} C - \bar{x}$ for all $\bar{x} \in \bar{C}$.

1.2.11 Lemma

Let M be a closed convex subset of $\mathbf{R}^m$. If $N_M(w) \subset N_0$ for all $w \in \partial M$, where N_0 is a closed convex cone, then

1) $\forall w \in M \quad T_M(w) \supset N_0^-$ and

2) $M + N_0^- = M$.

Proof:

1) is a consequence of the fact that if A and B are cones in $\mathbf{R}^m$ with $A \subset B$, then $A^- \supset B^-$.

Since $0 \in N_0^-$, $M + N_0^- \supset M$. Therefore, for 2) it is enough to show that $M + N_0^- \subset M$.

By contradiction, assume that there exists $\bar{m} \in M$ and $\bar{n} \in N_0^-$ such that $\bar{m} + \bar{n} \not\in M$.

Let m^* be the closest point to $\bar{m} + \bar{n}$ in M, and let $p := \bar{m} + \bar{n} - m^*$.

Suppose $m^* \in \operatorname{ri} M$, then there exists an $r \in (0, \|p\|)$ such that $(m^* + r\mathbf{B}) \cap \operatorname{aff} M \subset M$. Since $N_0^- \subset T_M(w) \subset \operatorname{aff} M - w$, $\bar{m} + \bar{n} \in \operatorname{aff} M$. Therefore $p \in \operatorname{aff} M - m^*$ and $m^* + r\frac{p}{\|p\|} \in M$.

But $\|\bar{m} + \bar{n} - (m^* + r\frac{p}{\|p\|})\| = \|p - r\frac{p}{\|p\|}\| = \|p\| - r < \|p\| = \|\bar{m} + \bar{n} - m^*\|$ which is a contradiction. So $m^* \in \partial M$.

We have that $\langle p, m - m^* \rangle \leq 0$ for each $m \in M$, so $p \in N_M(m^*)$. But $\langle p, \bar{n} \rangle \doteq \langle p, p \rangle - \langle p, \bar{m} - m^* \rangle \geq \|p\|^2 > 0$. This contradicts $\bar{n} \in T_M(m^*)$.

■

1.2.12 Corollary

Let M be a closed convex subset of $\mathbb{R}^m$. If $T_0 \subset T_M(w)$ for all $w \in \partial M$, where T_0 is a closed convex cone, then

1) $\forall w \in M \quad N_M(w) \subset T_0^-$ *and*

2) $M + T_0 \subset M$.

2. An exchange economy

2.1 Introduction

In this chapter we state our economic model and show the existence of a solution. The economic model consists of n consumers and a set of available commodities. We may think of it as a stock exchange market. For this reason we will use interchangeably the phrases "economic agent" and "consumer", and the phrases "commodity holding" and "consumption level". Each consumer holds a subset of the commodities and, by making transactions in a market, is able to change his holding. The market specifies prices at which commodities may be exchanged. This model differs from other approaches in two important aspects. First, it is dynamic, since the model intends to describe the behavior of the system over time, rather than to find a static equilibrium. Second, it is decentralized, since the economic agents have restricted information. At any time, each agent knows only his own current commodity holding and the current prices of all goods.

Each agent is described by his consumption set and his instantaneous demand function that, for each commodity holding and for each price vector, associates a vector that represents the desired change in the agent's commodity holding in a unit of time. The problem is then to find a function for each agent that will represent the agent's commodity holding at each moment, and a function that will represent the price vector at each moment. The variation of the commodity holding at each moment must equal the instantaneous demand function evaluated at the agent's current commodity holding and the current price vector. Also, each agent's commodity holding must remain in his consumption set and the total commodity holding must remain within the original set of available commodity.

Aubin [7] , to show the existence of a solution for such a model, requires that the instantaneous demand functions be linear in the prices. This requirement has proven unrealistic for some small scale problems we have tested. In section 2.2 we study three assumptions that allow the removal of this linearity requirement and show the existence of a solution for the economic model.

Our proof of the existence of a solution follows a different approach than the one given by Aubin [7] ; the study of the auxiliary problem (P") and the important paper by Filippov [11] are the main devices of our proof.

In section 2.3 we will compare our model to Aubin's model. We finish this section with a more detailed description of the economic model and the assumptions we make. Assumptions i) to viii) are basically the same assumptions Aubin makes, although assumptions i), iii) and vii) have been strengthened.

Let $\mathbf{R}^m$ be the commodity space and $M \subset \mathbf{R}^m$ be the set of available commodities.

Consumer i is described by his consumption set $L_i \subset \mathbf{R}^m$ and his instantaneous demand function $d_i : L_i \times S \to \mathbf{R}^m$, where

$$S := \left\{ p \in \mathbf{R}^m \mid p_j > 0 \quad j=1,..,m \quad \text{and} \quad \sum_{j=1}^{m} p_j = 1 \right\}.$$

At time t=0 consumer i starts with a level of consumption $x_i^0 \in L_i$ (the endowment). We shall try to construct n functions $x_i : [0,T] \to L_i$ i=1,..,n, where $x_i(t)$ represents the consumption of consumer i at time t, and a function $p : [0,T] \to S$ such that, at each moment, the variation of the consumption of consumer i is equal to $d_i(x_i(t),p(t))$ and the aggregate consumption level $\sum_{i=1}^{n} x_i(t)$ remains in the set of available commodities.

Let us define the linear function $A : (\mathbf{R}^m)^n \to \mathbf{R}^m$ by: $Ax := \sum_{i=1}^{n} x_i$ where $x = (x_1,..,x_n)$, $x_i \in \mathbf{R}^m$, and the set

$$K := \left\{ x=(x_1,..,x_n) \in \prod_{i=1}^{n} L_i \mid Ax \in M \right\}.$$

We will assume:

i) M is closed and strictly convex

ii) there exists $w \in M$ such that $M \cap (w + \mathbf{R}_+^m) = \{w\}$

iii) there exists $\varepsilon_0 > 0$ such that $N_M(w) \subset \bigcup_{\lambda \geq 0} \lambda \bar{S}_{\varepsilon_0}$ for all $w \in \partial M$, where:

$$S_\varepsilon := \left\{ p \in S \mid p_j > \varepsilon \quad j=1,..,m \right\} \quad \varepsilon \in [0,1/m)$$

(we actually only need $N_M(w) \subset \bigcup_{\lambda \geq 0} \lambda \bar{S}_{\varepsilon_0}$ for all $w \in \partial M \cap A(\prod_{i=1}^{n} L_i)$.)

iv) L_i is convex, closed and bounded below (i.e. there exists $\xi_i \in \mathbf{R}^m$ such that $L_i \subset \xi_i + \mathbf{R}^m_+$) for each $i=1,..,n$

v) $A(\prod_{i=1}^{n} L_i) \cap M \neq \phi$

vi) $d_i : L_i \times S \to \mathbf{R}^m$ is continuous for each $i=1,..,n$

vii) there exists $\bar{h} > 0$ such that $x_i + \bar{h}\, d_i(x_i,p) \in L_i$ for every $x_i \in L_i$ and $p \in S$

viii) $d : \prod_{i=1}^{n} L_i \times S \to (\mathbf{R}^m)^n$, defined by:

$$d(x,p) := \begin{bmatrix} d_1(x_1,p) \\ \cdot \\ d_n(x_n,p) \end{bmatrix}$$

satisfies the **Instantaneous Collective Walras Law,** i.e. $<p, \sum_{i=1}^{n} d_i(x_i,p)> \leq 0$ for every $x \in \prod_{i=1}^{n} L_i$ and every $p \in S$. It is clear that if each consumer's instantaneous demand function satisfies the **Instantaneous Walras Law** $<p, d_i(x_i,p)> \leq 0$ for every $x_i \in L_i$ and every $p \in S$, then the Instantaneous Collective Walras Law is satisfied.

The Instantaneous Walras Law is a budget constraint. It requires that the value of the commodity holding after making a transaction be no greater than the value of the commodity holding before the transaction. This law is the equivalent of the Walras Law for the Static Equilibrium Problem.

We will also require that d satisfy the assumptions I,II and III described in the following section.

Assumptions i) and iv) imply that K is convex and closed. Assumption ii) implies that $w \in \partial M$; hence, by assumption iii) we can take $q \in N_M(w) \subset \text{int}\, \mathbf{R}_+^m$.

Then $M \subset \left\{ z \in \mathbf{R}^m \mid <q,z> \leq <q,w> \right\}$ and therefore

$$K \subset \left\{ x=(x_1,..,x_n) \in (\mathbf{R}^m)^n \mid x_i \geq \xi_i \quad i=1,..,n \text{ and } <q,\sum_{i=1}^{n} x_i> \leq <q,w> \right\}.$$

Hence K is compact. Assumption v) is equivalent to saying that $K \neq \phi$.

Note:

It is possible to show that if M is closed, convex and satisfies assumption ii), and L_i is closed and bounded below for each i=1,..,n, then K is compact (see for example Aubin [5] pg. 86). So assumption iii) is not required to show that K is a compact convex set. However we will use this assumption to show, for example, the lemma 2.2.1.

Now, let us define the multifunctions

$$D(x) := \left\{ d(x,p) \mid p \in S \right\}$$

$$D_\varepsilon(x) := \left\{ d(x,p) \mid p \in \overline{S}_\varepsilon \right\} \quad \varepsilon \in [0, 1/m).$$

Since $\overline{S}_\varepsilon$ is compact and $d(x,.)$ is continuous, $D_\varepsilon(x)$ is compact for each $x \in \prod_{i=1}^{n} L_i$.

Then, the problem is to find $x : [0,T] \to (\mathbf{R}^m)^n$ absolutely continuous and $p : [0,T] \to S$, where $T>0$ is given, such that:

$$\begin{cases} \frac{dx}{dt}(t) = d(x(t),p(t)) & \text{a.e. } (0,T) \\ x(t) \in K & \forall t \in [0,T] \\ x(0) = x^0 & (x^0 \in K \text{ given}) \end{cases} \qquad \textbf{(P)}$$

where $\frac{dx}{dt}(t)$ is the derivative of x with respect to t in the sense of distributions. This problem is equivalent to finding a function $x : [0,T] \to (\mathbb{R}^m)^n$ absolutely continuous, such that:

$$\begin{cases} \frac{dx}{dt}(t) \in D(x(t)) & \text{a.e. } (0,T) \\ x(t) \in K & \forall t \in [0,T] \\ x(0) = x^0. \end{cases} \qquad (P')$$

To show the existence of a solution to problem (P') we will use the **Explicit Euler Method** and the following strategy:

1) show that $x + hD_{\varepsilon_0} \cap K \neq \phi$ for every $x \in K$ and $h \in (0,\bar{h}]$

2) show that the multifunction $D^*_{\varepsilon_0}$, defined by $D^*_{\varepsilon_0}(x) := \left\{ v \in D_{\varepsilon_0}(x) \mid x + \bar{h} v \in K \right\}$ for each $x \in K$, is continuous in the sense of the Hausdorff metric.

3) finally consider the problem of finding $x : [0,T] \to (\mathbb{R}^m)^n$ absolutely continuous such that:

$$\begin{cases} \frac{dx}{dt}(t) \in D^*_{\varepsilon_0}(x(t)) & \text{a.e. } [0,T] \\ x(0) = x^0. \end{cases} \qquad (P'')$$

Using Filippov's method [11] show that (P'') has a solution and that this is also a solution of (P').

2.2 Existence theorem

Assumption I

For $x_i \in L_i$, $p \in S$ and $h > 0$ write $z_i(x_i,p,h) := x_i + h\,d_i(x_i,p)$.

Define $\Phi(x) := \left\{ \sum_{i=1}^{n} z_i(x_i,p,\bar{h}) \mid p \in \bar{S}_{\varepsilon_0} \right\}$. We assume that:

$$x \in K \text{ and } \Phi(x) \cap M = \phi \implies \operatorname{conv}(\Phi(x)) \cap M = \phi.$$

Assumption I can be restated in the following way:

if $\sum_{i=1}^{n} z_i(x_i,p,\bar{h}) \not\in M \quad \forall p \in \bar{S}_{\varepsilon_0}$ then

$$\sum_{i=1}^{n} [\lambda z_i(x_i,p,\bar{h}) + (1-\lambda) z_i(x_i,q,\bar{h})] \not\in M \quad \forall \lambda \in (0,1) \text{ and } \forall p,q \in \bar{S}_{\varepsilon_0}.$$

2.2.1 Lemma

$x + h\,D_\varepsilon(x) \cap K \neq \phi$ for every $x \in K$, $h \in (0,\bar{h}]$ and $\varepsilon \in (0,\varepsilon_0]$.

Proof:

K is convex; hence if $x \in K$ and $v \in (\mathbf{R}^m)^n$ are such that $x + \bar{h}v \in K$, then $x + hv \in K$ for all $h \in (0,\bar{h}]$. Therefore it is enough to show that $x + \bar{h}D_\varepsilon(x) \cap K \neq \phi$ for each $x \in K$.

Also $D_\varepsilon(x) \supset D_{\varepsilon_0}(x)$ for each $\varepsilon \in (0,\varepsilon_0]$; therefore it is enough to show that $x + \bar{h}D_{\varepsilon_0}(x) \cap K \neq \phi$ for every $x \in K$.

By vii) we only need to show that:

$$\forall x \in K \;\exists p \in \bar{S}_{\varepsilon_0} \text{ such that } \sum_{i=1}^{n} z_i(x_i,p,\bar{h}) \in M.$$

By contradiction, assume $\sum_{i=1}^{n} z_i(x_i,p,\bar{h}) \not\subset M$ for all $p \in \bar{S}_{\varepsilon_0}$. Write $C := \text{conv}\,\Phi(x)$. Our assumption means that $\Phi(x) \cap M = \phi$. Thus by assumption I, $C \cap M = \phi$. Then, since C is compact, by the Hahn-Banach theorem, there exists $q \in \mathbf{R}^m$ and $\alpha \in \mathbf{R}$ such that $\langle q,y\rangle < \alpha < \langle q,c\rangle$ for all $y \in M$ and $c \in C$.

By lemma 1.2.11 $M + (\text{cone } S_{\varepsilon_0})^- = M$, i.e. $y + w \in M$ for all $y \in M$ and $w \in (\text{cone } S_{\varepsilon_0})^-$. Since $\langle q,y\rangle < \alpha$ for every $y \in M$, $\langle q,w\rangle \leq 0$ for all $w \in (\text{cone } S_{\varepsilon_0})^-$, which is equivalent to $q \in (\text{cone } S_{\varepsilon_0})^{--}$. Also $q \neq 0$, therefore $\bar{q} := q/\sum_{j=1}^{m} q_j \in \bar{S}_{\varepsilon_0}$. Then, without loss of generality, we can assume that $q \in \bar{S}_{\varepsilon_0}$.

Taking $y = \sum_{i=1}^{n} x_i$ and $p = q$ we get:

$$\langle q, \sum_{i=1}^{n} x_i\rangle < \alpha < \langle q, \sum_{i=1}^{n} z_i(x_i,q,\bar{h})\rangle .$$

But by the Instantaneous Walras Law, $\langle q, \sum_{i=1}^{n} d_i(x_i,q)\rangle \leq 0$, so :

$$\langle q, \sum_{i=1}^{n} z_i(x_i,q,\bar{h})\rangle = \langle q, \sum_{i=1}^{n} x_i\rangle + \bar{h}\langle q, \sum_{i=1}^{n} d_i(x_i,q)\rangle \leq \langle q, \sum_{i=1}^{n} x_i\rangle ,$$

which is a contradiction.

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Another consequence of Lemma 1.2.11 is that $M - \mathbf{R}_+^m \setminus \{0\} \subset \text{int}\,M$. To see this define $N_0 := \text{cone}\,\bar{S}_{\varepsilon_0} = \bigcup_{\lambda \geq 0} \lambda \bar{S}_{\varepsilon_0}$; then

$$N_0^- = \left\{ q \in \mathbf{R}^m \mid <q,p> \le 0 \quad \forall p \in \bar{S}_{\varepsilon_0} \right\}.$$

We show first that $-\mathbf{R}_+^m \setminus \{0\} \subset \operatorname{int} N_0^-$. Take any $q \in -\mathbf{R}_+^m \setminus \{0\}$ and write $\pi := \sum_{j=1}^{m} q_j$; then $\pi < 0$.

Let $\delta \in (0, -\varepsilon_0 \pi]$ and $z \in \delta \mathbf{B}$. For any $p \in N_0$ there exists $s \in \bar{S}_{\varepsilon_0}$ and $\lambda \ge 0$ such that $p = \lambda s$. Therefore

$$<q + z,p> = \lambda <q + z,s> = \lambda (<q,s> + <z,s>)$$
$$\le \lambda (\sum_{j=1}^{m} q_j \varepsilon_0 + \|z\| \|s\|) \le \lambda (\varepsilon_0 \pi + \delta) \le 0$$

so $q + z \in N_0^-$, i.e. $q + \delta \mathbf{B} \subset N_0^-$ and thus $q \in \operatorname{int} N_0^-$.

Then we have $M - \mathbf{R}_+^m \setminus \{0\} \subset M + \operatorname{int} N_0^- \subset M$. But $M + \operatorname{int} N_0^-$ is open, so $M - \mathbf{R}_+^m \setminus \{0\} \subset \operatorname{int} M$.

Assumption I is not convenient for isolating properties of the instantaneous demand function; it mixes the aggregated instantaneous demand with the set of available commodities. To overcome this drawback we can impose the following stronger assumption:

Assumption I'

$\Phi(x) + \mathbf{R}_+^m$ is a convex set for each x in K.

Then, since $\Phi(x) \subset \Phi(x) + \mathbf{R}_+^m$, $\operatorname{conv}(\Phi(x)) \subset \Phi(x) + \mathbf{R}_+^m$. Therefore $(\operatorname{conv}\Phi(x)) \cap M \subset (\Phi(x) + \mathbf{R}_+^m) \cap M$. But $\Phi(x) \cap M = \phi$ implies that $(\Phi(x) + \mathbf{R}_+^m) \cap M = \phi$ because if $\psi \in \Phi(x)$ and $z \in \mathbf{R}_+^m$ are such that $m := \psi + z \in M$, then $\psi = m - z \in M - \mathbf{R}_+^m = M$, which is a contradiction. Hence assumption I' implies assumption I.

2.2.2 Lemma

$D^{*}_{\varepsilon_0} : K \to (\mathbf{R}^m)^n$ *is an u.s.c. multifunction with compact images.*

Proof:

Since $x + \bar{h}D(x) \in \prod_{i=1}^{n} L_i$ for all $x \in \prod_{i=1}^{n} L_i$ (assumption vii)), we have:

$$D^{*}_{\varepsilon_0}(x) = \left\{ v \in D_{\varepsilon_0}(x) \mid \sum_{i=1}^{n}(x_i + \bar{h}v_i) \in M \right\} = D_{\varepsilon_0}(x) \cap A^{-1}\left[\frac{1}{\bar{h}}(M - Ax)\right].$$

Let us define the multifunction $T : (\mathbf{R}^m)^n \to (\mathbf{R}^m)^n$ by:

$$T(x) := A^{-1}\left[\frac{1}{\bar{h}}(M - Ax)\right] = \left\{ y \in (\mathbf{R}^m)^n \mid A(x + \bar{h}y) \in M \right\};$$

then

$$\mathrm{graph}(T) = \left\{ (x,y) \in (\mathbf{R}^m)^n \times (\mathbf{R}^m)^n \mid A(x + \bar{h}y) \in M \right\}.$$

First, we show that T is a closed multifunction. If $(x^k,y^k) \in \mathrm{graph}(T)$ for all k in $\mathbf{N}$ and $(x^k,y^k) \to (\bar{x},\bar{y})$ then $A(x^k + \bar{h}y^k) \in M$ for all $k \in \mathbf{N}$.

Since M is closed and A is continuous, $A(x^k + \bar{h}y^k) \to A(\bar{x} + \bar{h}\bar{y}) \in M$. Therefore $(\bar{x},\bar{y}) \in \mathrm{graph}(T)$ and graph(T) is closed.

Next we show that D_{ε_0} is u.s.c. Let $x \in \prod_{i=1}^{n} L_i$; then for each $p \in \bar{S}_{\varepsilon_0}$ and $\varepsilon > 0$, there exists $\delta(p,\varepsilon) > 0$ (δ also depends on x but x is fixed here) such that:

$$\left.\begin{array}{l} q\in \overline{S}_{\varepsilon_0},\ \|q-p\| < \delta(p,\varepsilon) \\ y\in \prod_{i=1}^{n} L_i,\ \|y-x\| < \delta(p,\varepsilon) \end{array}\right\} \Rightarrow \|d(x,p)-d(y,q)\| < \varepsilon.$$

But $\overline{S}_{\varepsilon_0}$ is compact; therefore there exist $p^1,..,p^r \in \overline{S}_{\varepsilon_0}$ such that $\overline{S}_{\varepsilon_0} \subset \bigcup_{\alpha=1}^{r} (p^\alpha+\delta(p^\alpha,\varepsilon)\mathbf{B})$.

Let $\delta^* := \min\left\{\delta(p^\alpha,\varepsilon) \mid \alpha=1,..,r\right\}$; then:

$$\left.\begin{array}{l} p\in \overline{S}_{\varepsilon_0} \text{ so } p\in p^\alpha+\delta(p^\alpha,\varepsilon)\mathbf{B} \text{ for some } \alpha \\ y\in \prod_{i=1}^{n} L_i,\ \|y-x\| < \delta^* \le \delta(p^\alpha,\varepsilon) \end{array}\right\} \Rightarrow \|d(x,p^\alpha)-d(y,p)\| < \varepsilon$$

i.e. if $y\in \prod_{i=1}^{n} L_i$ and $\|y-x\| < \delta^*$ then $D_{\varepsilon_0}(y) \subset D_{\varepsilon_0}(x) + \varepsilon\mathbf{B}$ and therefore D_{ε_0} is u.s.c. on $\prod_{i=1}^{n} L_i$.

By theorem 1.1.5 $D^*_{\varepsilon_0}$ is u.s.c.

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In order to show the lower semicontinuity of $D^*_{\varepsilon_0}$ we need two extra assumptions.

For u and v in $\mathbf{R}^m$ define $\Psi(u,v) := \bigcup_{\lambda\in[0,1]} (\lambda u + (1-\lambda) v - \mathbf{R}^m_+)$ and $[[u,v]] := \prod_{j=1}^{m} \left\{\lambda v_j + (1-\lambda) u_j \mid \lambda\in[0,1]\right\}$.

Assumption II

If $x \in \prod_{i=1}^{n} L_i$ and $p,q \in S$ are such that $d(x,p) \neq d(x,q)$ then $A\,d(x,p) \neq A\,d(x,q)$.

Assumption III

Let $x \in K$ and $p,q \in S$ be such that $A\,d(x,p) \neq A\,d(x,q)$; then for each $\varepsilon > 0$ there exists a p^{ε} in $[[p,q]]$ such that:

i) $A\,d(x,p^{\varepsilon}) \in \Psi(A\,d(x,p),A\,d(x,q))$

ii) $0 < \|A\,d(x,p) - A\,d(x,p^{\varepsilon})\| < \varepsilon$

(i.e. $A\,d(x,p^{\varepsilon}) \in \Psi(A\,d(x,p),A\,d(x,q)) \cap \overline{(A\,d(x,[[p,q]])\setminus\{A\,d(x,p)\})}$.).

2.2.3 Lemma

For each $\varepsilon > 0$ there exists a $\delta > 0$ such that:

$$\left.\begin{array}{l} \|d(x,p) - d(x,q)\| \geq \varepsilon \\ p,q \in \overline{S}_{\varepsilon_0},\ x \in \prod_{i=1}^{n} L_i \end{array}\right\} \Rightarrow \|A\,d(x,p) - A\,d(x,q)\| \geq \delta.$$

Proof:

The set Δ defined by:

$$\Delta := \left\{ (d(x,p),d(x,q)) \mid p,q \in \overline{S}_{\varepsilon_0},\ \|d(x,p) - d(x,q)\| \geq \varepsilon \right\}$$

is compact. The function from $(\mathbf{R}^m)^n \times (\mathbf{R}^m)^n$ to $\mathbf{R}$ defined by $(u,v) \to \|Au - Av\|$ is continuous and strictly positive on Δ.

■

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If $x \in \prod_{i=1}^{n} L_i$ and $p,q \in S$ are such that $d(x,p) \neq d(x,q)$ then $A\,d(x,p) \neq A\,d(x,q)$.

Assumption III

Let $x \in K$ and $p,q \in S$ be such that $A\,d(x,p) \neq A\,d(x,q)$; then for each $\varepsilon > 0$ there exists a p^{ε} in $[[p,q]]$ such that:

i) $A\,d(x,p^{\varepsilon}) \in \Psi(A\,d(x,p),A\,d(x,q))$

ii) $0 < \|A\,d(x,p) - A\,d(x,p^{\varepsilon})\| < \varepsilon$

(i.e. $A\,d(x,p^{\varepsilon}) \in \Psi(A\,d(x,p),A\,d(x,q)) \cap \overline{(A\,d(x,[[p,q]])\setminus\{A\,d(x,p)\})}$.).

2.2.3 Lemma

For each $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that:*

$$\left.\begin{array}{l} \|d(x,p) - d(x,q)\| \geq \varepsilon \\ p,q \in \overline{S}_{\varepsilon_0},\ x \in \prod_{i=1}^{n} L_i \end{array}\right\} \Rightarrow \|A\,d(x,p) - A\,d(x,q)\| \geq \delta.$$

Proof:

The set Δ defined by:

$$\Delta := \left\{ (d(x,p),d(x,q)) \mid p,q \in \overline{S}_{\varepsilon_0},\ \|d(x,p) - d(x,q)\| \geq \varepsilon \right\}$$

is compact. The function from $(\mathbf{R}^m)^n \times (\mathbf{R}^m)^n$ to $\mathbf{R}$ defined by $(u,v) \to \|Au - Av\|$ is continuous and strictly positive on Δ.

■

2.2.4 Lemma

Let $M \subset \mathbb{R}^m$ be strictly convex and such that $M - \mathbb{R}^m_+ \setminus \{0\} \subset int\, M$. If $u, v \in M$ then $\Psi(u,v) \setminus \{u,v\} \subset int\, M$.

Proof:

If $w \leq \lambda u + (1-\lambda) v$ for some $\lambda \in (0,1)$, then, since $\lambda u + (1-\lambda) v \in \operatorname{int} M$, $\lambda u + (1-\lambda) v - \mathbb{R}^m_+ \subset \operatorname{int} M$ and thus $w \in \operatorname{int} M$.

If $w \leq u$ and $w \neq u$ (or $w \leq v$ and $w \neq v$) then $w \in \operatorname{int} M$.

■

2.2.5 Lemma

$D^{\circ}_{\varepsilon_0} : K \to (\mathbb{R}^m)^n$ is l.s.c.

Proof:

For convenience, throughout this proof we will give $\mathbb{R}^m$ any of its (standard) equivalent norms, say $\| \; \|_2$. Then we define a norm on $(\mathbb{R}^m)^n$ by:

$$\|x\| := \sum_{i=1}^{n} \|x_i\|_2 .$$

We will denote by $\mathbf{B}$ the unit open ball of either $\mathbb{R}^m$ or $(\mathbb{R}^m)^n$.

Let W be any open set such that $W \cap D^{\circ}_{\varepsilon_0}(x) \neq \phi$. We will divide the proof into 2 parts. In the first case we show that if there exists $v \in W \cap D^{\circ}_{\varepsilon_0}(x)$ such that $\sum_{i=1}^{n} (x_i + \bar{h} v_i) \in \operatorname{int} M$ then $W \cap D^{\circ}_{\varepsilon_0}(y) \neq \phi$ for all y sufficiently close to x, so that $D^*_{\varepsilon_0}$ is l.s.c. at x.

In the second case we prove by contradiction that either there is a $v^0 \in W \cap D^*_{\varepsilon_0}(x)$ such that $\sum_{i=1}^{n}(x_i+\bar{h}v_i^0) \in \text{int}\, M$ or $D^*_{\varepsilon_0}$ is l.s.c. at x. In either case $D^*_{\varepsilon_0}$ is l.s.c. at x.

1) If $v \in W \cap D^*_{\varepsilon_0}(x)$ is such that $\sum_{i=1}^{n}(x_i+\bar{h}v_i) \in \text{int}\, M$, then, for all $i=1,..,n$, $v_i = d_i(x_i,p)$ for some $p \in \bar{S}_{\varepsilon_0}$ and there exists $\varepsilon > 0$ such that $v + \frac{\varepsilon}{2\bar{h}}\mathbf{B} \subset W$ and $\sum_{i=1}^{n}(x_i + \bar{h}v_i) + \varepsilon\mathbf{B} \subset M$.

Let $\delta \in (0, \frac{\varepsilon}{2n})$ be such that if $\|x_i - y_i\|_2 < \delta$ then $\bar{h}\|d_i(x_i,p) - d_i(y_i,p)\|_2 < \frac{\varepsilon}{2n}$ for all $i=1,..,n$. Then for any y in $\prod_{i=1}^{n} L_i$ with $\|y - x\| < \delta$

$$\|\sum_{i=1}^{n}(y_i + \bar{h}d_i(y_i,p)) - \sum_{i=1}^{n}(x_i + \bar{h}d_i(x_i,p))\|_2 \leq$$

$$\sum_{i=1}^{n}\|y_i - x_i\|_2 + \bar{h}\sum_{i=1}^{n}\|d_i(y_i,p) - d_i(x_i,p)\|_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $d(y,p) \in D^*_{\varepsilon_0}(y)$. Part of the inequality above also shows that $\|d(y,p) - v\| < \frac{\varepsilon}{2\bar{h}}$; thus $d(y,p) \in W \cap D^*_{\varepsilon_0}(y)$.

2) Assume $D^*_{\varepsilon_0}$ is not l.s.c. at x. Let $d(x,p^0) \in W \cap D^*_{\varepsilon_0}(x)$ and $\varepsilon > 0$ be such that $d(x,p^0) + \varepsilon\mathbf{B} \subset W$, and there is a sequence $\{y^k\} \subset K$ converging to x such that $D^*_{\varepsilon_0}(y^k) \cap (d(x,p^0) + \varepsilon\mathbf{B}) = \phi$.

But by lemma 2.2.1, $D^*_{\varepsilon_0}(y^k) \neq \phi$, thus there exists a $p^k \in \overline{S}_{\varepsilon_0}$ such that:

$$\begin{cases} A(y^k + \overline{h}\, d(y^k, p^k)) \in M \quad \text{and} \\ \| d(y^k, p^k) - d(x, p^0) \| \geq \varepsilon. \end{cases}$$

$\overline{S}_{\varepsilon_0}$ is compact; therefore there exists a subsequence $\left\{p^{k_\alpha}\right\}$ and a point $q \in \overline{S}_{\varepsilon_0}$ such that $p^{k_\alpha} \to q$.

By continuity $d(y^{k_\alpha}, p^{k_\alpha}) \to d(x,q)$, $\| d(x,q) - d(x,p^0) \| \geq \varepsilon$ and $A(x + \overline{h}\, d(x,q)) \in M$ (since M is closed). Then there is a $\delta > 0$ such that $\| A\, d(x,q) - A\, d(x,p^0) \| \geq \delta$.

By assumption III there exists a $p^\varepsilon \in [[p^0, q]] \subset \overline{S}_{\varepsilon_0}$ such that $A\, d(x,p^\varepsilon) \in (A\, d(x,p^0) + \varepsilon \mathbf{B}) \cap \Psi(A\, d(x,q), A\, d(x,p^0))$. Since $A(x + \overline{h}\, d(x,p^0))$ and $A(x + \overline{h}\, d(x,q))$ are in M, we can use lemma 2.2.4 to show that $A(x + \overline{h}\, d(x,p^\varepsilon)) \in \operatorname{int} M$. Therefore $d(x,p^\varepsilon) \in (d(x,p^0) + \varepsilon \mathbf{B}) \cap D^*_{\varepsilon_0}(x) \subset W \cap D^*_{\varepsilon_0}(x)$.

Thus we can use part 1) to show that $D^*_{\varepsilon_0}$ is l.s.c. at x, which is a contradiction.

■

2.2.6 Corollary

$D^*_{\varepsilon_0} : K \to (\mathbb{R}^m)^n$ is continuous in the sense of Hausdorff with $D^*_{\varepsilon_0}(x)$ compact for each $x \in K$.

Proof:

Use lemma 1.1.12.

∎

Next, consider the problem of finding an absolutely continuous function $x:[0,T] \to (\mathbf{R}^m)^n$ such that:

$$\begin{cases} \dfrac{dx}{dt}(t) \in D^{*}_{\varepsilon_0}(x(t)) \ \text{ a.e. } (0,T) \\ x(0) = x^0 . \end{cases} \qquad (P'')$$

Let $\mu := \max\left\{ \|d\| \mid d \in D^{*}_{\varepsilon_0}(x) \ , \ x \in K \right\}$ and let $\left\{ h_k \right\} \subset \mathbf{R}_+$ be any sequence such that:

$$\begin{cases} h_1 \in (0,\bar{h}] \ \text{ with } \ T/h_1 \in \mathbf{N} \\ h_k/h_{k+1} \in \mathbf{N} \quad \forall k \in \mathbf{N} \\ x,y \in K \quad \|x-y\| \le \mu h_k \ \Rightarrow \ \delta(D^{*}_{\varepsilon_0}(x), D^{*}_{\varepsilon_0}(y)) < 2^{-k}. \end{cases}$$

Such a sequence exists because K is compact and therefore $D^{\circ}_{\varepsilon_0}$ is uniformly continuous on K. Let $t^k_a := a h_k \quad a=0,1,..,T/h_k$, and define the following functions:

$$\kappa(a,k) := \min\left\{ q \mid t^q_b = t^k_a \ \text{ for some } \ b = 0,1,..,T/h_q \right\} - 1$$

$$\alpha(a,k) := \max\left\{ b = 0,1,..,T/h_{\kappa(a,k)} \mid t^{\kappa(a,k)}_b < t^k_a \right\}.$$

It is clear that for each $q \in \mathbf{N}$ with $\kappa(a,k) < q \le k$ there exists a $b \in \mathbf{N}$ such that $t^q_b = t^k_a$.

For each k, the points $\left\{t_a^k\right\}_{a=0}^{T/h_k}$ define a partition of [0,T). This k-th partition is composed of the T/h_k intervals of the form $[t_{a-1}^k, t_a^k)$ $a=1,..,T/h_k$. For each $q>k$, the q-th partition is a refinement of the k-th partition.

From now on, to simplify the notation, we will use κ for $\kappa(a,k)$ and α for $\alpha(a,k)$.

Then κ is the index of the last partition that does not contain t_a^k as an endpoint of one of its intervals, and t_α^κ is the left endpoint of the interval of this partition containing t_a^k (i.e. $t_\alpha^\kappa < t_a^k < t_{\alpha+1}^\kappa$).

Using Filippov's method [11] construct a sequence of approximate solutions $\{x^k\}$. x^k is a piecewise linear function on [0,T] such that:

$$\begin{cases} x^k(0) = x^0 \\ \dfrac{dx^k}{dt}(t) = d_a^k \quad \forall t \in (t_a^k, t_{a+1}^k), \text{ where } \quad d_a^k \in D_{\varepsilon_0}^*(x^k(t_a^k)), \quad \forall a = 0,1,..,T/h_k - 1 \end{cases}$$

i.e. x^k is affine on each interval of the k-th partition.

Assume we have constructed x^k on the interval $[0,t_a^k]$, where $a < T/h_k$. Then we define $x^k(t)$ on $[t_a^k, t_{a+1}^k]$ by the following two steps:

step 1: if $\kappa = 0$ then let d_a^k be any element of $D_{\varepsilon_0}^*(x^k(t_a^k))$. Otherwise, let $d_a^k \in D_{\varepsilon_0}^*(x^k(t_a^k))$ be such that: $\|d_a^k - d_\alpha^\kappa\| \le 2^{-k}$.

Actually, we have to show that step 1 can be carried out. First we note that x^k has already been defined on $[0,t_a^k]$ which contains the interval $[t_\alpha^\kappa, t_\alpha^\kappa + h_k]$.

Since $\|d\| \le \mu$ for all $x \in K$ and $d \in D^{*}_{\varepsilon_0}(x)$,

$$\|x^k(t^k_a) - x^k(t^{\kappa}_{\alpha})\| \le \mu(t^k_a - t^{\kappa}_{\alpha}) < \mu h_k.$$

Therefore $\delta(D^{*}_{\varepsilon_0}(x^k(t^k_a)), D^{*}_{\varepsilon_0}(x^k(t^{\kappa}_{\alpha}))) < 2^{-k}$, and thus, there exists $d^k_a \in D^{*}_{\varepsilon_0}(x^k(t^k_a))$ such that $\| d^k_a - d^{\kappa}_{\alpha}\| \le 2^{-k}$.

step 2: define $x^k(t) := x^k(t^k_a) + (t - t^k_a)\, d^k_a \quad t \in [t^k_a, t^k_{a+1}]$.

Filippov then shows that there exists a subsequence of $\{x^k\}$ converging uniformly to a solution x of P''. This solution has bounded and continuous derivatives everywhere on the interval [0,T] except at a countable set of points at which it has discontinuities of the first kind.

But given $x^k(t^k_a) \in K$, by definition of $D^{*}_{\varepsilon_0}$, $x^k(t^k_a) + \bar{h} D^{*}_{\varepsilon_0}(x^k(t^k_a)) \subset K$. Therefore, since $h_k \in (0,\bar{h}]$ and K is convex, $x^k(t^k_a) + h_k D^{*}_{\varepsilon_0}(x^k(t^{a^k})) \subset K$ and in particular $x^k(t^k_{a+1}) := x^k(t^k_a) + h_k\, d^k_a \in K$.

We have then that $x^k(t^k_a) \in K$ for $a = 0,1,..,T/h_k$. Also, for $t \in (t^k_a, t^k_{a+1})$:

$$\begin{aligned} x^k(t) &= x^k(t^k_a) + \frac{x^k(t^k_{a+1}) - x^k(t^k_a)}{h_k}(t - t^k_a) \\ &= \left[\frac{t^k_{a+1} - t}{t^k_{a+1} - t^k_a}\right] x^k(t^k_a) + \left[\frac{t - t^k_a}{t^k_{a+1} - t^k_a}\right] x^k(t^k_{a+1}) \\ &= \lambda_t x^k(t^k_a) + (1-\lambda_t)\, x^k(t^k_{a+1}) \quad \text{where} \quad \lambda_t := \left[\frac{t^k_{a+1} - t}{t^k_{a+1} - t^k_a}\right] \in (0,1). \end{aligned}$$

Therefore $x^k(t) \in K$ for each $t \in [0,T]$ and $k \in \mathbf{N}$. We conclude that $x(t) \in K$ for each $t \in [0,T]$, and thus x is in fact a solution of P'.

Finally, having defined a solution in the interval $[0,T]$, we can solve the problem again with initial condition $x(T)$ to get a solution on $[T,2T]$. This solution together with the solution on $[0,T]$ forms a solution on the interval $[0,2T]$. This process can be repeated again and again to define a solution on $[0,\infty)$.

2.3 Embedding Aubin's model in the model presented here

In the economic model developed by Aubin [7] the only instantaneous demand functions considered are those of the form:

$$d_i(x_i,p) = A^i(x_i)\,p + b^i(x_i) \quad i=1,..,n$$

where $A^i : L_i \to \mathbf{R}^{m\times m}$ and $b^i : L_i \to \mathbf{R}^m$ are continuous.

In fact, for this example, there is no loss of generality if we only consider instantaneous demand functions of the form:

$$d_i(x_i,p) = B^i(x_i)\,p \quad i=1,..,n$$

where $B^i : L_i \to \mathbf{R}^{m\times m}$ is continuous, because we can define:

$$B^i(x_i) := A^i(x_i) + [b^i(x_i),..,b^i(x_i)]$$

and

$$B^i(x_i)\,p = A^i(x_i)\,p + \sum_{j=1}^{m} p_j\, b^i(x_i)$$

$$= A^i(x_i)\,p + \Big(\sum_{j=1}^{m} p_j\Big)\, b^i(x_i) = A^i(x_i)\,p + b^i(x_i)$$

for all $p \in S$.

It is very simple to check that these demand functions satisfy assumption I'.

For assumption III, assume that $x \in K$ and $p,q \in S$ are such that $Ad(x,p) = (\sum_{i=1}^{n} B^i(x_i))\, p \neq (\sum_{i=1}^{n} B^i(x_i))\, q = Ad(x,q)$ and let $\varepsilon > 0$ be given. Take any $\lambda \in \left]0, \frac{\varepsilon}{\sqrt{m}\,\|\sum_{i=1}^{n} B^i(x_i)\|}\right[$ and define $p^{\varepsilon} := \lambda q + (1-\lambda) p$.

Then

$$Ad(x,p^{\varepsilon}) = \sum_{i=1}^{n} B^i(x_i)\, p^{\varepsilon} = \lambda \sum_{i=1}^{n} B^i(x_i)\, q + (1-\lambda) \sum_{i=1}^{n} B^i(x_i)\, p$$

$$= \lambda Ad(x,q) + (1-\lambda) Ad(x,p) \in \Psi(Ad(x,p), Ad(x,q))$$

(in particular this can be used to show that assumption I' is satisfied.).

Also, since $\lambda \neq 0$ we have

$$Ad(x,p^{\varepsilon}) = \lambda Ad(x,q) + (1-\lambda) Ad(x,p) \neq Ad(x,p).$$

Finally

$$\|Ad(x,p) - Ad(x,p^{\varepsilon})\| = \lambda \|Ad(x,p) - Ad(x,q)\|$$

$$\leq \lambda \|\sum_{i=1}^{n} B^i(x_i)\|\,\|p-q\| \leq \lambda \|\sum_{i=1}^{n} B^i(x_i)\| \sqrt{m} < \varepsilon.$$

However, assumption II is not found in Aubin's model. He only requires that the instantaneous demand functions satisfy the instantaneous collective Walras law:

$$<p, \sum_{i=1}^{n} d_i(x_i,p)> = <p, \sum_{i=1}^{n} B^i(x_i)\, p> \leq 0 \quad \forall p \in S \;\; \forall x \in \prod_{i=1}^{n} L_i.$$

This is equivalent to saying that $-\sum_{i=1}^{n} B^i(x_i)$ is copositive (Cottle, Habetler and Lemke [10]) for each $x \in \prod_{i=1}^{n} L_i$.

If instead we require that each consumer's demand function satisfy the instantaneous Walras law

$$<p,d_i(x_i,p)> = <p,B^i(x_i)\,p> \leq 0 \quad \forall x_i \in L_i \quad \forall p \in S$$

we have that $-B^i(x_i)$ is copositive for each $x_i \in L_i$.

It is clear that a positive semidefinite matrix is copositive. We will show that if we require $-B^i(x_i)$ to be positive semidefinite for each $x_i \in L_i$ and each i=1,..,n, then assumption II is equivalent to assumption II' stated below.

The following lemma is a well known result and we omit its proof.

2.3.1 Lemma

Let $B \in \mathbb{R}^{m \times m}$ be any symmetric positive semidefinite matrix. Then $<x,Bx> = 0$ if and only if $Bx = 0$.

2.3.2 Corollary

If $B^1, B^2 \in \mathbb{R}^{m \times m}$ are two symmetric,positive semidefinite matrices $ker(B^1+B^2) = ker(B^1) \cap ker(B^2)$.

Proof:

One implication is trivial. Let us show that $\ker(B^1+B^2) \subset \ker(B^i)$ $i=1,2$.

If $x \in \ker(B^1+B^2)$ then $\langle x,(B^1+B^2)x\rangle = 0$. But $\langle y,B^i y\rangle \geq 0$ for all $y \in \mathbf{R}^m$, therefore $\langle x,B^i x\rangle = 0$ $i=1,2$.

■

2.3.3 Lemma

Let $B \in \mathbf{R}^{m\times m}$ be any positive semidefinite matrix (not necessarily symmetric). Then $ker(B) = ker(S_B) + ker(K_B)$, where $S_B := ½(B + B^t)$, $K_B := ½(B - B^t)$ and B^t denotes the transpose of B.

Proof:

It is an immediate consequence of theorem 3 in Robinson [19].

■

It is easy to check that if $B^1,B^2 \in \mathbf{R}^{m\times m}$ are two matrices $S_{B^1+B^2} = S_{B^1} + S_{B^2}$ and $K_{B^1+B^2} = K_{B^1} + K_{B^2}$. Also $\langle x,K_B x\rangle = 0$ for any $x \in \mathbf{R}^m$ and any matrix $B \in \mathbf{R}^{m\times m}$.

Assumption II'

Define $H := \left\{ z \in \mathbf{R}^m \mid \sum_{j=1}^{m} z_j = 0 \right\}$. For $x \in \prod_{i=1}^{n} L_i$ let us write $S^i(x_i) = S_{B^i(x_i)}$ and $K^i(x_i) = K_{B^i(x_i)}$.

We will assume that for each $i_0 = 1,..,n$ and each $x \in \prod_{i=1}^{n} L_i$

$$H \cap [\bigcap_{i=1}^{n} \ker(S^i(x_i)) \cap \ker(\sum_{i=1}^{n} K^i(x_i))] \setminus [\ker(S^{i_0}(x_{i_0})) \cap \ker(K^{i_0}(x_{i_0}))] = \phi .$$

This is equivalent to saying that for each $x \in \prod_{i=1}^{n} L_i$

$$H \cap [\bigcap_{i=1}^{n} \ker(S^i(x_i)) \cap \ker(\sum_{i=1}^{n} K^i(x_i))] \setminus \bigcap_{i=1}^{n} [\ker(S^i(x_i)) \cap \ker(K^i(x_i))] = \phi .$$

2.3.4 Lemma

If $B^i(x_i)$ is negative semidefinite for each $x_i \in L_i$ and each $i = 1,..,n$, assumption II is equivalent to assumption II'.

Proof:

We will show that for $x \in \prod_{i=1}^{n} L_i$

1) there exists $p,q \in S$ such that $d(x,p) \neq d(x,q)$ and $Ad(x,p) = Ad(x,q)$

2) $H \cap [\bigcap_{i=1}^{n} \ker(S^i(x_i)) \cap \ker(\sum_{i=1}^{n} K^i(x_i))] \setminus \bigcap_{i=1}^{n} [\ker(S^i(x_i)) \cap \ker(K^i(x_i))] \neq \phi$

are equivalent.

Let $p,q \in S$ satisfy 1) at some $x \in \prod_{i=1}^{n} L_i$. Then, for some $i_0 \in \{1,..,n\}$ $B^{i_0}(x_{i_0})\, p \neq B^{i_0}(x_{i_0})\, q$ and $\sum_{i=1}^{n} B^i(x_i)\, p = \sum_{i=1}^{n} B^i(x_i)\, q$.

Write $C := \sum_{i \neq i_0} B^i(x_i)$, $D := B^{i_0}(x_{i_0})$ and $u := p-q$. Then $(C+D)u = 0$, $Du \neq 0$ and $u \in H$, i.e

$$u \in H \cap \ker(C+D) \setminus \ker(D) =$$
$$H \cap [\ker(S_{C+D}) \cap \ker(K_{C+D})] \setminus [\ker(S_D) \cap \ker(K_D)] =$$
$$H \cap [\ker(S_C) \cap \ker(S_D) \cap \ker(K_C+K_D)] \setminus [\ker(S_D) \cap \ker(K_D)]$$

therefore

$$u \in H \cap [\bigcap_{i=1}^{n} \ker(S^i(x_i) \cap \ker(\sum_{i=1}^{n} K^i(x_i))] \setminus [\ker(S^{i_0}(x_i)) \cap \ker(K^{i_0}(x_i))].$$

Conversely, it is easy to see that the argument can be made backwards because, for any $u \in H \setminus \{0\}$ there exist $p, q \in S$ and $\lambda > 0$ such that $u = \lambda(p-q)$. In fact, define $P := \left\{ j \mid u_j > 0 \right\}$ and $N := \left\{ j \mid u_j < 0 \right\}$. Then we can take

$$\lambda = \sum_{j \in P} u_j = -\sum_{j \in N} u_j \neq 0$$

$$p_j = \begin{cases} \frac{u_j}{\lambda} & \text{if } j \in P \\ 0 & \text{otherwise} \end{cases}$$

$$q_j = \begin{cases} \frac{-u_j}{\lambda} & \text{if } j \in N \\ 0 & \text{otherwise.} \end{cases}$$

■

One can show that

$$H \cap [\bigcap_{i=1}^{n} \ker(S^i(x_i)) \cap \ker(\sum_{i=1}^{n} K^i(x_i))] \setminus \bigcap_{i=1}^{n} [\ker(S^i(x_i)) \cap \ker(K^i(x_i))] =$$

$$H \cap [\bigcap_{i=1}^{n} \ker(S^i(x_i))] \cap [\ker(\sum_{i=1}^{n} K^i(x_i)) \setminus \bigcap_{i=1}^{n} \ker(K^i(x_i))].$$

We expect that if the number of consumers increases and their matrices $B^i(x_i)$ are not related, then $H \cap [\bigcap_{i=1}^{n} \ker(S^i(x_i))]$ has more and more chances of becoming empty. Therefore, we hope that assumption II' will be satisfied when the number of consumers in the economy is large.

3. Examples of instantaneous demand functions

3.1 Introduction

In this chapter we study a family of instantaneous demand functions that satisfy most of the conditions imposed in chapter 2. We construct these instantaneous demand functions by solving an optimization problem. In section 2.2 we give some small-scale examples obtained from the classical Cobb-Douglas and Constant Elasticity of Substitution utility functions. In section 3.3 we study continuity and differentiability properties of these instantaneous demand functions. Finally, in section 3.4, we study the solutions of the economic model as time goes to infinity in the case that each consumer has an instantaneous demand function in this family.

Let $L \subset \mathbf{R}^m$ denote the consumption set for a consumer. We assume L is closed, convex and bounded below.

In optimization there are two properties closely related to concavity (or convexity):

3.1.1 Definition

We say that $u : L \to \mathbf{R}$ is strictly quasiconcave on L if

$$\left.\begin{array}{l} x^1, x^2 \in L,\ x^1 \neq x^2 \\ u(x^2) \geq u(x^1) \end{array}\right\} \Rightarrow u(\lambda x^2 + (1-\lambda) x^1) > u(x^1) \qquad \forall \lambda \in (0,1).$$

3.1.2 Definition

Assume $u : L \to \mathbf{R}$ is differentiable at $\bar{x}$, then u is pseudoconcave at $\bar{x}$ with respect to L if

$$\left.\begin{array}{l} x \in L \\ \nabla u(\bar{x})\,(x-\bar{x}) \leq 0 \end{array}\right\} \Rightarrow u(x) \leq u(\bar{x}).$$

u is pseudoconcave on L if it is pseudoconcave at each $\bar{x} \in L$.

Suppose the consumer has a utility function $u : L \to \mathbf{R}$ to represent his preferences in L, which is continuous and strictly quasiconcave.

Given an endowment $x \in L$ and a price vector $p \in S$ consider the following problem :

$$\max u(x+\delta)$$
$$\text{s.t. } \langle p,\delta \rangle \leq 0$$
$$x+\delta \in L\,.$$

If δ^* denotes the optimum for this problem (the existance and uniqueness of the solution are guaranteed by theorem 3.3.5 below), we will assume that the consumer has an instantaneous demand function proportional to δ^*, i.e. there exists $\alpha > 0$ (independent of x and p) such that $d(x,p) = \alpha\,\delta^*$.

Note that $x + h\,d(x,p) \in L$ for all $h \in [0, 1/\alpha]$ and therefore assumption vii) is satisfied. Also, this instantaneous demand function satisfies the Instantaneous Walras Law.

3.2 Examples

In the following three examples we will take $L = \mathbf{R}^2_+$.

1) The **Cobb - Douglas** utility function (Varian [24]) is $u(x_1,x_2) = x_1^a x_2^{1-a}$ where $a \in (0,1)$. Consider then the problem:

$$\max \ (x_1+\delta_1)^a (x_2+\delta_2)^{1-a}$$
$$\text{s.t.} \ \ p_1\delta_1+p_2\delta_2 \leq 0$$
$$(x_1+\delta_1, x_2+\delta_2) \in \mathbf{R}^2_+ .$$

The utility function u can be replaced by any equivalent utility function of the form $\psi \circ u$ where $\psi : \mathbf{R} \to \mathbf{R}$ is strictly increasing. In particular if we take $\psi = \log$ we can write the last problem:

$$\max \ \{a \log(x_1+\delta_1) + (1-a) \log(x_2+\delta_2)\}$$
$$\text{s.t.} \ \ p_1\delta_1+p_2\delta_2 \leq 0$$
$$(x_1+\delta_1, x_2+\delta_2) \in \mathbf{R}^2_+ .$$

The solution to this problem is:

$$\delta_1^* = \frac{a\, p_2\, x_2-(1-a)\, p_1\, x_1}{p_1} \qquad \delta_2^* = \frac{(1-a)\, p_1\, x_1-a\, p_2\, x_2}{p_2}.$$

An important special case is when $a = \frac{1}{2}$. In that case,

$$\delta_1^* = \frac{p_2\, x_2-p_1\, x_1}{2\, p_1} \qquad \delta_2^* = \frac{p_1\, x_1-p_2\, x_2}{2\, p_2}.$$

2) The C.E.S. (Constant Elasticity of Substitution) utility function (Varian [24]) is $u(x_1,x_2) := (x_1^\rho+x_2^\rho)^{\frac{1}{\rho}} \quad \rho \neq 0$.

Again here we modify the utility function by applying the function $\psi(\xi) := \xi^{\rho}$ if $\rho > 0$ or $\psi(\xi) := -\xi^{\rho}$ if $\rho < 0$. The solution we get for the maximization problem is in this case:

$$\delta_1^* = \frac{p_2x_2-(p_1p_2^{-\rho})^{\frac{1}{1-\rho}}x_1}{p_1+(p_1p_2^{-\rho})^{\frac{1}{1-\rho}}} \qquad \delta_2^* = \frac{p_1x_1-(p_1^{-\rho}p_2)^{\frac{1}{1-\rho}}x_2}{p_2+(p_1^{-\rho}p_2)^{\frac{1}{1-\rho}}}.$$

An important special case here is when $\rho = -1$. Then $u(x_1,x_2) = -(\frac{1}{x_1}+\frac{1}{x_2})$ and:

$$\delta_1^* = \frac{p_2x_2-\sqrt{p_1p_2}x_1}{p_1+\sqrt{p_1p_2}} \qquad \delta_2^* = \frac{p_1x_1-\sqrt{p_1p_2}x_2}{p_2+\sqrt{p_1p_2}}.$$

One might think that all the instantaneous demand functions built in this way are linear in x. But this is not so.

3) Consider the utility function $u(x_1,x_2) := \log\left(\frac{x_1x_2}{1+x_1}\right)$ This utility function is concave on $\mathbf{R}_+^2$.

The solution of the maximization problem is in this case:

$$\delta_1^* := -(x_1+1) + \sqrt{\frac{p_1x_1+p_2x_2}{p_1}} \quad , \quad \delta_2^* := -\frac{p_1\delta_1^*}{p_2}.$$

3.3 Continuity and differentiability of the demand functions

3.3.1 Definition

We say that $u : L \to \mathbf{R}$ satisfies local nonsatiation if for every $\varepsilon>0$ and every $x \in L$, there exists a $y \in x+\varepsilon\mathbf{B} \cap L$ such that $u(y) > u(x)$.

3.3.2 Theorem

Let $L \subset \mathbf{R}^m$ be closed, convex and bounded below. Consider the set $F(x,p) := \left\{ \delta \in \mathbf{R}^m \mid x+\delta \in L,\ <p,\delta> \leq 0 \right\}$ *as a multifunction from* $L \times int\, \mathbf{R}^m_+$ *into* $\mathbf{R}^m$. *Assume that* $x^0 \in L$ *and* $p^0 \in int\, \mathbf{R}^m_+$ *satisfy:*

$$H(p^0) \cap (int\, L - x^0) \neq \phi \quad where \tag{1}$$

$$H(p^0) := \left\{ z \in \mathbf{R}^m \mid <p^0,z> \leq 0 \right\}.$$

Then F is l.s.c. at (x^0,p^0).

Note:

If $x^0 \in \text{int } L$, $0 \in H(p^0) \cap (\text{int } L - x^0)$. Thus, in particular, F is l.s.c. on $\text{int } L \times \text{int } \mathbf{R}^m_+$.

Proof:

Let $Q \subset \mathbf{R}^m$ be any open set such that $Q \cap F(x^0,p^0) \neq \phi$. We need to construct $N(x^0)$ and $M(p^0)$, neighborhoods of x^0 and p^0, such that $Q \cap F(x,p) \neq \phi$ for every $x \in N(x^0)$ and $p \in M(p^0)$.

Let $\delta_0 \in Q \cap F(x^0,p^0)$ and $\eta > 0$ be such that $\delta_0+\eta\mathbf{B} \subset Q$. Let $g \in H(p^0) \cap (\text{int}\, L-x^0)$ and write $(\delta_0,g] := \left\{(1-\lambda)\,\delta_0+\lambda g \mid \lambda\in(0,1]\right\}$. Then $k \in H(p^0) \cap (\text{int}\, L-x^0)$ for every $k \in (\delta_0,g]$, because $\delta_0 \in F(x^0,p^0) = H(p^0) \cap (L-x^0)$ and $H(p^0)$ and $(L-x^0)$ are convex sets.

Pick $k_0 \in \frac{\eta}{2}\mathbf{B}$ such that $\delta_0+k_0 \in (\delta_0,g]$. Then there exists an $\varepsilon \in (0,\eta/2]$ such that $\delta_0+k_0+2\varepsilon\,\mathbf{B} \subset (\text{int}\, L-x^0)$. Therefore $\delta_0+k_0+\varepsilon\,\mathbf{B} \subset (\text{int}\, L-x)$ for each $x \in x^0+\varepsilon\,\mathbf{B} =: N(x^0)$.

Let $\bar\gamma \in \left(0, \frac{\|p^0\|}{2}\right)$ be such that $p^0+\bar\gamma\,\mathbf{B} \subset \text{int}\, \mathbf{R}^m_+$. Since the function from $\mathbf{R}^m$ to $\mathbf{R}$ defined by $p \to \langle\delta_0+k_0,p\rangle$ is continuous and nonpositive at p^0, there exists $\gamma \in (0,\bar\gamma]$ such that:

$$\langle\delta_0+k_0,p\rangle < \frac{\varepsilon\|p^0\|}{2} \quad \forall p \in p^0+\gamma\mathbf{B} =: M(p^0).$$

For $p \in M(p^0)$ define:

$$k(p) := \begin{cases} k_0 & \text{if} \quad \langle\delta_0+k_0,p\rangle \le 0 \\ k_0 - \dfrac{\langle\delta_0+k_0,p\rangle}{\|p\|^2}p & \text{otherwise.} \end{cases}$$

Then

$$\langle\delta_0+k(p),p\rangle = \begin{cases} \langle\delta_0+k_0,p\rangle & \text{if} \quad \langle\delta_0+k_0,p\rangle \le 0 \\ \langle\delta_0+k_0,p\rangle - \dfrac{\langle\delta_0+k_0,p\rangle}{\|p\|^2}\langle p,p\rangle = 0 & \text{otherwise} \end{cases}$$

and

$$\|k(p)-k_0\| = \begin{cases} 0 & \text{if} \quad <\delta_0+k_0,p> \le 0 \\ \dfrac{<\delta_0+k_0,p>}{\|p\|} & \text{otherwise}. \end{cases}$$

But $\dfrac{<\delta_0+k_0,p>}{\|p\|} < \dfrac{\varepsilon\|p^0\|}{2\|p\|} \le \varepsilon$. Therefore, for each $p \in M(p^0)$ there exists a point $k(p)$ such that $\|k(p)-k_0\| < \varepsilon$ and $<\delta_0+k(p),p> \le 0$. Hence $\delta_0+k(p) \in \delta_0+k_0+\varepsilon\mathbf{B} \subset (\text{int}\, L-x)$ for each $x \in N(x^0)$. Thus $\delta_0+k(p) \in F(x,p)$ for each $x \in N(x^0)$ and $p \in M(p^0)$. But also $\|\delta_0+k(p)-\delta_0\| = \|k(p)\| \le \|k_0\| + \|k(p)-k_0\| < \frac{\eta}{2} + \varepsilon \le \frac{\eta}{2} + \frac{\eta}{2} = \eta$. Thus $\delta_0+k(p) \in \delta_0+\eta\mathbf{B} \subset Q$ and therefore $\delta_0+k(p) \in Q \cap F(x,p)$.

■

Note:

This theorem could be proved using theorem 1 in Robinson [17]. In order to do that we could define

$$\Sigma := \left\{ (\delta,z) \in \mathbf{R}^m \times L \mid \delta - z + x = 0 \text{ and } <p,\delta> \le 0 \right\}.$$

Then $F(x,p) = [I\ 0]\Sigma(x,p)$ and it is easy to see that if Σ is l.s.c. at (x,p) then F is l.s.c. at (x,p).

One can check that condition (1) is equivalent to the regularity condition stated in Robinson [17] for the system of inequalities defining the set Σ. Therefore Σ is l.s.c. at any point (x,p) satisfying (1).

3.3.3 Corollary

If $L = \mathbb{R}_+^m$ *then* F *is l.s.c. on* $L\setminus\{0\}\times int\,\mathbb{R}_+^m$.

Proof:

Let $x^0 \in L\setminus\{0\}$ and $p^0 \in \mathrm{int}\,\mathbb{R}_+^m$. We have to show that $H(p^0) \cap (\mathrm{int}\,L - x^0) \neq \phi$.

Since $x^0 \in \mathbb{R}_+^m\setminus\{0\}$, there exists $j \in \{1,..,m\}$ such that $x_j^0 > 0$. Define $x \in \mathbb{R}^m$ by

$$x_k := \begin{cases} x_k+\varepsilon & \text{if } k\neq j \\ \dfrac{x_j^0}{2} & \text{if } k=j \end{cases}$$

where ε is any number in the open interval $\left(0, \dfrac{p_j^0 x_j^0}{2\sum_{k\neq j} p_k^0}\right)$. Then $x = (x_1,..,x_m) \in \mathrm{int}\,\mathbb{R}_+^m$. Also

$$<p^0, x - x^0> = \varepsilon \sum_{k\neq j} p_k^0 - p_j^0\,\frac{x_j^0}{2} \leq 0.$$

Therefore $x - x^0 \in H(p^0) \cap (\mathrm{int}\,L - x^0)$.

■

3.3.4 Theorem

Let L, F *and* H *be defined as in the previous theorem. Then* F *is u.s.c. with convex compact images in* $L \times int\,\mathbb{R}_+^m$.

Proof:

Let $x_{min} \in \mathbf{R}^m$ such that $x_{min} \leq x$ for every $x \in L$ and define:

$$C(x,p) = \left\{ \delta \in \mathbf{R}^m \mid x+\delta \geq x_{min}, \ <p,\delta> \leq 0 \right\}.$$

It is easy to see that C(x,p) is closed, convex and nonempty for each $x \in L$ and each $p \in \mathbf{R}^m_+$ (since $0 \in C(x,p)$). For $p \in \operatorname{int} \mathbf{R}^m_+$, C(x,p) is compact.

Since $\quad F(x,p) = \left\{ \delta \in \mathbf{R}^m \mid x+\delta \in L, <p,\delta> \leq 0 \right\} = C(x,p) \cap (L-x),$

F(x,p) is closed convex and $0 \in F(x,p)$ for all $x \in L$ and $p \in \mathbf{R}^m_+$. Also F(x,p) is compact for $p \in \operatorname{int} \mathbf{R}^m_+$.

Take $x^0 \in L$ and $p^0 \in \operatorname{int} \mathbf{R}^m_+$. Define $\delta_* := x_{min} - x^0$ and let x and p in $\mathbf{R}^m$ be such that:

$$\begin{cases} x - x^0 \leq \mathbf{e} := (1,..,1) \in \mathbf{R}^m \\ p \in P(p^0) := \left\{ q \in \mathbf{R}^m \mid \frac{1}{2} p_j^0 \leq q_j \leq \frac{3}{2} p_j^0 \quad \forall j=1,..,m \right\}. \end{cases}$$

Let δ be in F(x,p). Then $x+\delta \in L$, so $x+\delta \geq x_{min}$. Thus $\delta \geq x_{min} - x \geq x_{min} - (x^0 + \mathbf{e})$ and therefore $\delta_j \geq \delta_{*j} - 1$.

Also $x^0 \in L$ implies $x^0 \geq x_{min}$, so $\delta_* = x_{min} - x^0 \leq 0$. Thus $\delta_{*j} - 1 < 0$ for all j=1,..,m. Hence

$$<p,\delta> \leq 0 \quad \Rightarrow \quad p_j \delta_j \leq -\sum_{i \neq j} p_i \delta_i$$

$$\left.\begin{array}{l} p_i \geq \frac{1}{2} p_i^0 > 0 \\ \delta_i \geq \delta_{*i} - 1 \end{array}\right\} \Rightarrow -p_i \delta_i \leq -p_i (\delta_{*i} - 1) \quad \forall i=1,..,m$$

$$-(\delta_{*i} - 1) > 0, \ p_i \leq \frac{3}{2} p_i^0 \quad \Rightarrow \quad 0 < -p_i (\delta_{*i} - 1) \leq -\frac{3}{2} p_i^0 (\delta_{*i} - 1) \quad \forall i=1,..,m$$

$$\Rightarrow\ p_j\delta_j \le -\frac{3}{2}\sum_{i\ne j} p_i^0(\delta_{*i}-1) < -\frac{3}{2}\sum_{i=1}^{m} p_i^0(\delta_{*i}-1) = \frac{3}{2}\sum_{i=1}^{m} p_i^0(1-\delta_{*i})$$

$$\Rightarrow\ \delta_j \le \frac{3}{2p_j}\sum_{i=1}^{m} p_i^0(1-\delta_{*i}) \le \frac{3}{p_j^0}\sum_{i=1}^{m} p_i^0(1-\delta_{*i}).$$

Therefore $F(x,p) \subset \left\{\delta\in\mathbf{R}^m \mid \delta_{*j}-1 \le \delta_j \le \frac{3}{p_j^0}\sum_{i=1}^{m} p_i^0(1-\delta_{*i})\right\} =: G$

for all $x \le x^0+\mathbf{e}$ and $p\in P(p^0)$. But G is a compact set, therefore there exists $R > 0$ such that $F(x,p) \subset R\mathbf{B}$ for all $x \le x^0+\mathbf{e}$ and all $p\in P(p^0)$, and we can write:

$$F(x,p) = (L-x) \cap H(p) = [(L-x)\cap H(p)] \cap R\mathbf{B}.$$

Since L is closed, it can be shown that the multifunction $(x,p) \to (L-x) \cap H(p)$ is closed. But the multifunction $(x,p) \to R\mathbf{B}$ is constant and therefore u.s.c. Hence, by theorem 1.1.6, the multifunction F is u.s.c.

■

3.3.5 Theorem

If $L \subset \mathbf{R}^m$ is closed, convex and bounded below, and $u{:}L \to \mathbf{R}$ is continuous and strictly quasiconcave, then the problem:

$$\begin{aligned} &\min\ u(x+\delta) \\ &s.t.\ \ \langle p,\delta\rangle \le 0 \\ &\qquad x+\delta \in L \end{aligned}$$

has a unique solution $\delta(x,p)$ for each $p\in int\,\mathbf{R}_+^m$ and each $x\in L$. In addition, δ is continuous at any $(x,p)\in L\times int\,\mathbf{R}_+^m$ satisfying (1). In *particular* δ *is continuous on int* $L \times$ *int* $\mathbf{R}_+^m$.

If we also assume that u satisfies local nonsatiation, then $<p,\delta(x,p)> = 0$ *for each* $x \in L$ *and each* $p \in int\,\mathbf{R}_+^m$.

Proof:

Since u is continuous, it has a maximun on F(x,p), say at $x + \delta^0$.

If there is another optimal point δ^1, i.e. $\delta^1 \in F(x,p)$, $\delta^1 \neq \delta^0$ and $u(x+\delta^0) = u(x+\delta^1)$ then:

$$u(x+(\lambda\delta^1+(1-\lambda)\delta^0)) = u(\lambda(x+\delta^1)+(1-\lambda)(x+\delta^0)) > u(x+\delta^0) \quad \forall\lambda\in(0,1).$$

But F(x,p) is convex, so $\lambda\delta^1+(1-\lambda)\delta^0 \in F(x,p)$ for all $\lambda\in(0,1)$, which is a contradiction. Therefore the maximization problem has a unique solution that from now on we denote by $\delta(x,p)$.

By the "maximun theorem" (see theorem 1 in Robinson and Day [16]), δ is continuous.

Now, assume that u satisfies local nonsatiation. Let $x \in L$ and $p \in \mathrm{int}\,\mathbf{R}_+^m$. If $<p,\delta(x,p)> < 0$ there exists $\varepsilon > 0$ such that $<p,\delta> < 0$ for all $\delta \in \delta(x,p) + \varepsilon B$. But then, by local nonsatiation, there exists a point $y \in (x + \delta(x,p) + \varepsilon B) \cap L$ such that $u(y) > u(x + \delta(x,p))$. Write $y = x + \delta$; then $\varepsilon > \|y - (x + \delta(x,p))\| = \|\delta - \delta(x,p)\|$ and $\delta \in F(x,p)$ which is a contradiction. Therefore $<p,\delta(x,p)> = 0$.

■

3.3.6 Corollary

If $L = \mathbf{R}_+^m$ *and* u *satisfies condition (1), then* $\delta(x,p)$ *is continuous on* $L \setminus \{0\} \times int\, \mathbf{R}_+^m$.

With little stronger assumptions we can show that $\delta(x,p)$ is locally Lipschitz continuous and therefore (by Rademacher's theorem) almost everywhere differentiable.

The following theorem is an immediate consequence of theorems 2.1 and 4.1 in Robinson [18].

Consider the problem:

$$
\begin{aligned}
P(a) \quad & \max u(x,a) \\
& \text{s.t. } g_i(x,a) \le 0 \quad i=1,..,r \\
& \qquad h_i(x,a) = 0 \quad i=1,..,s
\end{aligned}
$$

where $a \in A$, $A \subset \mathbf{R}^k$ is open and $u(.,a)$, $g(.,a)$ and $h(.,a)$ are Frechet differentiable functions from some open set $\Omega \subset \mathbf{R}^m$ into $\mathbf{R}$, $\mathbf{R}^r$ and $\mathbf{R}^s$, respectively, for each $a \in A$.

Let us denote $\nabla_x u(x,a)$ the vector $\left(\dfrac{\partial u(x,a)}{\partial x_i}\right)_{i=1,..,m}$ and $\nabla_x^2 u(x,a)$ the square matrix $\left(\dfrac{\partial^2 u(x,a)}{\partial x_i \partial x_j}\right)_{i,j=1,..,m}$.

The optimality conditions for P(a) can be written as the generalized equation:

$$
GE(a) \qquad 0 \in \begin{bmatrix} \nabla_x L(x,a,\lambda,\mu) \\ -g(x,a) \\ -h(x,a) \end{bmatrix} + N_{\mathbf{R}^m \times \mathbf{R}_+^r \times \mathbf{R}^s} \begin{bmatrix} x \\ \lambda \\ \mu \end{bmatrix}
$$

where $L(x,a,\lambda,\mu) = -u(x,a) + \sum_{i=1}^{r}\lambda_i g_i(x,a) + \sum_{i=1}^{s}\mu_i h_i(x,a)$.

For any point x satisfying the constraints of P(a) let us denote

$$I(x,a) = \left\{ i=1,..,r \mid g_i(x,a) = 0 \right\}$$

and, for any solution (x,λ,μ) of GE(a), let us denote

$$I^+(x,\lambda,a) = \left\{ i \in I(x,a) \mid \lambda_i > 0 \right\}$$

$$I^0(x,\lambda,a) = \left\{ i \in I(x,a) \mid \lambda_i = 0 \right\}.$$

3.3.7 Theorem

Let u,g and h be functions from $\Omega \times A$ *to* $\mathbb{R}$, $\mathbb{R}^r$ *and* $\mathbb{R}^s$, *respectively, which are twice differentiable with respect to their first argument at a point* $(\bar{x},\bar{a}) \in \Omega \times A$.

Suppose that $\bar{x}$, *together with points* $\bar{\lambda} \in \mathbb{R}^r$ *and* $\bar{\mu} \in \mathbb{R}^s$ *solves* $GE(\bar{a})$. *Assume that:*

i) $\nabla_x^2 u$, $\nabla_x^2 g$ *and* $\nabla_x^2 h$ *are continuous at* $(\bar{x},\bar{a})$

ii) there is a $\nu > 0$ *and neighborhoods V of* $\bar{x}$ *and W of* $\bar{a}$ *such that for each x in V and for each* a^1 *and* a^2 *in W:*

$$\|\nabla_x u(x,a^1) - \nabla_x u(x,a^2)\| \le \nu \|a^1 - a^2\|$$
$$\|\nabla_x g_i(x,a^1) - \nabla_x g_i(x,a^2)\| \le \nu \|a^1 - a^2\| \quad \forall i=1,..,r$$
$$\|g_i(x,a^1) - g_i(x,a^2)\| \le \nu \|a^1 - a^2\| \quad \forall i=1,..,r$$
$$\|\nabla_x h_i(x,a^1) - \nabla_x h_i(x,a^2)\| \le \nu \|a^1 - a^2\| \quad \forall i=1,..,s$$
$$\|h_i(x,a^1) - h_i(x,a^2)\| \le \nu \|a^1 - a^2\| \quad \forall i=1,..,s$$

iii) the gradients of the binding constraints are linearly independent, i.e.

$$\left\{ \nabla_x g_i(\bar{x},\bar{a}) \mid i \in I(\bar{x},\bar{a}) \right\} \cup \left\{ \nabla_x h_i(\bar{x},\bar{a}) \mid i=1,..,s \right\} \text{ is linearly independent}$$

iv) the strong second order sufficient condition:

$$\left.\begin{array}{l} y \in \mathbf{R}^m \setminus \{0\} \\ <\nabla_x g_i(\bar{x},\bar{a}),y> = 0 \quad \forall i \in I^+(\bar{x},\bar{\lambda},\bar{a}) \\ <\nabla_x h_i(\bar{x},\bar{a}),y> = 0 \quad \forall i = 1,..,s \end{array}\right\} \Rightarrow y^t \nabla_x^2 L(\bar{x},\bar{a})\, y > 0$$

is satisfied.

Then there exist $\alpha > 0$, neighborhoods N of $\bar{x}$ and U of $\bar{a}$ and a single-valued function $x : U \to N$ such that for each a in U, x(a) is the unique solution of GE(a) in N. Furthermore, for each a^1 and a^2 in U one has:

$$\| x(a^1) - x(a^2) \| \leq \alpha \| a^1 - a^2 \| .$$

We can get as a corollary a result that is a little stronger than the one stated by Cornet and Laroque [9].

3.3.8 Corollary

Let u,g and h be defined as in the previous theorem. Suppose that $\bar{x}$ solves $P(\bar{a})$ and that u, g and h satisfy the properties i), ii) and iii) of the previous theorem at $(\bar{x},\bar{a})$.

Also assume that:

iv') there is a neighborhood V of $\bar{x}$ and W of $\bar{a}$ such that for each x in V and for each a in W:

$$\begin{cases} g_i(.,a) \text{ is convex in } V \\ h_i(.,a) \text{ is affine in } V \\ y \in \mathbb{R}^m \setminus \{0\} \text{ and } <\nabla_x u(x,a),y> = 0 \Rightarrow y^t \nabla_x^2 u(x,a) y < 0. \end{cases}$$

Then, there is a neighborhood U of $\bar{a}$ and a function $x : U \to \mathbb{R}^m$ such that:

a) $x(\bar{a}) = \bar{x}$

b) *$x(a)$ is the maximizer of $P(a)$ for each a in U*

c) *x is Lipschitz continuous on U.*

Proof:

Because of iii), the optimality conditions are necessary at $\bar{x}$. Therefore there exists $\bar{\lambda}$ in $\mathbb{R}^r$ and $\bar{\mu}$ in $\mathbb{R}^s$ such that $(\bar{x},\bar{\lambda},\bar{\mu})$ solves GE($\bar{a}$).

In the other hand, condition iv') implies u(.,a) is strictly quasiconcave for each a in W (look at lemma 3.3.12 below). Therefore, if $a \in W$ and (x,λ,μ) solves GE(a), then x solves P(a) and, by convexity of $g_i(.,a)$ 1=1,..,r and $h_i(.,a)$ i=1,..,s, the solution of P(a) is unique.

But condition iv') also implies that the strong second order sufficient conditions for P($\bar{a}$) are satisfied at $\bar{x}$:

$$\nabla_x u(\bar{x},\bar{a}) = \sum_{i=1}^{r} \bar{\lambda}_i \nabla_x g_i(\bar{x},\bar{a}) + \sum_{i=1}^{s} \bar{\mu}_i \nabla_x h_i(\bar{x},\bar{a}) \quad \text{so}$$

$$\left.\begin{array}{ll} <\nabla_x g_i(\bar{x},\bar{a}),y> = 0 & \forall i \in I^+(\bar{x},\bar{\lambda},\bar{a}) \\ <\nabla_x h_i(\bar{x},\bar{a}),y> = 0 & \forall i = 1,..,s \end{array}\right\} \Rightarrow <\nabla_x u(\bar{x},\bar{a}),y> = 0.$$

But

$$<\nabla_x u(\bar{x},\bar{a}),y> = 0 \Rightarrow y^t \nabla_x^2 u(\bar{x},\bar{a}) y < 0$$

$g_i(.,\bar{a})$ is convex $\Rightarrow$ $\nabla_x^2 g_i(\bar{x},\bar{a})$ is positive semidefinite for each $i=1,..,r$

$h_i(.,\bar{a})$ is affine $\Rightarrow$ $\nabla_x^2 h_i(\bar{x},\bar{a}) = 0$ for each $i=1,..,s$.

Therefore:

$$y^t \nabla_x^2 L(\bar{x},\bar{a})\, y = -y^t \nabla_x^2 u(\bar{x},\bar{a})\, y + \sum_{i=1}^{r} y^t \nabla_x^2 g_i(\bar{x},\bar{a})\, y + \sum_{i=1}^{s} y^t \nabla_x^2 h_i(\bar{x},\bar{a})\, y$$

$$\geq -y^t \nabla_x^2 u(\bar{x},\bar{a})\, y > 0.$$

■

In our case we are interested in the problem:

$$\begin{aligned} P(x,p) \quad & \max \; u(x+\delta) \\ & \text{s.t.} \; \langle p,\delta\rangle \leq 0 \\ & \qquad x+\delta \in L. \end{aligned}$$

3.3.9 Lemma

Let L be a closed convex subset of R^m. Assume that $int\, L \neq \phi$ and that $x \in L$ and $p \in R^m$ satisfy the following condition:

$$-p \notin N_L(x). \qquad (2)$$

Then there is a $\delta^0 \in R^m$ such that $\langle p,\delta^0\rangle < 0$ and $\delta^0 \in int\,(L-x)$.

Note:

Condition (2) is satisfied if $x \in int\, L$ and $p \neq 0$.

Proof:

Let $x^0 \in \operatorname{int} L$. Since $-p \notin N_L(x)$, there exists a point $x^1 \in L$ such that $<p, x^1 - x> < 0$.

Define $x^\lambda := \lambda x^1 + (1-\lambda) x^0$, then $x^\lambda \in \operatorname{int} L$ for each $\lambda \in [0,1)$.

Also, there exists $\lambda^* \in [0,1)$ such that $<p, x^\lambda - x> < 0$ for all $\lambda \in [\lambda^*, 1)$.

Take any λ in $[\lambda^*, 1)$ and define $\delta^0 := x^\lambda - x$. Then $<p, \delta^0> < 0$ and $\delta^0 \in \operatorname{int} L - x$.

■

Assume $L := \left\{ x \in \mathbf{R}^m \mid g_i(x) \leq 0 \quad i=1,..,r \right\}$ where $g_i : \mathbf{R}^m \to \mathbf{R}$ is twice continuously differentiable, convex and locally Lipschitz with ∇g_i locally Lipschitz on $\mathbf{R}^m$ for all $i=1,..,r$.

Also assume that $u : \mathbf{R}^m \to \mathbf{R}$ is twice continuously differentiable, with $\nabla_x u$ locally Lipschitz continuous, and that

$$h \in \mathbf{R}^m,\ h \neq 0 \text{ and } \nabla u(x) h = 0 \quad \Rightarrow \quad h^t \nabla^2 u(x) h < 0.$$

In particular this last assumption implies that u is strictly quasiconcave (see lemma 3.3.12 below) .

Then the problem P(x,p) can be written:

$$\begin{aligned} P(x,p) \quad & \max\ u(\delta,x,p) \\ & \text{s.t. } g_0(\delta,x,p) \leq 0 \\ & \qquad g_i(\delta,x,p) \leq 0 \end{aligned}$$

where $u(\delta,x,p) := u(x+\delta)$, $g_0(\delta,x,p) := <p,\delta>$, and $g_i(\delta,x,p) := g_i(x+\delta)$ $i=1,..,r$.

Finally assume

iii') $\left\{ \nabla_x g_i(\delta,(x,p)) \mid i \in I(\delta,(x,p)) \right\}$ is linearly independent for any δ satisfying the constraints of P(x,p) and for any x in L and p in $\operatorname{int} \mathbf{R}^m_+$, where

$$I(\delta,(x,p)) := \left\{ i=0,1,..,r \mid g_i(\delta,x,p) = 0 \right\}.$$

Then $\delta(x,p)$ is locally Lipschitz continuous for any $(x,p) \in L \times \operatorname{int} \mathbf{R}^m_+$ satisfying (2).

There is an interesting case where i) and iii') are satisfied trivially. If $L = \mathbf{R}^m_+$ then $g_i(x) = -x_i \quad i=1,..,m$. Any m vectors from $\left\{ p, -e_i \ \ i=1,..,m \right\}$ where $p \in \operatorname{int} \mathbf{R}^m_+$ and $e_i^t := (0,..,0,1,0,..,0) \in \mathbf{R}^m$, are linearly independent.

If $x \in \mathbf{R}^m_+$, $x \neq 0$ and $p \in \operatorname{int} \mathbf{R}^m_+$ then the system:

$$\begin{cases} <p,\delta> = 0 \\ x_i + \delta_i = 0 \quad i=1,..,m \end{cases}$$

has no solution because if $\delta_i = -x_i \ \ i=1,..,m$, then $<p,\delta> < 0$. Therefore the conditions i) and iii') are satisfied if we take $A = \mathbf{R}^m_+ \setminus \{0\} \times \operatorname{int} \mathbf{R}^m_+$ and $\delta : A \to \mathbf{R}^m$ is locally Lipschitz continuous. Moreover, it is easy to see that δ is also continuous at (0,p) for any $p \in \operatorname{int} \mathbf{R}^m_+$.

For the following three lemmas assume $L \subset \mathbf{R}^m$ is convex, $u : L \to \mathbf{R}$ is twice continuously differentiable and that:

$$\nabla u(x)\, h = 0 \quad \Rightarrow \quad h^t \nabla^2 u(x)\, h < 0 \quad \forall x \in L \quad \forall h \in T_L(x) \setminus \{0\} . \tag{3}$$

3.3.10 Lemma

Assume that the points $\bar{x}, x \in L$ *are such that* $\nabla u(\bar{x})(x-\bar{x}) \leq 0$ *and* $\bar{x} \neq x$. *Then* $\nabla u(\bar{x}+t(x-\bar{x}))(x-\bar{x}) < 0$ *a.e.* $t \in [0,1]$.

Proof:

We will show that for each $\bar{t} \in [0,1)$ such that $\nabla u(\bar{x}+\bar{t}(x-\bar{x}))(x-\bar{x}) = 0$ there exists a $\delta > 0$ (depending on $\bar{t}$, x and $\bar{x}$) such that:

$$\nabla u(\bar{x}+t(x-\bar{x}))(x-\bar{x}) < 0 \quad \forall t \in (\bar{t}, \bar{t}+\delta).$$

Define $g(t) := \nabla u(\bar{x}+t(x-\bar{x}))(x-\bar{x})$. Then

$$g'(t) = (x-\bar{x})^t \nabla^2 u(\bar{x}+t(x-\bar{x}))(x-\bar{x}). \text{ and}$$

$$\begin{aligned} g(t) &= g(\bar{t}) + g'(\bar{t})(t-\bar{t}) + o(|t-\bar{t}|) \\ &= (x-\bar{x})^t \nabla^2 u(\bar{x}+\bar{t}(x-\bar{x}))(x-\bar{x})(t-\bar{t}) + o(|t-\bar{t}|). \end{aligned}$$

But $x - \bar{x} = \frac{1}{1-\bar{t}}(x - (\bar{x} + \bar{t}(x-\bar{x}))) \in T_L(\bar{x} + \bar{t}(x - \bar{x}))$ and $\nabla u(\bar{x} + \bar{t}(x - \bar{x}))(x - \bar{x}) = 0$, therefore $(x - \bar{x})^t \nabla^2 u(\bar{x} + \bar{t}(x - \bar{x}))(x - \bar{x}) < 0$.

Since $\lim_{t \to \bar{t}} \frac{o(|t-\bar{t}|)}{t-\bar{t}} = 0$, there exists $\delta > 0$ such that $\frac{|o(|t-\bar{t}|)|}{t-\bar{t}} < -(x - \bar{x})^t \nabla^2 u(\bar{x} + \bar{t}(x - \bar{x}))(x - \bar{x})$ for each $t \in (\bar{t}, \bar{t}+\delta)$ and therefore $g(t) < 0$ in the same interval.

■

3.3.11 Lemma

If $\bar{x}, x \in L$, $\bar{x} \neq x$ *and* $\nabla u(\bar{x})(x-\bar{x}) \leq 0$ *then* $u(x) < u(\bar{x})$. *In particular, u is pseudoconcave (in fact u is strictly pseudoconcave).*

Proof:

We have that:

$$u(x) = u(\bar{x}) + \int_0^1 \nabla u(\bar{x}+t(x-\bar{x}))(x-\bar{x})\,dt$$

and by the previous lemma $\nabla u(\bar{x}+t(x-\bar{x}))(x-\bar{x}) < 0$ a.e. $t \in [0,1]$, therefore $u(x) < u(\bar{x})$.

■

3.3.12 Lemma

$u : L \to \mathbf{R}$ *is strictly quasiconcave.*

Proof:

This proof is a modification of the proof of theorem 9.3.5 in Mangasarian [13] pg 143.

For x^0 and x^1 in $\mathbf{R}^m$ let us denote:

$$[x^0,x^1] := \left\{ \lambda x^1 + (1-\lambda)x^0 \mid \lambda \in [0,1] \right\}$$

$$(x^0,x^1) := \left\{ \lambda x^1 + (1-\lambda)x^0 \mid \lambda \in (0,1) \right\} \quad \text{and}$$

$x^\lambda := \lambda x^1 + (1-\lambda)x^0$ for $\lambda \in [0,1]$.

The proof is by contradiction. If u is not strictly quasiconcave then there exist $x^0, x^1 \in L$ and $\lambda \in (0,1)$ such that $u(x^1) \geq u(x^0)$ and $u(x^\lambda) \leq u(x^0)$.

Therefore, there exists $\bar{x} \in (x^0,x^1)$ such that $u(\bar{x}) = \min_{x \in [x^0,x^1]} u(x)$. Hence $\nabla u(\bar{x})(x^0 - \bar{x}) \geq 0$ and $\nabla u(\bar{x})(x^1 - \bar{x}) \geq 0$.

Since $\bar{x} \in (x^0,x^1)$, there exists a $\lambda \in (0,1)$ such that $\bar{x} = x^\lambda$. Thus

$$0 \leq \nabla u(\bar{x})(x^0 - \bar{x}) = \lambda \nabla u(\bar{x})(x^0 - x^1) \quad \text{and}$$
$$0 \leq \nabla u(\bar{x})(x^1 - \bar{x}) = -(1-\lambda)\nabla u(\bar{x})(x^0 - x^1)$$

therefore $\nabla u(\bar{x})(x^0 - x^1) = 0$ and $\nabla u(\bar{x})(x^1 - \bar{x}) = 0$.

But by the previous lemma, it follows that $u(x^1) < u(\bar{x})$, which is a contradiction.

■

We shall prove the following theorem for $L = \mathbf{R}^m_+$, even though it is true for more general feasible sets.

3.3.13 Theorem

Let $L = \mathbf{R}^m_+$ and $u : L \to \mathbf{R}$ be twice continuously differentiable satisfying local nonsatiation and the property (3):

$$\nabla u(x)h = 0 \quad \Rightarrow \quad h^t \nabla^2 u(x)h < 0 \quad \forall x \in L \quad \forall h \in T_L(x) \setminus \{0\}.$$

Assume that for any $b \in \mathbf{R}$ *with* $b>0$ *and for any* $p \in int\,\mathbf{R}_+^m$, *the solution* $\bar{x}$ *to the problem:*

$$\max\ u(x)$$
$$\text{s.t.}\quad \langle p,x\rangle \le b$$
$$x \in L$$

is such that $\bar{x} > 0$. *(For example if for each* $x \in int\,\mathbf{R}_+^m$ *there is a level surface of u,* $\left\{ y \in L \mid u(y) = c \right\}$, *separating* x *from* ∂L, *i.e. for each continuous function* $y:[0,1] \to \mathbf{R}^m$, *with* $y(0) = x$ *and* $y(1) \in \partial L$ *there is a* $t \in (0,1)$ *such that* $y(t) \in int\,\mathbf{R}_+^m$ *and* $u(y(t)) = c$.*) Then* $\delta(x,p)$ *is continuously differentiable on* $int\,\mathbf{R}_+^m \times int\,\mathbf{R}_+^m$.

Proof:

We are dealing with the problem:

$$\text{(M)}\quad \max\ u(x+\delta)$$
$$\text{s.t.}\quad \langle p,\delta\rangle \le 0$$
$$\delta \ge -x.$$

Since $x \in \text{int}\,\mathbf{R}_+^m$ and $p \in \text{int}\,\mathbf{R}_+^m$, the feasible set $\left\{ \delta \in \mathbf{R}^m \mid \delta \ge -x \text{ and } \langle p,\delta\rangle \le 0 \right\}$ satisfies Slater's constraint qualification (Mangasarian [13]). Therefore the Kuhn - Tucker conditions are necessary. They are also sufficient because u is pseudoconcave.

Also condition (3) implies that u is strictly quasiconcave, so we have all the assumptions of Theorem 3.3.5. Therefore $\delta(x,p)$ is well defined and at least continuous.

By local nonsatiation, if $\bar{\delta}$ denotes the solution of (M) (for a fixed $\bar{x}$ and $\bar{p}$), $<\bar{p},\bar{\delta}> = 0$. Also, by assumption, $\bar{x}_j + \bar{\delta}_j > 0$ for all j=1,..,m, therefore the Kuhn - Tucker conditions are:

$$\begin{cases} -\nabla u(\bar{x}+\bar{\delta})+\bar{\lambda}\,\bar{p} = 0 \\ <\bar{p},\bar{\delta}> = 0 \\ \bar{\lambda} \geq 0 . \end{cases}$$

Disregard the last inequality to get $H(\bar{\delta},\bar{\lambda},\bar{x},\bar{p}) = 0$ where $H : \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^{m+1}$.

The matrix

$$\nabla_{(\delta,\lambda)} H(\bar{\delta},\bar{\lambda},\bar{x},\bar{p}) = \begin{bmatrix} -\nabla^2 u(\bar{x}+\bar{\delta}) & \bar{p} \\ \bar{p}^t & 0 \end{bmatrix}$$

is nonsingular because if

$$\nabla_{(\delta,\lambda)} H(\bar{\delta},\bar{\lambda},\bar{x},\bar{p}) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0 \quad \text{then} \quad \begin{cases} -\nabla^2 u(\bar{x}+\bar{\delta})\,\xi + \eta\,\bar{p} = 0 \\ \bar{p}^t \xi = 0 . \end{cases}$$

But, from the first equality, since $\bar{p} \in \operatorname{int} \mathbf{R}^m_+$, if $\xi = 0$ then $\eta = 0$. Otherwise $\bar{p}^t\xi = 0$, hence $\bar{\lambda}\,\bar{p}^t\xi = \nabla u(\bar{x}+\bar{\delta})\,\xi = 0$ and therefore $\xi^t \nabla^2 u(\bar{x}+\bar{\delta})\,\xi < 0$ (note that since $x \in \operatorname{int} L$ we have $T_L(x) = \mathbf{R}^m$.) Premultiply the first equality by ξ^t to get:

$$-\xi^t \nabla^2 u(\bar{x}+\bar{\delta})\,\xi + \eta\,\bar{p}^t\xi = 0 \quad \Rightarrow \quad \eta\,\bar{p}^t\xi = \xi^t \nabla^2 u(\bar{x}+\bar{\delta})\,\xi < 0$$

which is a contradiction because $\bar{p}^t\xi = 0$.

By the Implicit Function Theorem, there is an open set $U \subset \mathbf{R}^m \times \mathbf{R}^m$, containing $(\bar{x},\bar{p})$ and a function $(\delta,\lambda) : U \to \mathbf{R}^m \times \mathbf{R}$ such that:

$$\begin{cases} H(\delta(x,p),\lambda(x,p),x,p) = 0 \quad \forall (x,p) \in U \\ \delta(\bar{x},\bar{p}) = \bar{\delta} \quad \lambda(\bar{x},\bar{p}) = \bar{\lambda} \\ \delta \text{ and } \lambda \text{ are continuously differentiable in } U. \end{cases}$$

Since $\nabla u(\bar{x}+\bar{\delta}) \neq 0$, $\bar{\lambda} > 0$. By shrinking U if necessary, we can ensure that $\lambda(x,p) > 0$ for all $(x,p) \in U$; then $\delta(x,p)$ is the solution to the problem (M) for each $(x,p) \in U$.

■

3.4 Liapunov functions and stability of Pareto maxima

In this section we review some results that appear in Aubin, Cellina and Nohel [2].

Let us recall some of the notation we used in chapter 2 and the previous sections of this chapter that we will be using here.

We have n consumers; consumer i is characterized by its consumption set $L_i \subset \mathbf{R}^m$, which we assume closed convex and bounded below, and its demand function $d_i : L_i \times S \to \mathbf{R}^m$.

The set of available commodities $M \subset \mathbf{R}^m$ is closed convex and satisfies conditions that guarantee that the set

$$K := \left\{ x=(x_1,..,x_n) \in \prod_{i=1}^{n} L_i \mid Ax \in M \right\}.$$

is compact and nonempty (for example conditions ii and v of chapter 2).

For $x = (x_1,..,x_n) \in \prod_{i=1}^{n} L_i$ and $p \in S$ we write:

$$d(x,p) := \begin{bmatrix} d_1(x_1,p) \\ \cdot \\ \cdot \\ d_n(x_n,p) \end{bmatrix} \quad \text{and} \quad D(x) := \left\{ d(x,p) \mid p \in S \right\}.$$

Aubin, Cellina and Nohel work with loss functions rather than utility functions, but our functions are the same as theirs up to a change of sign. (Also they use the implicit Euler method while we use the explicit Euler method in the proof of the existence theorem.).

3.4.1 Definition:

The solutions of $0 \in D(\bar{x})$ are called the critical points of the multifunction D. If $\bar{x} \in K$ is such a critical point, the constant trajectory $x(t) = \bar{x}$ is obviously a solution to problem P' of chapter 2 (with initial condition $x_0 = \bar{x}$).

Let $V : K \to \mathbf{R}^n$ be a function. A point $\bar{x} \in K$ is a Pareto maximum of V on K if there is no $y \in K$ such that $V(y) \geq V(\bar{x})$ and $V(y) \neq V(\bar{x})$, i.e if we set

$$\Pi(\bar{x}) := \left\{ y \in K \mid V(y) \geq V(\bar{x}) \right\}$$

$\bar{x} \in K$ is a Pareto maximum of V if and only if

$$\Pi(\bar{x}) = V^{-1}V(\bar{x}).$$

3.4.2 Theorem:

If V is continuous on K, then for every $\bar{x} \in K$ the set $\Pi(\bar{x})$ contains a Pareto maximal point.

Proof:

Let $\bar{x} \in K$ and define the following partial ordering on $\Pi(\bar{x})$:

$$x,y \in \Pi(\bar{x}) \quad \text{then} \quad x \prec y \iff \begin{cases} V(x) \leq V(y) \\ V(x) \neq V(y) . \end{cases}$$

By the Hausdorff Maximal Principle, there is a maximal totally ordered subset T of $\Pi(\bar{x})$.

Let $x_1,..,x_s \in T$, then we can assume that $x_1 \prec x_2 .. \prec x_s$. Therefore $x_s \in \bigcap_{i=1}^{s} \Pi(x_i)$. But K is compact and $\Pi(x) \subset K$ is closed for each $x \in K$ because V is continuous, therefore $\bigcap_{x \in T} \Pi(x) \neq \phi$.

T has a maximal element $\bar{z}$ for if not let $\bar{z} \in \bigcap_{x \in T} \Pi(x)$. Then, for each $x \in T$ there exists $y \in T$ such that $x \prec y$. Therefore $V(\bar{z}) \geq V(y) \geq V(x)$ and $V(y) \neq V(x)$; so $V(\bar{z}) \geq V(x)$ and $V(\bar{z}) \neq V(x)$, i.e. $x \prec \bar{z}$. Thus $\bar{z} \succ x$ for all $x \in T$ and $\bar{z} \in T$ because T is maximal. This is equivalent to saying that $\bar{z}$ is maximal for T which is a contradiction.

Now we show that in fact $\bar{z}$ is a Pareto maximal point for V on K. By contradiction, assume that there exists $z^* \in K$ such that $z^* \succ \bar{z}$. Then $z^* \succ x$ for all $x \in T$ and, since T is maximal, $z^* \in T$. But $\bar{z}$ is maximal for T; therefore $\bar{z} \succ z^*$ which is a contradiction because the relation $\prec$ does not permit the existence of two points $x,y \in K$ such that $x \prec y$

and $y \prec x$.

■

3.4.3 Definition:

Let $u : \mathbf{R}^s \to \mathbf{R}$ and $x, h \in \mathbf{R}^s$ be such that the limit:

$$Du(x)(h) := \lim_{\vartheta \to 0^+} \frac{u(x + \vartheta h) - u(x)}{\vartheta}$$

exists. Then we say that u has a right derivative, $Du(x)(h)$, at x in the direction of h.

Definition:

Let $u : \mathbf{R}^s \to [-\infty, +\infty)$ be a concave function. The domain of u is defined by:

$$\operatorname{dom} u := \left\{ x \in \mathbf{R}^s \mid u(x) > -\infty \right\}.$$

We will denote by H_u the subspace parallel to $\operatorname{aff} \operatorname{dom} u$, i.e $H_u = -x + \operatorname{aff} \operatorname{dom} u$, where x is any point in $\operatorname{aff} \operatorname{dom} u$ (in particular, x can be any point in $\operatorname{dom} u$).

3.4.4 Theorem

Let $u : \mathbf{R}^s \to [-\infty, +\infty)$ be a concave function with a nonempty domain. Let $x \in \operatorname{dom} u$ and $h \in \mathbf{R}^s$ be such that the segment $[x - \overline{\vartheta} h, x + \overline{\vartheta} h]$ is contained in the domain of u for some $\overline{\vartheta} > 0$. Then $Du(x)(h)$ exists and satisfies:

1) $\frac{u(x)-u(x-\bar{\vartheta}h)}{\bar{\vartheta}} \geq Du(x)(h) \geq \frac{u(x+\bar{\vartheta}h)-u(x)}{\bar{\vartheta}}$

2) $h \to Du(x)(h)$ *is concave and positively homogeneous.*

Note:

If $x \in \operatorname{ri} \operatorname{dom} u$ and $h \in H_u$, then there exists a positive number $\bar{\vartheta}$ such that the interval $[x-\bar{\vartheta}h, x+\bar{\vartheta}h]$ is contained in the domain of u.

Proof:

See for example theorem 2.1 in Aubin [3] or theorem 23.1 in Rockafellar [20].

■

3.4.5 Lemma

Let $u : \mathbf{R}^s \to [-\infty, +\infty)$ *be concave and* $x : \mathbf{R}^r \to \mathbf{R}^s$ *be Gateaux differentiable at* $t \in \mathbf{R}^r$.

If $x(t) \in \operatorname{ri} \operatorname{dom} u$ *and there is a neighborhood N of t such that* $x(N) \subset \operatorname{dom} u$, *then* $u \circ x$ *is right differentiable at t on any direction* $h \in \mathbf{R}^r$ *and*

$$D(u \circ x)(t)(h) = Du(x(t))(x'(t)h).$$

Proof:

Since $x(N) \subset \operatorname{dom} u$ for some neighborhood N of t, it is easy to see that $x'(t)h \in T_{\operatorname{dom} u}(x(t))$ for all $h \in \mathbf{R}^r$. It is also possible to show that $T_{\operatorname{dom} u}(z) \subset H_u$ for all $z \in \operatorname{dom} u$. Therefore $Du(x(t))(x'(t)h)$ exists for all $h \in \mathbf{R}^r$.

It is known (see for example theorem 10.4 in Rockafellar [20]) that u, restricted to dom u, is locally Lipschitz continuous at any point $z \in \operatorname{ri}\operatorname{dom} u$. Then, if L is the Lipschitz constant for u near x(t), for ϑ small enough

$$-L\left\| \frac{x(t+\vartheta h) - x(t)}{\vartheta} - x'(t)h \right\| + \frac{u(x(t) + \vartheta x'(t)h) - u(x(t))}{\vartheta}$$

$$\leq \frac{u(x(t+\vartheta h)) - u(x(t))}{\vartheta}$$

$$\leq \frac{u(x(t) + \vartheta x'(t)h) - u(x(t))}{\vartheta} + L\left\| \frac{x(t+\vartheta h) - x(t)}{\vartheta} - x'(t)h \right\|.$$

Taking limits, we conclude that: $Du(x(t))(x'(t)h) \leq D(u \circ x)(t)(h) \leq Du(x(t))(x'(t)h)$.

■

3.4.6 Definition:

Let $v : (\mathbf{R}^m)^n \to [-\infty, +\infty)$ be a concave function with $K \subset \operatorname{ri}\operatorname{dom} v$. Assume that $D(x) \subset H_v$ for all $x \in K$. Then v has a derivative $Dv(x)(z)$ from the right at each point $x \in K$, in any direction $z \in D(x)$. Let us define

$$B_v(x) := \inf_{z \in D(x)} Dv(x)(z).$$

We say that v is a Liapunov function on K for the multifunction D if $B_v(x) \geq 0$ for each $x \in K$.

3.4.7 Lemma

Let $L \subset \mathbb{R}^m$ *be closed, convex and bounded below, and* $u : \mathbb{R}^m \to [-\infty, +\infty)$ *be a strictly concave function with* $L \subset ri\ dom\ u$.

For each $x \in L$ *and each* $p \in S$ *denote by* $\delta(x,p)$ *the solution of the problem:*

$$\max u(x + \delta)$$
$$s.t.\ \ <p,\delta> \leq 0$$
$$x + \delta \in L.$$

Then $Du(x)(\delta(x,p)) \geq 0$ *for each* $x \in L$ *and each* $p \in S$.

Note:

Since strict concavity is a stronger condition than strict quasiconcavity, we have all the conditions of theorem 3.3.5. Therefore $\delta(x,p)$ is well defined for all $(x,p) \in L \times S$ and continuous at each point (x,p) satisfying (1).

Proof:

By definition we have that $\delta(x,p) \in -x + L \subset \text{aff dom}\, u$ and $u(x) \leq u(x + \delta(x,p))$. Therefore, since $x \in L \subset \text{ri dom}\, u$, there exists a positive number $\bar{\vartheta}$ such that the interval $[x - \bar{\vartheta}\delta(x,p), x + \bar{\vartheta}\delta(x,p)]$ is contained in ri dom u.

Also, by strict concavity:

$$u(x + \lambda\delta(x,p)) = u(\lambda(x + \delta(x,p)) + (1-\lambda)x) > u(x) \quad \forall \lambda \in (0,1).$$

Thus

$$Du(x)(\delta(x,p)) = \lim_{\lambda \to 0^+} \frac{u(x + \lambda\delta(x,p)) - u(x)}{\lambda} \geq 0.$$

■

3.4.8 Theorem

For $i=1,..,n$ let the utility function $u_i : \mathbb{R}^m \to [-\infty, +\infty)$ be strictly concave with $L_i \subset ri\ dom\ u_i$. For $z \in L_i$ and $p \in S$ denote by $\delta_i(z,p)$ the solution of the problem:

$$\max u_i(z + \delta)$$
$$s.t. \quad \langle p,\delta \rangle \leq 0$$
$$z + \delta \in L_i.$$

Assume that for each $i=1,..,n$ there exists a constant $\alpha_i > 0$ such that $d_i(z,p) = \alpha_i\, \delta_i(z,p)$ for all $z \in L_i$ and all $p \in S$.

For $x = (x_1,..,x_n) \in (\mathbb{R}^m)^n$ define $v_i(x) := u_i(x_i)$ and the function $V : (\mathbb{R}^m)^n \to \mathbb{R}^n$ by $V(x) := (v_1(x),..,v_n(x))$. Then each v_i is a Liapunov function on K for the multifunction D and, for every $\gamma \in \mathbb{R}^n_+$ $\sum_{i=1}^{n} \gamma_i v_i$ is also a Liapunov function for the multifunction D.

Proof:

We only need to mention here that for all $x \in \prod_{i=1}^{n} L_i$

$$D(x) \subset \prod_{i=1}^{n} \alpha_i(-x_i + L_i) \subset \prod_{i=1}^{n} H_{u_i} = H_v$$

and that $K \subset \prod_{i=1}^{n} L_i \subset \prod_{i=1}^{n} \operatorname{ri}\operatorname{dom} u_i \subset \operatorname{ri}\operatorname{dom} v_i$. Therefore v_i is a Liapunov function on K.

The inequality:

$$\sum_{i=1}^{n} \gamma_i B_{v_i}(x) \le B_{\sum_{i=1}^{n} \gamma_i v_i}(x)$$

holds for any $\gamma \in \mathbf{R}_+^n$, so $\sum_{i=1}^{n} \gamma_i v_i$ is also a Liapunov function.

■

3.4.9 Theorem

With u_i, v_i and d_i defined as in the previous theorem, let us suppose that the assumptions i - vii and I - III of chapter 2 are satistfied.

If $x : [0,T] \to K$ is a solution to the problem P' of chapter 2, then $t \to w_i(t) := u_i(x_i(t))$ is an absolutely continuous nondecreasing function for each $i=1,..,n$.

Proof:

The solution $x : [0,T] \to K$ is absolutely continuous; therefore x is differentiable almost everywhere in $(0,T)$. Also $K \subset \mathrm{ri\,dom}\, v_i$, therefore the right derivative $D(v_i \circ x)(t)(h)$ exists for almost all t in $(0,T)$.

On the other hand, by lemma 3.4.7 $D(v_i)(x((t))(d(x(t),p(t))) \geq 0$ for all $t \in [0,T]$.

But $x'(t) = d(x(t),p(t))$ almost everywhere in $(0,T)$, therefore $D(v_i)(x((t))(x'(t)) = Dw_i(t)(1) \geq 0$ for almost all t in $(0,T)$.

Finally, the function v_i being Lipschitz continuous on K (recall again theorem 10.4 of Rockafellar) and the function x being absolutely continuous implies that $v_i \circ x$ is absolutely continuous. In fact, if L is the Lipschitz constant of v_i on K:

$$0 \leq \alpha_1 \leq \beta_1 \leq \alpha_2 \leq \ldots\ldots \leq \beta_N \leq T \quad \Longrightarrow$$

$$\sum_{i=1}^{N} | v_i \circ x(\beta_i) - v_i \circ x(\alpha_i) | \leq L \sum_{i=1}^{N} | x(\beta_i) - x(\alpha_i) | < L\varepsilon$$

whenever $\sum_{i=1}^{N} (\beta_i - \alpha_i) < \delta$. Therefore there exists a measurable function $g : [0,T] \to \mathbf{R}$ such that:

$$v_i \circ x(\tau) = v_i \circ x(0) + \int_0^{\tau} g(t)dt \qquad \forall \tau \in [0,T].$$

It is possible to show that $g(t) = D(v_i \circ x)(t)(1)$ almost everywhere in $(0,T)$ (see for example theorem 9 on pg 103 of Royden [21]). So, if $0 \leq \sigma < \tau \leq T$

$$w_i(\tau) = w_i(\sigma) + \int_\sigma^\tau D(w_i)(t)(1)dt \geq w_i(\sigma).$$

■

3.4.10 Corollary

Under the same conditions of the above theorem, each Pareto maximal point $\bar{x}$ of V on K is a critical point of D.

Proof:

Let us first note that the strict concavity of the functions: u_i $i=1,..,n$ implies that $\Pi(\bar{x}) = V^{-1}V(\bar{x}) = \{\bar{x}\}$, for if $x \in K$ is another Pareto maximal point,

$$V(\lambda x + (1-\lambda)\bar{x}) > \lambda V(x) + (1-\lambda)V(\bar{x}) = V(\bar{x}) \qquad \forall \lambda \in (0,1)$$

which is a contradiction because $\lambda x + (1-\lambda)\bar{x} \in K$ for all $\lambda \in [0,1]$.

If we take $x^0 = \bar{x}$, $V(x(t)) \geq V(\bar{x})$ for all $t>0$ by the theorem above. So $x(t) = \bar{x}$ for all $t>0$ and

$$0 = x'(t) = d(x(t),p(t)) = d(\bar{x},p(t)) \quad \forall t \geq 0.$$

Therefore $0 \in D(\bar{x})$.

■

3.4.11 Corollary

Under the same conditions of the above theorem, for each $x \in K$, $\Pi(x)$ *contains a critical point of D.*

Proof:

It is an immediate consequence of theorem 3.4.2. and corollary 3.4.10.

■

3.4.12 Definition:

$Q \subset K$ is stable for the problem P' of chapter 2 if for any neighborhood M of Q, there exists a neighborhood N of Q such that, for any initial value $x^0 \in N$, all the solutions of P' starting at x^0 remain in M.

3.4.13 Theorem

Suppose we have the same assumptions of the previous theorem. Then, for any Pareto maximum $\bar{x} \in K$, *the subset* $\Pi(\bar{x}) := V^{-1}V(\bar{x})$ *is stable.*

Proof:

See theorem 3.3 on pages 110-111 and remark on page 112 of Aubin, Cellina and Nohel [2].

■

We saw in the proof of theorem 3.4.8 that the inequality:

$$\sum_{i=1}^{n}\lambda_i B_{v_i}(x) \leq B_{\sum_{i=1}^{n}\lambda_i v_i}(x)$$

holds for any $\lambda \in \mathbf{R}_+^n$. If we assume that whenever x is not a Pareto maximum, there is a $\lambda \in \mathbf{R}_+^n \setminus \{0\}$ and $\varepsilon > 0$ such that:

$$B_{\sum_{i=1}^{n}\lambda_i v_i}(y) \geq \varepsilon \quad \text{for every y satisfying} \quad v_i(y) \geq v_i(x) \quad \forall i=1,..,n,$$

we can show the following:

3.4.14 Theorem

Suppose the assumptions of the last two theorems hold. Let $x : \mathbf{R}_+ \to K$ *be any solution of P'. Then, the set* $\left\{ x(t) \mid t \in \mathbf{R}_+ \right\}$ *has accumulation points and any such accumulation point is a Pareto maximum of V.*

Proof:

This is a modification of the proof of the last remark on pg. 112 of Aubin, Cellina and Nohel [2].

As before, define $w_i(t) := u_i(x_i(t))$. We have shown that w_i is a nondecreasing absolutely continuous function and we also have:

$$w'_i(t) = Du_i(x_i(t))(d_i(x_i(t),p(t))) \quad \text{a.e. } t > 0 \quad \forall i=1,..,n.$$

Since K is compact, the functions w_i $i=1,..,n$ are bounded above. Therefore, for each $i=1,..,n$, there exists $c_i \in \mathbf{R}$ such that

$w_i(t) \to c_i$ as $t \to \infty$.

Also $x(t) \in K$ for each $t \in \mathbf{R}_+$. Therefore, for any increasing sequence $\left\{t_\alpha\right\} \subset \mathbf{R}_+$ with $t_\alpha \to \infty$, there exists a subsequence of $\left\{x(t_\alpha)\right\}$ converging to some point $\bar{x} \in K$. This point $\bar{x}$ satisfies: $u_i(\bar{x}_i) \geq \sup_\alpha u_i(x_i(t_\alpha))$.

Let us assume that $\bar{x}$ is not a Pareto maximum. Then, there exists $\lambda \in \mathbf{R}_+^n \setminus \{0\}$ and $\varepsilon \geq 0$ such that:

$$B_{\sum_{i=1}^{n} \lambda_i v_i}(x(t)) \geq \varepsilon \quad \forall t \geq 0.$$

Therefore

$$\sum_{i=1}^{n} \lambda_i w'_i(t) = \sum_{i=1}^{n} \lambda_i Du_i(x_i(t))(d_i(x_i(t),p(t))) \geq \varepsilon \quad \text{a.e. } t > 0$$

so

$$\sum_{i=1}^{n} \lambda_i w_i(\tau) - \sum_{i=1}^{n} \lambda_i w_i(0) \geq \int_0^\tau \sum_{i=1}^{n} \lambda_i Du_i(x_i(t))(d_i(x_i(t),p(t)))\, dt \geq \varepsilon \tau .$$

The right-hand side goes to $+\infty$ as $\tau \to +\infty$ which is a contradiction.

■

4. Numerical analysis of the economic model

4.1 Introduction

In this chapter we study numerically the economic model described in chapter 2. We will discuss the use of both the explicit and implicit Euler Methods. Some small-scale examples are provided and solved with both methods. The programs are provided in appendix 2.

Both methods require the solution of a system of inequalities at each iteration. We have decided to use a method due to Robinson [15] for solving these systems. In general we will not meet the requirements that guarantee the convergence of this method, so a different technique may be required. It is not the purpose of this chapter to address this problem. For the examples we have tested, the method we are using has performed adequately.

4.2 The explicit Euler method

By the proof given in chapter 2 we cannot expect more than to construct an approximate solution that is an element of a sequence having a convergent subsequence.

In the proof of the existence theorem we use the explicit Euler method. We are concerned here only with the solution of the technical problems posed by the method.

Specifically, suppose we have a $\bar{h}>0$ such that $x_i + \bar{h}\, d_i(x_i,p) \in L_i$ for each $x_i \in L_i$, $p \in S$ and $i=1,..,n$, and an $\bar{\varepsilon} > 0$ such that $N_M(x) \subset \operatorname{cone} S_{\bar{\varepsilon}}$ for all $x \in M$. We are given $x^0 \in K$ and we choose $h \in (0,\bar{h}]$. Having $x^k \in K$ we construct $x^{k+1} \in K$ by solving:

$$x^{k+1} \in x^k + h\, D^{*}_{\bar{\varepsilon}}(x^k).$$

This means we have to find a price vector $p^k \in S$ such that:

$$\begin{cases} p_j^k \geq \bar{\varepsilon} \quad \forall j=1,..,m \\ \sum_{i=1}^{n} x_i^k + h\, d_i(x_i^k,p^k) \in M. \end{cases}$$

Let us assume that there exists a convex function $f : \mathbf{R}^m \to \mathbf{R}^k$ such that: $M = \left\{ z \in \mathbf{R}^m \mid f(z) \leq 0 \right\}$. Then we have to solve the problem of finding a price vector $p^k \in \mathbf{R}^m$ such that:

$$\begin{cases} \text{i)} \quad \sum_{j=1}^{m} p_j^k - 1 = 0 \\ \text{ii)} \quad \bar{\varepsilon} - p_j^k \leq 0 \quad \forall j=1,..,m \\ \text{iii)} \quad f\left(\sum_{i=1}^{n} (x_i^k + h\, d_i(x_i^k,p^k))\right) \leq 0. \end{cases} \qquad \text{(P1)}$$

In section 4.4 below we describe an algorithm to deal with this problem.

4.3 The implicit Euler Method

Here we will provide sufficient conditions for the use of the implicit Euler method, i.e. we will furnish conditions that guarantee the existence of solution for the system

$$\frac{x^{k+1} - x^k}{h} \in D_\varepsilon(x^{k+1})$$

at each iteration k, for some fixed stepsize h. No proof is provided that this method will generate a sequence of approximate solutions, as we let h go to 0, having a convergent subsequence. However, it is a common feeling among people working in differential equations that the implicit Euler method is in general more stable than the explicit Euler method, and in fact more stable than most of the methods derived from finite difference schemes.

We will make use of the topological degree theory. Some of the definitions and properties of this theory are summarized in appendix 1.

As always, let L_i represent the consumption set and d_i the demand function of consumer i. We make the following assumptions:

i) L_i is a closed convex set

ii) there exists a closed convex cone $K_i \subset \mathbf{R}^m_+$ and a point $x_i^* \in \mathbf{R}^m$ such that $L_i \subset x_i^* + K_i$

iii) d_i (usually defined only on $L_i \times S$) is a continuous function from $(x_i^* + K_i) \times S$ into $\mathbf{R}^m$ (or can be extended continuously over $(x_i^* + K_i) \times S$)

iv) there is an $\bar{h}>0$ such that $x+\bar{h}d_i(x,p) \in L_i$ for each $x \in L_i$, $p \in S$ and $i=1,..,n$

v) $\|d_i(x_i^*+\lambda k,p)\| \le \lambda \|d_i(x_i^*+k,p)\|$ for each $\lambda>1$, $k \in K_i$ and $p \in S$.

4.3.1 Lemma

Let $k \in T_{L_i}(z)$ and $z \in \partial L_i$. Then $z-\lambda k \notin int\, L_i$ for all $\lambda>0$.

Note:

If $z \in L_i$ and $z+k \in L_i$ then $k \in T_{L_i}(z)$.

Proof:

If $z-\lambda k \in \text{int}\, L_i$ then there exists $r>0$ such that $z-\lambda k+r\mathbf{B} \subset L_i$ and therefore $-\lambda k+r\mathbf{B} \subset T_{L_i}(z)$. But $k \in T_{L_i}(z)$ and $T_{L_i}(z)$ is a convex cone, therefore $\mathbf{B} \subset T_{L_i}$ and thus $\mathbf{R}^m = T_{L_i}(z)$. However $N_{L_i}(z) \neq \{0\}$, therefore $T_{L_i}(z) \neq \mathbf{R}^m$ and we have a contradiction.

■

4.3.2 Lemma

Under the above conditions there exists an $h^* \in (0,\bar{h}]$ such that, for every $i=1,..,n$, $x \in int\, L_i$, $p \in S$ and $h \in (0,h^*]$, there exists a $z \in L_i$ solving the nonlinear system $z-h\, d_i(z,p)=x$.

Proof:

Consider the homotopy $H_h \in C([0,1]\times L_i, \mathbb{R}^m)$ defined by: $H_h(t,z) := z - t\,h\,d_i(z,p)$.

Let $\mu_i := \max\left\{ \|d_i(x_i^* + k, p)\| \mid k \in K_i \cap \mathbf{B} \text{ and } p \in \overline{S}_\varepsilon \right\}$ and take $h_i^* \in (0,\overline{h}]$ such that $\mu_i h_i^* \leq \frac{1}{2}$. Define $h^* := \min\left\{ h_i^* \mid i=1,..,n \right\}$. Choose $x \in \operatorname{int} L_i$ and let $R>1$ be such that $\frac{R}{2} > \|x - x_i^*\|$, then for all $h \in (0,h^*]$:

$$H_h(t,z) \neq x \quad \forall t \in [0,1] \ \ \forall z \in \partial(L_i \cap (x_i^* + R\mathbf{B})).$$

In fact, $\partial(L_i \cap x_i^* + R\mathbf{B}) \subset \partial L_i \cup [\partial(x_i^* + R\mathbf{B}) \cap L_i]$ and if $\|z - x_i^*\| = R$ with $z \in L_i$ then:

$$\|z - t\,h\,d_i(z,p) - x\| \geq \|z - x_i^*\| - t\,h\|d_i(z,p)\| - \|x - x_i^*\|$$
$$\geq R - t\,h\,R\mu_i - \|x - x_i^*\| \geq \frac{R}{2} - \|x - x_i^*\| > 0$$

since by v) $\|d_i(z,p)\| \leq R\mu_i$. On the other hand, if $z \in \partial L_i$:

$$z + h\,d_i(z,p) \in L_i \implies z - h\,d_i(z,p) \notin \operatorname{int} L_i \implies z - h\,d_i(z,p) \neq x\,.$$

By homotopy invariance:

$$1 = \deg(id, (x_i^* + R\mathbf{B}) \cap L_i, x) = \deg(H_h(0,.), (x_i^* + R\mathbf{B}) \cap L_i, x)$$
$$= \deg(H_h(1,.), (x_i^* + R\mathbf{B}) \cap L_i, x)$$

therefore $z - h\,d_i(z,p) = x$ has a solution $z \in x_i^* + R\mathbf{B} \cap L_i$.

■

Condition iii) above may appear a little confusing, however in the important case when $L_i = x_i^* + K_i$ (for example $L_i = \mathbb{R}_+^m$) it reduces to requiring that $d_i : L_i \times S \to \mathbf{R}^m$ be continuous.

Also in this case we can give an interpretation for condition v); proportionally, the more a consumer has the less he wants to change.

We do not want to exclude the case where $x \in \partial L$; therefore we need to strengthen our assumptions. Besides iv) above we will require

vi) $-d_i(z,p) \notin T_{L_i}(z) \quad \forall z \in \partial L_i \quad \forall p \in S$.

However there are cases for which this condition is too restrictive. Consider for example the case where L_i is a polyhedral convex cone. If z is in the relative interior of a face F of L_i then it is possible that $z + \bar{h}\, d_i(z,p) \in F$ for some $p \in S$; therefore $-d_i(z,p) \in T_{L_i}(z)$.

4.3.3 Lemma

If $z \in \partial L_i$ *and* $k \in int\, T_{L_i}(z)$ *then* $-k \notin T_{L_i}(z)$ *and therefore, for all* $\lambda > 0$, $z - \lambda k \notin L_i$.

Proof:

Let r>0 such that $k + r\mathbf{B} \subset T_{L_i}(z)$. If we assume that $-k \in T_{L_i}(z)$ then, since $T_{L_i}(z)$ is a convex cone, $\mathbf{B} \subset T_{L_i}(z)$ and therefore $\mathbf{R}^m = T_{L_i}(z)$. But $N_{L_i}(z) \neq \phi$, therefore $T_{L_i}(z) \neq \mathbf{R}^m$ which is a contradiction.

Now, if for some $\lambda > 0$ $z - \lambda k \in L_i$ then $-k \in T_{L_i}(z)$. Therefore $z - \lambda k \notin L_i$ for all $\lambda > 0$.

■

Thus a stronger condition than vi) is to require that:

vi') $d_i(z,p) \in \text{int}\, T_{L_i}(z)$ for each $z \in \partial L_i$ and $p \in S$.

4.3.4 Corollary

If the assumption i) to vi) are satisfied there exists a $h^* \in (0,\bar{h}]$ such that, for each $i=1,..,n$, $x \in L_i$, $p \in S$ and $h \in (0,h^*]$, there exists $z \in L_i$ solving the system $z - h\, d_i(z,p) = x$.

4.3.5 Lemma

Assume that $d(.,p) : \prod_{i=1}^{n} L_i \to (\mathrm{R}^m)^n$ is Lipschitz continuous with constant α. Then for all $h \in \left[0, \min\left\{h^*, \frac{1}{\alpha}\right\}\right]$ and $x \in \prod_{i=1}^{n} L_i$, $z - h\, d(z,p) = x$ has a unique solution $z(x,p,h) \in \prod_{i=1}^{n} L_i$, which is Lipschitz continuous in x with constant $(1 - \alpha h)^{-1}$.

Proof:

If $z - h d(z,p) = x$ and $\bar{z} - h d(\bar{z},p) = x$ $z - \bar{z} = h(d(z,p) - d(\bar{z},p))$. Therefore $\|z - \bar{z}\| = h\|d(z,p) - d(\bar{z},p)\| \le \alpha h \|z - \bar{z}\|$. But $\alpha h < 1$, so $z = \bar{z}$.

If $z - h d(z,p) = x$ and $\bar{z} - h d(\bar{z},p) = \bar{x}$ then $\|z - \bar{z}\| \le \|x - \bar{x}\| + h\|d(z,p) - d(\bar{z},p)\| \le \|x - \bar{x}\| + \alpha h \|z - \bar{z}\|$. Thus

$$\|z - \bar{z}\| \le \frac{1}{1 - \alpha h} \|x - \bar{x}\|.$$

■

4.3.6 Lemma

Suppose that assumptions i) to vi) are satisfied and that $d : \prod_{i=1}^{n} L_i \times \overline{S}_{\varepsilon_0} \to (\mathbb{R}^m)^n$ *is Lipschitz continuous with constant* β. *Let* $h^{**} := \min\left\{h^{*}, \frac{1}{\beta}\right\}$.

Then, for every $x \in \prod_{i=1}^{n} L_i$ *and* $h \in (0,h^{**})$, *the set* $\Phi(x,h)$ *defined by*

$$\Phi(x,h) := \left\{ \sum_{i=1}^{n} z_i \mid z \in \prod_{i=1}^{n} L_i \text{ and } z - h\,d(z,p) = x \text{ for some } p \in \overline{S}_{\varepsilon_0} \right\}$$

is nonempty and compact.

Proof:

Let $x \in \prod_{i=1}^{n} L_i$ and $h \in (0,h^{**})$. By corollary 4.3.4 $\Phi(x,h) \neq \phi$.

If $z^k - h\,d(z^k,p^k) = x$ with $p^k \in \overline{S}_{\varepsilon_0}$, k=1,2, then

$$\|z^1 - z^2\| = h\|d(z^1,p^1) - d(z^2,p^2)\| \leq \beta h(\|z^1 - z^2\| + \|p^1 - p^2\|).$$

For instance, if we use the euclidian norm, $\|p^1 - p^2\| \leq \sqrt{2}$ for each $p^1,p^2 \in S$; therefore $\|z^1 - z^2\| \leq \beta h(\|z^1 - z^2\| + \sqrt{2})$, so

$$\|z^1 - z^2\| \leq \frac{\sqrt{2}}{1 - \beta h}$$

and in particular $\Phi(x,h)$ is bounded.

Assume that the sequence $\{z^k\} \subset \overline{\Phi}(x,h)$ is such that $z^k \to z^*$. Let $p^k \in \overline{S}_{\varepsilon_0}$ such that $z^k - h\,d(z^k,p^k) = x$. Without loss of generality we can assume that $\{p^k\}$ is convergent, i.e. there exists $p^* \in \overline{S}_{\varepsilon_0}$ such that $p^k \to p^*$.

Since d is continuous, $z^* - h\,d(z^*,p^*) = x$ and therefore $z^* \in \overline{\Phi}(x,h)$; so $\overline{\Phi}(x,h)$ is closed.

■

Now we make the following

Assumption

There exists an $\tilde{h} \in (0,h^{**})$ such that:

$$x \in K,\ h \in (0,\tilde{h}] \text{ and } \overline{\Phi}(x,h) \cap M = \phi \implies \operatorname{conv}(\overline{\Phi}(x,h)) \cap M = \phi .$$

This assumption plays the same role as assumption I in chapter 2, and in fact:

4.3.7 Theorem

Suppose that assumptions i) to vi) are satisffied and that $d : \prod_{i=1}^{n} L_i \times \overline{S}_{\varepsilon_0} \to (\mathbf{R}^m)^n$ *is Lipschitz continuous with constant* β. *Then, for every* $x \in K$ *and* $h \in (0,\tilde{h}]$ $\overline{\Phi}(x,h) \cap K \neq \phi$.

Proof:

By contradiction; if $\Phi(x,h) \cap K = \phi$ then by corollary 4.3.4 $\Phi(x,h) \cap M = \phi$. By assumption above $\operatorname{conv}(\Phi(x,h)) \cap M = \phi$.

But $\Phi(x,h)$ is compact, hence $\operatorname{conv}(\Phi(x,h))$ is also compact. Therefore there is a $q \in \overline{S}_{\varepsilon_0}$ such that $<q,w> \; < \; <q,\varphi>$ for each $w \in M$ and $\varphi \in \Phi(x,h)$.

In the other hand, for each $p \in \overline{S}_{\varepsilon_0}$ there exists $z \in \prod_{i=1}^{n} L_i$ such that $z - h\,d(z,p) = x$, i.e. $\sum_{i=1}^{n} z_i \in \Phi(x,h)$. By Walras law $<p, \sum_{i=1}^{n} z_i> \; \leq \; <p, \sum_{i=1}^{n} x_i>$.

In particular, if we take p=q we can find $z \in \prod_{i=1}^{n} L_i$ such that $<q, \sum_{i=1}^{n} z_i> \; \leq \; <q, \sum_{i=1}^{n} x_i>$ and $\sum_{i=1}^{n} z_i \in \Phi(x,h)$. This is a contradiction because $\sum_{i=1}^{n} x_i \in M$.

■

The problem is now : given $x^0 \in K$ we choose $h \in (0,\tilde{h}]$ and $\gamma \in (0,\varepsilon_0)$. Having $x^k \in K$ we construct $x^{k+1} \in K$ by solving:

$$x^k \in x^{k+1} - h D_\gamma(x^{k+1})$$

i.e. we have to find a price $p^{k+1} \in S$ and a consumption level x^{k+1} such that:

$$\begin{cases} p_j^{k+1} \geq \gamma \quad \forall j=1,..,m \\ x_i^{k+1} \in L_i \quad \forall i=1,..,n \\ \sum_{i=1}^{n} x_i^{k+1} \in M \\ x^{k+1} - h\,d(x^{k+1},p^{k+1}) = x^k . \end{cases}$$

Again, if we assume that there exist convex functions $f_0 : \mathbf{R}^m \to \mathbf{R}^{k_0}$, $f_i : \mathbf{R}^m \to \mathbf{R}^{k_i}$ $i=1,..,n$ such that $M = \left\{ z \in \mathbf{R}^m \mid f_0(z) \leq 0 \right\}$ and $L_i = \left\{ z \in \mathbf{R}^m \mid f_i(z) \leq 0 \right\}$, then the problem is to find $p^{k+1} \in \mathbf{R}^m$ and $x^{k+1} \in (\mathbf{R}^m)^n$ such that

$$\begin{cases} \text{i)} \quad \sum_{j=1}^{m} p_j^{k+1} - 1 = 0 \\ \text{ii)} \quad \gamma - p_j^{k+1} \leq 0 \quad \forall j=1,..,m \\ \text{iii)} \quad f_i(x_i^{k+1}) \leq 0 \quad \forall i=1,..,n \\ \text{iv)} \quad f_0(\sum_{i=1}^{n} x_i^{k+1}) \leq 0 \\ \text{v)} \quad x^{k+1} - h\,d(x^{k+1},p^{k+1}) - x^k = 0 . \end{cases} \qquad \text{(P2)}$$

4.4 An algorithm

Problems (P1) and (P2) can be stated as: find $\bar{z} \in \mathbf{R}^s$ such that $g(\bar{z}) \in \Delta$, where $g : \mathbf{R}^s \to \mathbf{R}^t$ and Δ is a nonempty closed convex cone in $\mathbf{R}^t$. In particular Δ is of the form $(-\mathbf{R}_+^{t_1}) \times \{0\} \subset \mathbf{R}^{t_1} \times \mathbf{R}^{t_2}$ where $t_1 + t_2 = t$.

Robinson [15] describes an algorithm for solving this problem in the case where g is continuously differentiable and g' is Lipschitz continuous on $X_0 \subset \mathbf{R}^s$ (g' denotes the jacobian matrix of g at x).

If L is the Lipschitz constant of g', assume that there is a point $z_0 \in X_0$ and real numbers B and ρ with the following properties:

a) for each $y \in \mathbf{R}^t$ there exists $\xi \in \mathbf{R}^s$ such that $y \in g'(z_0)\xi - \Delta$

b) $B \geq \sup \left\{ \inf \left\{ \|\xi\| \mid g'(z_0)\xi \in \eta + \Delta \right\} \mid \eta \in \mathbf{R}^t \text{ and } \|\eta\| \leq 1 \right\}$

c) $\rho \geq \min \left\{ \|z - z_0\| \mid g(z_0) + g'(z_0)(z - z_0) \in \Delta \right\}$

d) $0 < h \leq \frac{1}{2}$ where $h := BL\rho$.

Let $r := \dfrac{\rho(1 - \sqrt{1 - 2h})}{h}$ and $\Omega := \left\{ z \mid \|z - z_0\| \leq r \right\}$. If $\Omega \subset X_0$, then he shows that his algorithm generates at least a sequence. Any sequence thus generated remains in Ω and converges at least linearly to some $\bar{z} \in \Omega$ such that $g(\bar{z}) \in \Delta$.

In our case, even if we assume that f (in the case of (P1)) or f_i $i=0,1,..,n$ (in the case of (P2)) are continuously differentiable, we cannot expect g to be continuously differentiable because, in general, the instantaneous demand function is only Lipschitz continuous.

In the examples we will provide, the assumptions of theorem 3.3.13 are met. In this case the instantaneous demand function is differentiable but still we cannot guarantee its gradient is Lipschitz continuous.

The question of whether a modification of this algorithm can be used in the general case remains open. Perhaps we could still formulate this problem as a nonlinear nondifferentiable optimization problem.

There are a number of algorithms in nondifferentiable optimization but we have chosen not to use them here because they are not very efficient.

Starting from a point z^0, Robinson's algorithm generates a sequence $\left\{z^\alpha\right\}$ in the following way: having z^α, choose $z^{\alpha+1}$ to be any solution of

$$\min\left\{ \|z - z^\alpha\| \mid g(z^\alpha) + \nabla g(z^\alpha)(z - z^\alpha) \in \Delta \right\}. \tag{N}$$

This is a Newton type algorithm.

We usually have a good starting point for this algorithm; in the case of (P1) we can take $z^0 = p^{k-1}$ and in the case of (P2) we can take $z^0 = (x^k + d(x^k, p^k), p^k)$.

When Δ is a polyhedral convex cone (which is our case), (N) can be expressed as a linear program if we take, for example, the l_∞ or the l_1 norm in $\mathbf{R}^s$.

4.5 Example 1

Let us consider an economy with only two consumers and two goods, where each consumer is characterized by the Cobb-Douglas instantaneous demand function, i.e. for i=1,2:

$$\begin{cases} L_i = \mathbf{R}_+^2 \qquad a_i \in (0,1) \\ \\ \forall z \in L_i \;\; \forall p \in S \quad d_i(z,p) = (a_i p_2 z_2 - (1-a_i) p_1 z_1) \begin{bmatrix} \frac{1}{p_1} \\ \frac{-1}{p_2} \end{bmatrix} . \end{cases}$$

It is easy to check that

$$Ad(x,p) = d_1(ADx,p) \qquad \forall x = (x_1,x_2) \in L_1 \times L_2 \;\; \forall p \in S \tag{*}$$

where, as always, $Ax = x_1 + x_2$ and $D := \begin{bmatrix} 1-a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & 1-a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{bmatrix}$.

Also $d_1(0,p) = 0$ for all $p \in S$ and , if $z \in \mathbf{R}_+^2 \setminus \{0\}$ then $d_1(z,.) : S \to \mathbf{R}^2$ is injective.

For $z \in \mathbf{R}_+^2$ define:

$$g(z,t) := \frac{z_2 - t(z_1 + z_2)}{2} \begin{bmatrix} \frac{1}{t} \\ \frac{1}{t-1} \end{bmatrix} \quad t \in (0,1).$$

Then

$$\frac{\partial g_1}{\partial t}(z,t) = \frac{-1}{2t^2}, \quad \frac{\partial^2 g_1}{\partial t^2}(z,t) = \frac{1}{t^3}$$

$$\frac{\partial g_2}{\partial t}(z,t) = \frac{z_1}{2(t-1)^2}, \quad \frac{\partial^2 g_2}{\partial t^2}(z,t) = \frac{z_1}{(1-t)^3};$$

therefore $g_i(z,.) : (0,1) \to \mathbf{R}$ is convex for each $z \in \mathbf{R}_+^2$, for i=1,2.

Let us show that for this example, assumptions I', II and III of chapter 2 are satisfied.

Assumption I'

Since $Dx \in \mathbf{R}^2_+$ for each $x \in \mathbf{R}^2_+$ and since (*) above holds, it is enough to show that for each $t_0, t_1 \in (0,1)$, $z \in \mathbf{R}^2_+$ and $\lambda \in [0,1]$, there is a $t_\lambda \in \mathrm{conv}\left\{t_0, t_1\right\}$ such that:

$$\lambda g(z,t_1) + (1-\lambda) g(z,t_0) \in g(z,t_\lambda) + \mathbf{R}^2_+.$$

We have shown that $g(z,.)$ is convex, therefore we can take $t_\lambda = \lambda t_1 + (1-\lambda) t_0$. In fact we can show more, namely that

$$\lambda g(z,t_1) + (1-\lambda) g(z,t_0) = \begin{bmatrix} \dfrac{z_2 - t_*(z_1+z_2)}{2t_*} \\ \dfrac{z_2 - t^*(z_1+z_2)}{2(t^*-1)} \end{bmatrix}$$

where

$$t_* := \frac{t_0 t_1}{\lambda t_0 + (1-\lambda) t_1} \qquad t^* := \frac{t_0 t_1 - \lambda t_1 - (1-\lambda) t_0}{\lambda t_0 + (1-\lambda) t_1 - 1}$$

Then $\lambda g(z,t_1) + (1-\lambda) g(z,t_0) \in g(z,s) + \mathbf{R}^2_+$ for each $s \in [t_*, t^*] \subset \mathrm{conv}\left\{t_0, t_1\right\}$.

Assumption II

If $d(x,p) \neq d(x,q)$ then $x \in (\mathbf{R}^2_+ \times \mathbf{R}^2_+) \setminus \{0\}$ and $p \neq q$. Therefore $ADx \in \mathbf{R}^2_+ \setminus \{0\}$ and, since $d_1(z,.)$ is injective whenever $z \neq 0$,

$$Ad(x,p) = d_1(ADx,p) \neq d_1(ADx,q) = Ad(x,q).$$

Assumption III

Let $x \in R_+^2 \times R_+^2$. Let $p,q \in S$, then there exist $t,s \in (0,1)$ such that:

$$p = \begin{bmatrix} t \\ 1-t \end{bmatrix} \quad q = \begin{bmatrix} s \\ 1-s \end{bmatrix}.$$

For $\lambda \in [0,1]$ let $p_\lambda := \lambda p + (1-\lambda) q$, then:

$$p_\lambda = \begin{bmatrix} \lambda t + (1-\lambda) s \\ 1 - (\lambda t + (1-\lambda) s) \end{bmatrix} =: \begin{bmatrix} t_\lambda \\ 1-t_\lambda \end{bmatrix}$$

and

$$\begin{aligned} Ad(x,p_\lambda) &= d_1(ADx,p_\lambda) = g(ADx,t_\lambda) \leq \lambda g(ADx,t) + (1-\lambda) g(ADx,s) \\ &= \lambda d_1(ADx,p) + (1-\lambda) d_1(ADx,q) = \lambda Ad(x,p) + (1-\lambda) Ad(x,q) \end{aligned}$$

therefore

$$Ad(x,p_\lambda) \in \lambda Ad(x,p) + (1-\lambda) Ad(x,q) - R_+^2 \subset \Psi(Ad(x,p),Ad(x,q))$$

for all $\lambda \in [0,1]$. But $Ad(x,p) \neq Ad(x,q)$ implies $x \neq 0$ and $p \neq q$, therefore:

$Ad(x,p_\lambda) \neq Ad(x,p)$ for all $\lambda \in [0,1)$ and, by continuity,

$\|Ad(x,p_\lambda) - Ad(x,p)\| < \varepsilon$ for λ close to 1.

Finally, $p_\lambda \in [[p,q]]$ for all $\lambda \in (0,1]$.

It should be clear for this example that, due to property (*), if we have n consumers ($n > 2$) behaving according to the Cobb-Douglas instantaneous demand function, the same arguments we used above shows that assumptions I', II and III are satisfied. However, the nice convexity property of the function g(z,.) is destroyed if we have more than two goods.

Now we need to choose a set of available commodities. Let us take for example:

$$M := \left\{ z \in \mathbf{R}^2 \mid (z_1+1)^2 + (z_2+1)^2 \leq 100 \right\} = (-1,-1) + 10\mathbf{B}.$$

It is easy to see that all the required conditions are satisfied, in particular we have that:

$$N_M(w) \subset \text{cone}\left\{ \begin{bmatrix} \sqrt{99} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{99} \end{bmatrix} \right\} = \text{cone}\,\overline{S}_\gamma \quad \forall w \in \partial M \cap A(L_1 \times L_2)$$

$$\text{where } \gamma = \frac{1}{\sqrt{99}+1} = 0.091325$$

because $A(L_1 \times L_2) = \mathbf{R}^2_+$ and any point in the normal cone to M at $w \in \partial M \cap A(L_1 \times L_2)$ can be expresed as a positive linear combination of the exterior normals to M at the points $(0, \sqrt{99}-1)$ and $(\sqrt{99}-1, 0)$.

4.6 Example 2

We consider now an economy with 3 goods and two consumers, where each consumer is characterized by the Cobb-Douglas instantaneous demand function. For i=1,2:

$$L_i = \mathbf{R}^3_+ \qquad \alpha_i, \beta_i, \gamma_i \in (0,1) \qquad \alpha_i + \beta_i + \gamma_i = 1$$

$$\forall z \in L_i \ \ \forall p \in S \quad d_i(z,p) = \begin{bmatrix} \dfrac{<p,z>\alpha_i - p_1 z_1}{p_1} \\ \dfrac{<p,z>\beta_i - p_2 z_2}{p_2} \\ \dfrac{<p,z>\gamma_i - p_3 z_3}{p_3} \end{bmatrix}.$$

The set of available commodities is defined by

$$M = \left\{ z \in \mathbf{R}^3_+ \mid (z_1+1)^2 + (z_2+1)^2 + (z_3+1)^2 \le 200 \right\} = (-1,-1,-1) + \sqrt{200}\,\mathbf{B}.$$

We have not checked for this example if assumptions I' and III are satisfied. We can use exactly the same argument we used above in example 1 to show that assumption II is satisfied.

4.7 Example 3

Again let us take an economy with only two goods and two consumers. Assume that consumer i is characterized by

$$\begin{cases} L_i = \mathbf{R}^2_+ \\ \forall z \in L_i \ \ \forall p \in S \quad d_i(z,p) = B^i(z)\,p \end{cases}$$

where $B^i : \mathbf{R}^2_+ \to \mathbf{R}^{2\times 2}$ is continuous (i.e. it is an Aubin type of consumer).

Write $B^i(z) := (B^i_{k,j}(z))^m_{k,j=1}$ and let us study the following two conditions:

1) Instantaneous Walras law: $<p, B^i(z)\,p> \le 0 \quad \forall p \in S \quad \forall z \in L_i$

2) $B^i(z)\,p \in T_{L_i}(z) \quad \forall p \in S \quad \forall z \in L_i$.

Note that we can think of B^i as being defined only on $M \cap \mathbf{R}^2_+$, where M is the space of available commodities.

In general, since we have assumed each L_i to be bounded below, say by ξ_i, we can think of d_i as being defined only on $(M - \sum_{k \neq i} \xi_k) \cap L_i$. Therefore we can relax conditions 1) and 2) accordingly.

Condition 1) is equivalent to saying that the matrices $-B^1(z)$ and $-B^2(z)$ are copositive. By theorem 4.2 in Cottle, Habetler and Lemke [10] this is equivalent to

$$\begin{cases} B^i_{1,1}(z) \leq 0 \quad B^i_{2,2}(z) \leq 0 \\ \text{if } B^i_{1,2}(z) + B^i_{2,1}(z) > 0 \text{ then } 4B^i_{1,1}(z)B^i_{2,2}(z) \geq (B^i_{1,2}(z) + B^i_{2,1}(z))^2 \end{cases}$$

for each $z \in M \cap \mathbf{R}^2_+$ and i=1,2 (note that this says that either $S_{B^i(z)}$ is negative semidefinite or has only non positive entries).

Condition 2) implies condition vii) of chapter 2 (in this particular case, where L_i is a polyhedron, they are in fact equivalent). Also, it is not difficult to see that condition 2) is equivalent to

$$\left.\begin{array}{l} z_1 = 0 \implies B^i_{1,1}(z) \geq 0 \text{ and } B^i_{1,2}(z) \geq 0 \\ z_2 = 0 \implies B^i_{2,1}(z) \geq 0 \text{ and } B^i_{2,2}(z) \geq 0 \end{array}\right\} \quad i=1,2 .$$

In particular, we have that $B^i(0) = 0 \quad i=1,2$. Take for example

$$M := \left\{ z \in \mathbf{R}^2 \mid z_2 + 0.1 z_1 \leq 5 \text{ and } z_1 + 0.1 z_2 \leq 5 \right\} \text{ and}$$

$$B^1(z) =$$

$$\begin{bmatrix} -2z_1 + 0.3z_1^2 & -z_1 + z_2 - 1.2z_1^2 + z_1z_2 - 0.2z_2^2 \\ 0.5z_1 - 2z_2 + z_1^2 + 0.5z_1z_2 - z_2^2 & -z_2 + 0.2z_2^2 \end{bmatrix}$$

$$B^2(z) =$$

$$\begin{bmatrix} -2z_1 + 0.4z_1^2 - 0.3z_1z_2 & -1.5z_1 + 2z_2 + 0.1z_1^2 + 0.5z_1z_2 - 0.4z_2^2 \\ 1.8z_1 - 2z_2 - 0.36z_1^2 + 0.3z_1z_2 + 0.4z_2^2 & -z_2 - 0.1z_2^2 \end{bmatrix}$$

It has been checked numerically that the matrices $B^1(z)$ and $B^2(z)$ are copositive at any point $z \in \{0,1,2,3,4,5\} \times \{0,1,2,3,4,5\}$.

Assumption II' has not been checked for this example so we do not know if assumption II is satisfied. The set M is convex but not strictly convex and therefore assumption i) is violated.

4.8 Numerical results

We tested each of the three examples above with the explicit Euler method and the implicit Euler method. The first hundred iterations are listed below for each case.

We encountered some difficulties when using the implicit Euler method with the first example. We believe they may be overcome by, for example, using double precision. By perturbing the current iteration by a small amount and restarting the algorithm, we were able to continue the method. Even with these difficulties, the method yielded results comparable to those achieved with the explicit Euler method. In fact, in all three examples, the results achieved with the two methods are comparable.

We can observe in the numerical results for example 3, an "abnormal behavior" of the market. Each consumer loses all of his assets in a short period of time. This has been characteristic for a few examples we have constructed with consumers satisfying the linearity condition in the demand functions. We suspect this behavior is intrinsic to this condition.

Example 1 with parameters a1 = 0.3 and a2 = 0.6
Solved with the explicit Euler method.

	consumer 1		consumer 2		prices	
time	x1	x2	x1	x2	p1	p2
0.0	4.0000	2.0000	2.1000	4.0000	0.6000	0.4000
0.1	3.7660	2.3049	2.2001	3.8695	0.5659	0.4341
0.2	3.5740	2.4904	2.3526	3.7223	0.4913	0.5087
0.3	3.4020	2.6548	2.4921	3.5889	0.4888	0.5112
0.4	3.2478	2.8010	2.6195	3.4681	0.4867	0.5133
0.5	3.1097	2.9311	2.7356	3.3587	0.4850	0.5150
0.6	2.9859	3.0471	2.8414	3.2597	0.4837	0.5163
0.7	2.8749	3.1506	2.9374	3.1701	0.4826	0.5174
0.8	2.7754	3.2431	3.0246	3.0891	0.4816	0.5184
0.9	2.6862	3.3256	3.1038	3.0157	0.4808	0.5192
1.0	2.6062	3.3995	3.1756	2.9495	0.4801	0.5199
1.1	2.5344	3.4657	3.2405	2.8896	0.4797	0.5203
1.2	2.4699	3.5250	3.2993	2.8355	0.4792	0.5208
1.3	2.4121	3.5782	3.3524	2.7867	0.4789	0.5211
1.4	2.3601	3.6260	3.4003	2.7427	0.4789	0.5211
1.5	2.3136	3.6686	3.4438	2.7028	0.4782	0.5218
1.6	2.2717	3.7069	3.4831	2.6668	0.4782	0.5218
1.7	2.2341	3.7414	3.5184	2.6345	0.4782	0.5218
1.8	2.2002	3.7725	3.5501	2.6054	0.4782	0.5218
1.9	2.1702	3.7999	3.5795	2.5786	0.4771	0.5229
2.0	2.1432	3.8245	3.6058	2.5546	0.4771	0.5229
2.1	2.1189	3.8467	3.6296	2.5329	0.4771	0.5229
2.2	2.0971	3.8666	3.6509	2.5134	0.4771	0.5229
2.3	2.0774	3.8846	3.6702	2.4959	0.4771	0.5229
2.4	2.0597	3.9007	3.6875	2.4801	0.4771	0.5229
2.5	2.0438	3.9153	3.7030	2.4659	0.4771	0.5229
2.6	2.0294	3.9283	3.7171	2.4531	0.4771	0.5229
2.7	2.0165	3.9401	3.7297	2.4416	0.4771	0.5229
2.8	2.0049	3.9507	3.7410	2.4312	0.4771	0.5229
2.9	1.9944	3.9603	3.7512	2.4219	0.4771	0.5229
3.0	1.9850	3.9688	3.7604	2.4135	0.4771	0.5229
3.1	1.9766	3.9766	3.7687	2.4059	0.4771	0.5229
3.2	1.9689	3.9835	3.7762	2.3991	0.4771	0.5229
3.3	1.9621	3.9898	3.7829	2.3930	0.4771	0.5229
3.4	1.9559	3.9954	3.7889	2.3875	0.4771	0.5229
3.5	1.9503	4.0005	3.7943	2.3825	0.4771	0.5229
3.6	1.9453	4.0051	3.7992	2.3781	0.4771	0.5229
3.7	1.9408	4.0092	3.8036	2.3741	0.4771	0.5229
3.8	1.9368	4.0129	3.8076	2.3705	0.4771	0.5229
3.9	1.9331	4.0162	3.8111	2.3672	0.4771	0.5229
4.0	1.9299	4.0192	3.8143	2.3643	0.4771	0.5229
4.1	1.9269	4.0219	3.8172	2.3617	0.4771	0.5229
4.2	1.9243	4.0243	3.8198	2.3593	0.4771	0.5229
4.3	1.9219	4.0265	3.8222	2.3571	0.4771	0.5229
4.4	1.9197	4.0285	3.8243	2.3552	0.4771	0.5229
4.5	1.9178	4.0302	3.8262	2.3535	0.4771	0.5229
4.6	1.9160	4.0318	3.8279	2.3519	0.4771	0.5229

4.7	1.9145	4.0332	3.8294	2.3505	0.4771	0.5229
4.8	1.9130	4.0345	3.8308	2.3493	0.4771	0.5229
4.9	1.9118	4.0357	3.8320	2.3482	0.4771	0.5229
5.0	1.9106	4.0367	3.8331	2.3471	0.4771	0.5229
5.1	1.9096	4.0377	3.8341	2.3462	0.4771	0.5229
5.2	1.9087	4.0385	3.8351	2.3454	0.4771	0.5229
5.3	1.9078	4.0393	3.8359	2.3446	0.4771	0.5229
5.4	1.9071	4.0400	3.8366	2.3440	0.4771	0.5229
5.5	1.9064	4.0406	3.8373	2.3434	0.4771	0.5229
5.6	1.9058	4.0411	3.8379	2.3428	0.4771	0.5229
5.7	1.9053	4.0416	3.8384	2.3423	0.4771	0.5229
5.8	1.9048	4.0421	3.8389	2.3419	0.4771	0.5229
5.9	1.9043	4.0425	3.8393	2.3415	0.4771	0.5229
6.0	1.9039	4.0429	3.8397	2.3412	0.4771	0.5229
6.1	1.9036	4.0432	3.8400	2.3408	0.4771	0.5229
6.2	1.9032	4.0435	3.8404	2.3405	0.4771	0.5229
6.3	1.9030	4.0437	3.8406	2.3403	0.4771	0.5229
6.4	1.9027	4.0440	3.8409	2.3401	0.4771	0.5229
6.5	1.9025	4.0442	3.8411	2.3398	0.4771	0.5229
6.6	1.9022	4.0444	3.8413	2.3397	0.4771	0.5229
6.7	1.9021	4.0446	3.8415	2.3395	0.4771	0.5229
6.8	1.9019	4.0447	3.8417	2.3393	0.4771	0.5229
6.9	1.9017	4.0449	3.8418	2.3392	0.4771	0.5229
7.0	1.9016	4.0450	3.8420	2.3391	0.4771	0.5229
7.1	1.9015	4.0451	3.8421	2.3390	0.4771	0.5229
7.2	1.9013	4.0452	3.8422	2.3389	0.4771	0.5229
7.3	1.9012	4.0453	3.8423	2.3388	0.4771	0.5229
7.4	1.9012	4.0454	3.8424	2.3387	0.4771	0.5229
7.5	1.9011	4.0455	3.8425	2.3386	0.4771	0.5229
7.6	1.9010	4.0455	3.8426	2.3385	0.4771	0.5229
7.7	1.9009	4.0456	3.8426	2.3385	0.4771	0.5229
7.8	1.9009	4.0456	3.8427	2.3384	0.4771	0.5229
7.9	1.9008	4.0457	3.8427	2.3384	0.4771	0.5229
8.0	1.9008	4.0457	3.8428	2.3383	0.4771	0.5229
8.1	1.9007	4.0458	3.8428	2.3383	0.4771	0.5229
8.2	1.9007	4.0458	3.8429	2.3383	0.4771	0.5229
8.3	1.9007	4.0458	3.8429	2.3382	0.4771	0.5229
8.4	1.9006	4.0459	3.8429	2.3382	0.4771	0.5229
8.5	1.9006	4.0459	3.8430	2.3382	0.4771	0.5229
8.6	1.9006	4.0459	3.8430	2.3382	0.4771	0.5229
8.7	1.9005	4.0459	3.8430	2.3381	0.4771	0.5229
8.8	1.9005	4.0460	3.8430	2.3381	0.4771	0.5229
8.9	1.9005	4.0460	3.8430	2.3381	0.4771	0.5229
9.0	1.9005	4.0460	3.8431	2.3381	0.4771	0.5229
9.1	1.9005	4.0460	3.8431	2.3381	0.4771	0.5229
9.2	1.9005	4.0460	3.8431	2.3381	0.4771	0.5229
9.3	1.9004	4.0460	3.8431	2.3380	0.4771	0.5229
9.4	1.9004	4.0460	3.8431	2.3380	0.4771	0.5229
9.5	1.9004	4.0460	3.8431	2.3380	0.4771	0.5229
9.6	1.9004	4.0461	3.8431	2.3380	0.4771	0.5229
9.7	1.9004	4.0461	3.8431	2.3380	0.4771	0.5229
9.8	1.9004	4.0461	3.8431	2.3380	0.4771	0.5229

Example 1 with parameters a1 = 0.3 and a2 = 0.6 .
Solved with the implicit Euler method.

	consumer 1		consumer 2		prices	
time	x1	x2	x1	x2	p1	p2
0.0	4.0000	2.0000	2.1000	4.0000	0.6000	0.4000
0.1	3.7866	2.2831	2.1881	3.8831	0.5702	0.4298
0.2	3.6100	2.4537	2.3277	3.7482	0.4915	0.5085
0.3	3.4501	2.6072	2.4562	3.6249	0.4896	0.5104
0.4	3.3053	2.7450	2.5747	3.5122	0.4876	0.5124
0.5	3.1741	2.8689	2.6838	3.4091	0.4859	0.5141
0.6	3.0554	2.9805	2.7840	3.3149	0.4845	0.5155
0.7	2.9478	3.0812	2.8760	3.2288	0.4834	0.5166
0.8	2.8504	3.1719	2.9604	3.1501	0.4823	0.5177
0.9	2.7622	3.2539	3.0378	3.0783	0.4815	0.5185
1.0	2.6822	3.3279	3.1086	3.0127	0.4809	0.5191
1.1	2.6097	3.3949	3.1734	2.9528	0.4803	0.5197
1.2	2.5441	3.4554	3.2327	2.8982	0.4797	0.5203
1.3	2.4845	3.5103	3.2868	2.8484	0.4795	0.5205
1.4	2.4305	3.5599	3.3363	2.8029	0.4790	0.5210
1.5	2.3816	3.6049	3.3814	2.7614	0.4788	0.5212
1.6	2.3372	3.6456	3.4226	2.7236	0.4785	0.5215
1.7	2.2969	3.6825	3.4602	2.6891	0.4782	0.5218
1.8	2.2604	3.7160	3.4945	2.6577	0.4781	0.5219
1.9	2.2271	3.7465	3.5256	2.6292	0.4782	0.5218
2.0	2.1972	3.7739	3.5543	2.6030	0.4776	0.5224
2.1	2.1698	3.7989	3.5802	2.5793	0.4778	0.5222
2.2	2.1450	3.8216	3.6038	2.5577	0.4777	0.5223
2.3	2.1227	3.8419	3.6257	2.5377	0.4771	0.5229
2.4	2.1024	3.8605	3.6454	2.5198	0.4774	0.5226
2.5	2.0838	3.8775	3.6632	2.5034	0.4774	0.5226
2.6	2.0671	3.8927	3.6797	2.4884	0.4771	0.5229
2.7	2.0504	3.9081	3.6928	2.4763	0.4803	0.5197
2.8	2.0363	3.9210	3.7059	2.4642	0.4781	0.5219
2.9	2.0240	3.9323	3.7186	2.4527	0.4770	0.5230

*** error *** 30 minor iterations done without converging

2.9	2.0240	3.9325	3.7185	2.4525	0.4770	0.5230
3.0	2.0128	3.9427	3.7300	2.4421	0.4770	0.5230
3.1	2.0026	3.9520	3.7404	2.4326	0.4770	0.5230
3.2	1.9933	3.9605	3.7498	2.4239	0.4770	0.5230
3.3	1.9849	3.9681	3.7584	2.4161	0.4770	0.5230
3.4	1.9773	3.9751	3.7663	2.4089	0.4770	0.5230
3.5	1.9703	3.9815	3.7734	2.4024	0.4770	0.5230
3.6	1.9640	3.9872	3.7798	2.3965	0.4770	0.5230
3.7	1.9582	3.9925	3.7857	2.3912	0.4770	0.5230
3.8	1.9530	3.9973	3.7911	2.3863	0.4770	0.5230
3.9	1.9482	4.0016	3.7959	2.3819	0.4770	0.5230
4.0	1.9439	4.0055	3.8003	2.3779	0.4770	0.5230
4.1	1.9400	4.0091	3.8044	2.3742	0.4770	0.5230
4.2	1.9364	4.0124	3.8080	2.3709	0.4770	0.5230

4.3	1.9332	4.0153	3.8113	2.3678	0.4770	0.5230
4.4	1.9302	4.0180	3.8143	2.3651	0.4770	0.5230
4.5	1.9275	4.0205	3.8171	2.3626	0.4770	0.5230
4.6	1.9251	4.0227	3.8196	2.3603	0.4770	0.5230
4.7	1.9229	4.0247	3.8218	2.3582	0.4770	0.5230
4.8	1.9209	4.0266	3.8239	2.3564	0.4770	0.5230
4.9	1.9190	4.0282	3.8258	2.3547	0.4770	0.5230
5.0	1.9174	4.0297	3.8275	2.3531	0.4770	0.5230
5.1	1.9159	4.0311	3.8290	2.3517	0.4770	0.5230
5.2	1.9145	4.0324	3.8304	2.3504	0.4770	0.5230
5.3	1.9132	4.0335	3.8317	2.3492	0.4770	0.5230
5.4	1.9121	4.0346	3.8329	2.3482	0.4770	0.5230
5.5	1.9111	4.0355	3.8339	2.3472	0.4770	0.5230
5.6	1.9101	4.0364	3.8349	2.3463	0.4770	0.5230
5.7	1.9093	4.0371	3.8358	2.3455	0.4770	0.5230
5.8	1.9085	4.0379	3.8366	2.3448	0.4770	0.5230
5.9	1.9078	4.0385	3.8373	2.3442	0.4770	0.5230
6.0	1.9071	4.0391	3.8379	2.3436	0.4770	0.5230
6.1	1.9066	4.0396	3.8385	2.3430	0.4770	0.5230
6.2	1.9060	4.0401	3.8391	2.3425	0.4770	0.5230
6.3	1.9055	4.0405	3.8396	2.3421	0.4770	0.5230
6.4	1.9051	4.0409	3.8400	2.3417	0.4770	0.5230
6.5	1.9047	4.0413	3.8404	2.3413	0.4770	0.5230
6.6	1.9043	4.0416	3.8408	2.3410	0.4770	0.5230
6.7	1.9040	4.0419	3.8411	2.3406	0.4770	0.5230
6.8	1.9037	4.0422	3.8414	2.3404	0.4770	0.5230
6.9	1.9034	4.0425	3.8417	2.3401	0.4770	0.5230
7.0	1.9032	4.0427	3.8420	2.3399	0.4770	0.5230
7.1	1.9030	4.0429	3.8422	2.3397	0.4770	0.5230
7.2	1.9028	4.0431	3.8424	2.3395	0.4770	0.5230
7.3	1.9026	4.0433	3.8426	2.3393	0.4770	0.5230
7.4	1.9024	4.0434	3.8428	2.3392	0.4770	0.5230
7.5	1.9023	4.0435	3.8429	2.3390	0.4770	0.5230
7.6	1.9021	4.0437	3.8431	2.3389	0.4770	0.5230
7.7	1.9020	4.0438	3.8432	2.3388	0.4770	0.5230
7.8	1.9019	4.0439	3.8433	2.3387	0.4770	0.5230
7.9	1.9018	4.0440	3.8434	2.3386	0.4770	0.5230
8.0	1.9017	4.0441	3.8435	2.3385	0.4770	0.5230
8.1	1.9016	4.0442	3.8436	2.3384	0.4770	0.5230
8.2	1.9015	4.0442	3.8437	2.3383	0.4770	0.5230
8.3	1.9014	4.0443	3.8438	2.3382	0.4770	0.5230
8.4	1.9014	4.0444	3.8438	2.3382	0.4770	0.5230
8.5	1.9013	4.0444	3.8439	2.3381	0.4770	0.5230
8.6	1.9014	4.0443	3.8441	2.3379	0.4767	0.5233

*** error *** 30 minor iterations done without converging

Example 2 with parameters alpha1 = 0.2 beta1 = 0.5 gamma1 = 0.3
alpha2 = 0.5 beta2 = 0.3 gamma2 = 0.2
Solved with the explicit Euler method.

	consumer 1			consumer 2			prices		
time	x1	x2	x3	x1	x2	x3	p1	p2	p3
0.0	4.0000	1.0000	3.2000	2.0000	3.0000	7.0000	0.5000	0.2000	0.3000
0.1	3.7264	1.6900	3.1960	2.1700	3.2550	6.5467	0.5000	0.2000	0.3000
0.2	3.4802	2.3110	3.1924	2.3230	3.4845	6.1387	0.5000	0.2000	0.3000
0.3	3.2627	2.7955	3.1869	2.4778	3.6454	5.7730	0.4808	0.2192	0.3000
0.4	3.0743	3.1488	3.1807	2.6443	3.7371	5.4460	0.4531	0.2469	0.3000
0.5	2.9083	3.4395	3.1752	2.8060	3.8014	5.1525	0.4420	0.2580	0.3000
0.6	2.7620	3.6820	3.1707	2.9613	3.8456	4.8889	0.4331	0.2669	0.3000
0.7	2.6331	3.8862	3.1673	3.1091	3.8752	4.6522	0.4260	0.2740	0.3000
0.8	2.5194	4.0595	3.1649	3.2486	3.8937	4.4394	0.4203	0.2797	0.3000
0.9	2.4191	4.2076	3.1635	3.3796	3.9041	4.2481	0.4156	0.2844	0.3000
1.0	2.3305	4.3348	3.1629	3.5019	3.9085	4.0760	0.4119	0.2881	0.3000
1.1	2.2522	4.4446	3.1630	3.6155	3.9084	3.9212	0.4088	0.2912	0.3000
1.2	2.1830	4.5398	3.1636	3.7207	3.9051	3.7820	0.4064	0.2936	0.3000
1.3	2.1217	4.6226	3.1647	3.8177	3.8995	3.6567	0.4043	0.2957	0.3000
1.4	2.0673	4.6949	3.1660	3.9070	3.8924	3.5439	0.4027	0.2973	0.3000
1.5	2.0191	4.7581	3.1676	3.9889	3.8843	3.4425	0.4014	0.2986	0.3000
1.6	1.9763	4.8136	3.1693	4.0638	3.8756	3.3511	0.4003	0.2997	0.3000
1.7	1.9383	4.8624	3.1710	4.1323	3.8667	3.2689	0.3994	0.3006	0.3000
1.8	1.9044	4.9054	3.1728	4.1947	3.8577	3.1949	0.3987	0.3013	0.3000
1.9	1.8743	4.9434	3.1746	4.2516	3.8489	3.1283	0.3982	0.3018	0.3000
2.0	1.8474	4.9770	3.1764	4.3033	3.8404	3.0683	0.3977	0.3023	0.3000
2.1	1.8234	5.0068	3.1781	4.3503	3.8323	3.0143	0.3973	0.3027	0.3000
2.2	1.8020	5.0333	3.1797	4.3929	3.8245	2.9657	0.3970	0.3030	0.3000
2.3	1.7828	5.0568	3.1813	4.4315	3.8173	2.9220	0.3968	0.3032	0.3000
2.4	1.7657	5.0777	3.1827	4.4665	3.8105	2.8826	0.3966	0.3034	0.3000
2.5	1.7504	5.0963	3.1841	4.4982	3.8041	2.8472	0.3964	0.3036	0.3000
2.6	1.7368	5.1130	3.1852	4.5270	3.7983	2.8151	0.3962	0.3037	0.3001
2.7	1.7246	5.1279	3.1862	4.5530	3.7928	2.7862	0.3960	0.3038	0.3002
2.8	1.7137	5.1412	3.1871	4.5767	3.7879	2.7601	0.3958	0.3039	0.3003
2.9	1.7039	5.1532	3.1879	4.5980	3.7833	2.7366	0.3958	0.3040	0.3003
3.0	1.6952	5.1638	3.1886	4.6173	3.7791	2.7155	0.3957	0.3040	0.3003
3.1	1.6874	5.1734	3.1893	4.6347	3.7752	2.6965	0.3956	0.3041	0.3003
3.2	1.6803	5.1819	3.1899	4.6504	3.7717	2.6793	0.3956	0.3041	0.3003
3.3	1.6740	5.1896	3.1905	4.6645	3.7685	2.6639	0.3956	0.3041	0.3003
3.4	1.6683	5.1964	3.1910	4.6773	3.7656	2.6500	0.3956	0.3042	0.3003
3.5	1.6632	5.2026	3.1915	4.6888	3.7629	2.6376	0.3955	0.3042	0.3003
3.6	1.6587	5.2081	3.1919	4.6992	3.7605	2.6263	0.3955	0.3042	0.3003
3.7	1.6546	5.2130	3.1923	4.7086	3.7584	2.6162	0.3955	0.3042	0.3003
3.8	1.6509	5.2175	3.1927	4.7170	3.7564	2.6071	0.3955	0.3042	0.3003
3.9	1.6476	5.2215	3.1930	4.7246	3.7546	2.5989	0.3955	0.3042	0.3003
4.0	1.6446	5.2250	3.1933	4.7315	3.7530	2.5915	0.3955	0.3042	0.3003
4.1	1.6419	5.2283	3.1935	4.7377	3.7515	2.5849	0.3955	0.3042	0.3003
4.2	1.6395	5.2311	3.1938	4.7432	3.7502	2.5789	0.3955	0.3043	0.3003
4.3	1.6373	5.2337	3.1940	4.7482	3.7490	2.5735	0.3955	0.3043	0.3003
4.4	1.6354	5.2361	3.1942	4.7527	3.7479	2.5687	0.3955	0.3043	0.3003
4.5	1.6336	5.2382	3.1944	4.7568	3.7469	2.5643	0.3955	0.3043	0.3003

4.6	1.6321	5.2401	3.1946	4.7605	3.7460	2.5604	0.3955	0.3043	0.3003
4.7	1.6306	5.2417	3.1947	4.7638	3.7452	2.5568	0.3955	0.3043	0.3003
4.8	1.6294	5.2433	3.1948	4.7667	3.7445	2.5537	0.3955	0.3043	0.3003
4.9	1.6282	5.2446	3.1950	4.7694	3.7438	2.5508	0.3955	0.3043	0.3003
5.0	1.6272	5.2459	3.1951	4.7718	3.7433	2.5482	0.3955	0.3043	0.3003
5.1	1.6263	5.2470	3.1952	4.7740	3.7427	2.5459	0.3955	0.3043	0.3003
5.2	1.6254	5.2480	3.1953	4.7759	3.7423	2.5438	0.3955	0.3043	0.3003
5.3	1.6247	5.2489	3.1953	4.7777	3.7418	2.5420	0.3955	0.3043	0.3003
5.4	1.6240	5.2497	3.1954	4.7792	3.7414	2.5403	0.3955	0.3043	0.3003
5.5	1.6234	5.2504	3.1955	4.7807	3.7411	2.5388	0.3954	0.3043	0.3003
5.6	1.6229	5.2511	3.1955	4.7820	3.7408	2.5374	0.3954	0.3043	0.3003
5.7	1.6224	5.2516	3.1956	4.7831	3.7405	2.5362	0.3954	0.3043	0.3003
5.8	1.6219	5.2522	3.1956	4.7842	3.7402	2.5350	0.3954	0.3043	0.3003
5.9	1.6215	5.2526	3.1957	4.7851	3.7400	2.5340	0.3954	0.3043	0.3003
6.0	1.6212	5.2531	3.1957	4.7859	3.7398	2.5331	0.3954	0.3043	0.3003
6.1	1.6208	5.2534	3.1958	4.7867	3.7396	2.5323	0.3954	0.3043	0.3003
6.2	1.6206	5.2538	3.1958	4.7874	3.7394	2.5316	0.3954	0.3043	0.3003
6.3	1.6203	5.2541	3.1958	4.7880	3.7393	2.5310	0.3954	0.3043	0.3003
6.4	1.6201	5.2544	3.1958	4.7885	3.7391	2.5304	0.3954	0.3043	0.3003
6.5	1.6199	5.2546	3.1959	4.7890	3.7390	2.5298	0.3954	0.3043	0.3003
6.6	1.6197	5.2548	3.1959	4.7895	3.7389	2.5294	0.3954	0.3043	0.3003
6.7	1.6195	5.2551	3.1959	4.7899	3.7388	2.5289	0.3954	0.3043	0.3003
6.8	1.6193	5.2552	3.1959	4.7903	3.7387	2.5285	0.3954	0.3043	0.3003
6.9	1.6192	5.2554	3.1959	4.7906	3.7386	2.5282	0.3954	0.3043	0.3003
7.0	1.6191	5.2555	3.1959	4.7909	3.7386	2.5279	0.3954	0.3043	0.3003
7.1	1.6190	5.2557	3.1960	4.7911	3.7385	2.5276	0.3954	0.3043	0.3003
7.2	1.6189	5.2558	3.1960	4.7914	3.7384	2.5273	0.3954	0.3043	0.3003
7.3	1.6188	5.2559	3.1960	4.7916	3.7384	2.5271	0.3954	0.3043	0.3003
7.4	1.6187	5.2560	3.1960	4.7918	3.7383	2.5269	0.3954	0.3043	0.3003
7.5	1.6186	5.2561	3.1960	4.7920	3.7383	2.5267	0.3954	0.3043	0.3003
7.6	1.6186	5.2562	3.1960	4.7921	3.7382	2.5266	0.3954	0.3043	0.3003
7.7	1.6185	5.2562	3.1960	4.7923	3.7382	2.5264	0.3954	0.3043	0.3003
7.8	1.6184	5.2563	3.1960	4.7924	3.7382	2.5263	0.3954	0.3043	0.3003
7.9	1.6184	5.2564	3.1960	4.7925	3.7382	2.5262	0.3954	0.3043	0.3003
8.0	1.6183	5.2564	3.1960	4.7926	3.7381	2.5260	0.3954	0.3043	0.3003
8.1	1.6183	5.2565	3.1960	4.7927	3.7381	2.5260	0.3954	0.3043	0.3003
8.2	1.6183	5.2565	3.1960	4.7928	3.7381	2.5259	0.3954	0.3043	0.3003
8.3	1.6182	5.2565	3.1960	4.7929	3.7381	2.5258	0.3954	0.3043	0.3003
8.4	1.6182	5.2566	3.1960	4.7929	3.7380	2.5257	0.3954	0.3043	0.3003
8.5	1.6182	5.2566	3.1960	4.7930	3.7380	2.5256	0.3954	0.3043	0.3003
8.6	1.6182	5.2566	3.1960	4.7930	3.7380	2.5256	0.3954	0.3043	0.3003
8.7	1.6181	5.2567	3.1961	4.7931	3.7380	2.5255	0.3954	0.3043	0.3003
8.8	1.6181	5.2567	3.1961	4.7931	3.7380	2.5255	0.3954	0.3043	0.3003
8.9	1.6181	5.2567	3.1961	4.7932	3.7380	2.5254	0.3954	0.3043	0.3003
9.0	1.6181	5.2567	3.1961	4.7932	3.7380	2.5254	0.3954	0.3043	0.3003
9.1	1.6181	5.2567	3.1961	4.7932	3.7380	2.5254	0.3954	0.3043	0.3003
9.2	1.6181	5.2568	3.1961	4.7933	3.7380	2.5253	0.3954	0.3043	0.3003
9.3	1.6181	5.2568	3.1961	4.7933	3.7380	2.5253	0.3954	0.3043	0.3003
9.4	1.6180	5.2568	3.1961	4.7933	3.7379	2.5253	0.3954	0.3043	0.3003
9.5	1.6180	5.2568	3.1961	4.7933	3.7379	2.5253	0.3954	0.3043	0.3003
9.6	1.6180	5.2568	3.1961	4.7934	3.7379	2.5252	0.3954	0.3043	0.3003
9.7	1.6180	5.2568	3.1961	4.7934	3.7379	2.5252	0.3954	0.3043	0.3003

Example 2 with parameters alpha1 = 0.2 beta1 = 0.5 gamma1 = 0.3
alpha2 = 0.5 beta2 = 0.3 gamma2 = 0.2
Solved with the implicit Euler method.

	consumer 1			consumer 2			prices		
time	x1	x2	x3	x1	x2	x3	p1	p2	p3
0.0	4.0000	1.0000	3.2000	2.0000	3.0000	7.0000	0.5000	0.2000	0.3000
0.1	3.7551	1.6826	3.1744	2.1751	3.2851	6.5763	0.4866	0.1868	0.3267
0.2	3.5235	2.2341	3.2076	2.2872	3.4640	6.2208	0.5261	0.2048	0.2691
0.3	3.3275	2.6586	3.1868	2.4561	3.6054	5.8740	0.4578	0.2268	0.3154
0.4	3.1683	2.9896	3.1341	2.6889	3.7154	5.5413	0.3950	0.2470	0.3580
0.5	3.0174	3.2718	3.1045	2.8743	3.7942	5.2508	0.4124	0.2553	0.3324
0.6	2.8748	3.4985	3.1048	3.0189	3.8369	5.0056	0.4296	0.2699	0.3005
0.7	2.7530	3.6970	3.0892	3.1779	3.8733	4.7716	0.4066	0.2745	0.3190
0.8	2.6404	3.8650	3.0864	3.3148	3.8953	4.5671	0.4123	0.2815	0.3062
0.9	2.5455	4.0098	3.0728	3.4630	3.9099	4.3723	0.3925	0.2872	0.3203
1.0	2.4597	4.1338	3.0647	3.5980	3.9167	4.1983	0.3921	0.2922	0.3157
1.1	2.3828	4.2406	3.0593	3.7235	3.9179	4.0414	0.3898	0.2963	0.3139
1.2	2.3191	4.3319	3.0484	3.8546	3.9134	3.8929	0.3760	0.3013	0.3227
1.3	2.2634	4.4098	3.0387	3.9797	3.9045	3.7576	0.3714	0.3054	0.3232
1.4	2.2154	4.4759	3.0298	4.1000	3.8919	3.6337	0.3663	0.3095	0.3243
1.5	2.1746	4.5314	3.0217	4.2160	3.8759	3.5204	0.3611	0.3135	0.3254
1.6	2.1404	4.5773	3.0143	4.3283	3.8568	3.4165	0.3559	0.3176	0.3264
1.7	2.1123	4.6144	3.0079	4.4374	3.8346	3.3214	0.3507	0.3221	0.3272
1.8	2.0898	4.6432	3.0028	4.5437	3.8093	3.2345	0.3456	0.3269	0.3276
1.9	2.0727	4.6640	2.9996	4.6474	3.7808	3.1555	0.3405	0.3322	0.3273
2.0	2.0601	4.6773	2.9988	4.7481	3.7493	3.0844	0.3361	0.3379	0.3260
2.1	2.0517	4.6831	3.0013	4.8458	3.7146	3.0211	0.3320	0.3444	0.3237
2.2	2.0446	4.6849	3.0067	4.9356	3.6801	2.9657	0.3314	0.3478	0.3207
2.3	2.0388	4.6831	3.0146	5.0182	3.6459	2.9174	0.3310	0.3513	0.3178
2.4	2.0342	4.6779	3.0254	5.0944	3.6118	2.8759	0.3304	0.3550	0.3145
2.5	2.0304	4.6697	3.0389	5.1641	3.5780	2.8409	0.3303	0.3586	0.3111
2.6	2.0271	4.6588	3.0553	5.2276	3.5445	2.8121	0.3306	0.3623	0.3071
2.7	2.0238	4.6453	3.0752	5.2845	3.5115	2.7897	0.3315	0.3660	0.3025
2.8	2.0200	4.6299	3.0986	5.3340	3.4795	2.7738	0.3335	0.3691	0.2974
2.9	2.0152	4.6195	3.1165	5.3767	3.4539	2.7573	0.3351	0.3649	0.3000
3.0	2.0102	4.6079	3.1365	5.4141	3.4292	2.7454	0.3364	0.3671	0.2965
3.1	2.0048	4.6001	3.1523	5.4463	3.4093	2.7332	0.3376	0.3640	0.2984
3.2	1.9994	4.5908	3.1699	5.4747	3.3898	2.7248	0.3385	0.3661	0.2954
3.3	1.9939	4.5843	3.1842	5.4994	3.3739	2.7161	0.3394	0.3640	0.2966
3.4	1.9886	4.5767	3.1998	5.5211	3.3583	2.7104	0.3402	0.3657	0.2942
3.5	1.9833	4.5709	3.2130	5.5400	3.3452	2.7047	0.3407	0.3643	0.2950
3.6	1.9781	4.5642	3.2275	5.5563	3.3325	2.7016	0.3416	0.3657	0.2927
3.7	1.9731	4.5588	3.2401	5.5706	3.3216	2.6985	0.3420	0.3649	0.2930
3.8	1.9682	4.5545	3.2511	5.5832	3.3123	2.6954	0.3423	0.3643	0.2934
3.9	1.9637	4.5511	3.2606	5.5942	3.3043	2.6923	0.3426	0.3637	0.2937
4.0	1.9593	4.5485	3.2689	5.6040	3.2975	2.6894	0.3429	0.3631	0.2940
4.1	1.9553	4.5455	3.2774	5.6126	3.2909	2.6874	0.3431	0.3638	0.2931
4.2	1.9515	4.5430	3.2849	5.6203	3.2852	2.6856	0.3433	0.3635	0.2933
4.3	1.9480	4.5410	3.2915	5.6271	3.2802	2.6838	0.3434	0.3632	0.2934
4.4	1.9448	4.5389	3.2978	5.6332	3.2755	2.6824	0.3434	0.3635	0.2931
4.5	1.9417	4.5372	3.3035	5.6386	3.2715	2.6812	0.3436	0.3632	0.2932

4.6	1.9389	4.5354	3.3090	5.6434	3.2676	2.6803	0.3437	0.3634	0.2929
4.7	1.9363	4.5336	3.3144	5.6475	3.2640	2.6799	0.3439	0.3637	0.2924
4.8	1.9341	4.5313	3.3198	5.6517	3.2602	2.6798	0.3436	0.3643	0.2921
4.9	1.9316	4.5300	3.3244	5.6545	3.2576	2.6797	0.3445	0.3633	0.2922
5.0	1.9292	4.5286	3.3290	5.6569	3.2550	2.6800	0.3445	0.3637	0.2918
5.1	1.9271	4.5269	3.3336	5.6591	3.2525	2.6806	0.3446	0.3640	0.2915
5.2	1.9252	4.5251	3.3381	5.6611	3.2499	2.6814	0.3446	0.3643	0.2911
5.3	1.9234	4.5233	3.3425	5.6628	3.2475	2.6824	0.3447	0.3644	0.2909
5.4	1.9217	4.5213	3.3469	5.6644	3.2451	2.6836	0.3447	0.3647	0.2905
5.5	1.9202	4.5193	3.3513	5.6658	3.2427	2.6850	0.3448	0.3650	0.2902
5.6	1.9179	4.5184	3.3552	5.6652	3.2418	2.6867	0.3463	0.3637	0.2900
5.7	1.9157	4.5175	3.3590	5.6646	3.2410	2.6886	0.3465	0.3638	0.2897
5.8	1.9136	4.5164	3.3629	5.6637	3.2402	2.6906	0.3467	0.3640	0.2893
5.9	1.9115	4.5153	3.3667	5.6628	3.2394	2.6927	0.3469	0.3641	0.2890
6.0	1.9096	4.5141	3.3705	5.6618	3.2385	2.6950	0.3470	0.3643	0.2887
6.1	1.9077	4.5129	3.3744	5.6605	3.2378	2.6974	0.3473	0.3644	0.2883
6.2	1.9058	4.5117	3.3782	5.6591	3.2372	2.7000	0.3476	0.3644	0.2879
6.3	1.9039	4.5104	3.3821	5.6574	3.2366	2.7027	0.3479	0.3645	0.2876
6.4	1.9015	4.5116	3.3834	5.6548	3.2382	2.7039	0.3488	0.3620	0.2892
6.5	1.8993	4.5128	3.3846	5.6523	3.2398	2.7049	0.3489	0.3619	0.2892
6.6	1.8972	4.5140	3.3857	5.6499	3.2413	2.7059	0.3491	0.3617	0.2892
6.7	1.8952	4.5152	3.3867	5.6475	3.2429	2.7068	0.3492	0.3616	0.2892
6.8	1.8933	4.5163	3.3875	5.6452	3.2444	2.7077	0.3493	0.3615	0.2892
6.9	1.8915	4.5174	3.3882	5.6431	3.2459	2.7085	0.3494	0.3614	0.2892
7.0	1.8898	4.5184	3.3892	5.6408	3.2473	2.7095	0.3497	0.3614	0.2889
7.1	1.8881	4.5193	3.3901	5.6386	3.2486	2.7105	0.3498	0.3614	0.2888
7.2	1.8865	4.5200	3.3910	5.6365	3.2498	2.7115	0.3499	0.3614	0.2887
7.3	1.8851	4.5207	3.3919	5.6345	3.2509	2.7125	0.3499	0.3614	0.2887
7.4	1.8837	4.5213	3.3929	5.6326	3.2519	2.7136	0.3500	0.3614	0.2886
7.5	1.8823	4.5219	3.3938	5.6308	3.2529	2.7146	0.3501	0.3614	0.2885
7.6	1.8811	4.5223	3.3947	5.6290	3.2537	2.7156	0.3502	0.3614	0.2884
7.7	1.8799	4.5227	3.3957	5.6273	3.2546	2.7167	0.3503	0.3614	0.2883
7.8	1.8788	4.5231	3.3966	5.6257	3.2553	2.7177	0.3504	0.3614	0.2882
7.9	1.8779	4.5236	3.3970	5.6245	3.2561	2.7182	0.3501	0.3613	0.2886
8.0	1.8774	4.5241	3.3970	5.6240	3.2566	2.7182	0.3496	0.3613	0.2891
8.1	1.8770	4.5245	3.3970	5.6235	3.2570	2.7182	0.3496	0.3613	0.2891
8.2	1.8766	4.5249	3.3970	5.6231	3.2574	2.7182	0.3496	0.3613	0.2891
8.3	1.8762	4.5253	3.3969	5.6228	3.2578	2.7181	0.3496	0.3613	0.2891
8.4	1.8759	4.5256	3.3969	5.6224	3.2582	2.7181	0.3496	0.3613	0.2891
8.5	1.8756	4.5259	3.3969	5.6221	3.2585	2.7181	0.3496	0.3613	0.2891
8.6	1.8753	4.5262	3.3968	5.6219	3.2588	2.7180	0.3496	0.3613	0.2891
8.7	1.8751	4.5265	3.3968	5.6216	3.2590	2.7180	0.3496	0.3613	0.2891
8.8	1.8749	4.5267	3.3968	5.6214	3.2592	2.7180	0.3496	0.3613	0.2891
8.9	1.8747	4.5269	3.3967	5.6212	3.2595	2.7180	0.3496	0.3613	0.2891
9.0	1.8745	4.5271	3.3967	5.6210	3.2596	2.7179	0.3496	0.3613	0.2891
9.1	1.8744	4.5273	3.3967	5.6209	3.2598	2.7179	0.3495	0.3613	0.2891
9.2	1.8742	4.5275	3.3967	5.6207	3.2600	2.7179	0.3495	0.3613	0.2891
9.3	1.8741	4.5276	3.3966	5.6206	3.2601	2.7178	0.3495	0.3613	0.2891
9.4	1.8740	4.5278	3.3966	5.6205	3.2602	2.7178	0.3495	0.3613	0.2891
9.5	1.8739	4.5279	3.3966	5.6204	3.2604	2.7178	0.3495	0.3613	0.2891
9.6	1.8738	4.5280	3.3965	5.6203	3.2605	2.7178	0.3495	0.3613	0.2891
9.7	1.8737	4.5281	3.3965	5.6203	3.2606	2.7177	0.3495	0.3613	0.2891

Example 3
Solved with the explicit Euler method.

time	consumer 1 x1	consumer 1 x2	consumer 2 x1	consumer 2 x2	prices p1	prices p2
0.0	3.0000	1.5000	1.5000	3.0000	0.6000	0.4000
0.1	2.3417	1.6829	1.5579	2.7696	0.3368	0.6632
0.2	1.9831	1.6906	1.6102	2.5604	0.3368	0.6632
0.3	1.7413	1.6285	1.6544	2.3708	0.3368	0.6632
0.4	1.5587	1.5359	1.6892	2.1991	0.3368	0.6632
0.5	1.4109	1.4308	1.7138	2.0435	0.3368	0.6632
0.6	1.2856	1.3225	1.7280	1.9026	0.3368	0.6632
0.7	1.1762	1.2160	1.7319	1.7747	0.3368	0.6632
0.8	1.0787	1.1137	1.7260	1.6587	0.3368	0.6632
0.9	0.9908	1.0171	1.7109	1.5532	0.3368	0.6632
1.0	0.9107	0.9267	1.6876	1.4572	0.3368	0.6632
1.1	0.8374	0.8428	1.6570	1.3695	0.3368	0.6632
1.2	0.7700	0.7652	1.6200	1.2893	0.3368	0.6632
1.3	0.7079	0.6938	1.5776	1.2158	0.3368	0.6632
1.4	0.6507	0.6283	1.5307	1.1483	0.3368	0.6632
1.5	0.5978	0.5684	1.4803	1.0859	0.3368	0.6632
1.6	0.5490	0.5136	1.4272	1.0282	0.3368	0.6632
1.7	0.5040	0.4637	1.3722	0.9746	0.3368	0.6632
1.8	0.4624	0.4183	1.3160	0.9247	0.3368	0.6632
1.9	0.4239	0.3771	1.2593	0.8780	0.3368	0.6632
2.0	0.3885	0.3397	1.2026	0.8342	0.3368	0.6632
2.1	0.3558	0.3057	1.1464	0.7930	0.3368	0.6632
2.2	0.3257	0.2750	1.0910	0.7540	0.3368	0.6632
2.3	0.2980	0.2473	1.0368	0.7171	0.3368	0.6632
2.4	0.2725	0.2222	0.9841	0.6821	0.3368	0.6632
2.5	0.2490	0.1996	0.9330	0.6488	0.3368	0.6632
2.6	0.2274	0.1793	0.8837	0.6171	0.3368	0.6632
2.7	0.2076	0.1609	0.8362	0.5868	0.3368	0.6632
2.8	0.1894	0.1444	0.7908	0.5579	0.3368	0.6632
2.9	0.1727	0.1295	0.7474	0.5302	0.3368	0.6632
3.0	0.1574	0.1161	0.7059	0.5038	0.3368	0.6632
3.1	0.1434	0.1041	0.6665	0.4785	0.3368	0.6632
3.2	0.1305	0.0933	0.6289	0.4544	0.3368	0.6632
3.3	0.1188	0.0837	0.5933	0.4313	0.3368	0.6632
3.4	0.1080	0.0750	0.5596	0.4092	0.3368	0.6632
3.5	0.0982	0.0672	0.5277	0.3881	0.3368	0.6632
3.6	0.0892	0.0602	0.4974	0.3679	0.3368	0.6632
3.7	0.0811	0.0539	0.4688	0.3486	0.3368	0.6632
3.8	0.0736	0.0483	0.4418	0.3303	0.3368	0.6632
3.9	0.0668	0.0433	0.4163	0.3128	0.3368	0.6632
4.0	0.0606	0.0388	0.3922	0.2961	0.3368	0.6632
4.1	0.0550	0.0348	0.3695	0.2802	0.3368	0.6632
4.2	0.0498	0.0312	0.3481	0.2650	0.3368	0.6632
4.3	0.0451	0.0279	0.3279	0.2507	0.3368	0.6632
4.4	0.0409	0.0250	0.3088	0.2370	0.3368	0.6632
4.5	0.0370	0.0224	0.2908	0.2240	0.3368	0.6632
4.6	0.0335	0.0201	0.2739	0.2117	0.3368	0.6632

4.7	0.0303	0.0180	0.2579	0.1999	0.3368	0.6632
4.8	0.0274	0.0162	0.2429	0.1888	0.3368	0.6632
4.9	0.0248	0.0145	0.2287	0.1783	0.3368	0.6632
5.0	0.0224	0.0130	0.2154	0.1683	0.3368	0.6632
5.1	0.0203	0.0116	0.2028	0.1589	0.3368	0.6632
5.2	0.0183	0.0104	0.1909	0.1499	0.3368	0.6632
5.3	0.0166	0.0094	0.1797	0.1415	0.3368	0.6632
5.4	0.0150	0.0084	0.1692	0.1335	0.3368	0.6632
5.5	0.0135	0.0075	0.1593	0.1259	0.3368	0.6632
5.6	0.0122	0.0068	0.1500	0.1187	0.3368	0.6632
5.7	0.0110	0.0061	0.1412	0.1119	0.3368	0.6632
5.8	0.0099	0.0055	0.1329	0.1055	0.3368	0.6632
5.9	0.0090	0.0049	0.1251	0.0995	0.3368	0.6632
6.0	0.0081	0.0044	0.1178	0.0938	0.3368	0.6632
6.1	0.0073	0.0039	0.1109	0.0884	0.3368	0.6632
6.2	0.0066	0.0035	0.1044	0.0833	0.3368	0.6632
6.3	0.0059	0.0032	0.0982	0.0785	0.3368	0.6632
6.4	0.0054	0.0029	0.0925	0.0740	0.3368	0.6632
6.5	0.0048	0.0026	0.0870	0.0697	0.3368	0.6632
6.6	0.0043	0.0023	0.0819	0.0656	0.3368	0.6632
6.7	0.0039	0.0021	0.0771	0.0618	0.3368	0.6632
6.8	0.0035	0.0019	0.0726	0.0582	0.3368	0.6632
6.9	0.0032	0.0017	0.0683	0.0549	0.3368	0.6632
7.0	0.0029	0.0015	0.0643	0.0517	0.3368	0.6632
7.1	0.0026	0.0014	0.0605	0.0487	0.3368	0.6632
7.2	0.0023	0.0012	0.0569	0.0458	0.3368	0.6632
7.3	0.0021	0.0011	0.0536	0.0432	0.3368	0.6632
7.4	0.0019	0.0010	0.0504	0.0406	0.3368	0.6632
7.5	0.0017	0.0009	0.0475	0.0383	0.3368	0.6632
7.6	0.0015	0.0008	0.0447	0.0360	0.3368	0.6632
7.7	0.0014	0.0007	0.0420	0.0339	0.3368	0.6632
7.8	0.0012	0.0006	0.0396	0.0319	0.3368	0.6632
7.9	0.0011	0.0006	0.0372	0.0301	0.3368	0.6632
8.0	0.0010	0.0005	0.0350	0.0283	0.3368	0.6632
8.1	0.0009	0.0005	0.0330	0.0266	0.3368	0.6632
8.2	0.0008	0.0004	0.0310	0.0251	0.3368	0.6632
8.3	0.0007	0.0004	0.0292	0.0236	0.3368	0.6632
8.4	0.0007	0.0003	0.0275	0.0222	0.3368	0.6632
8.5	0.0006	0.0003	0.0259	0.0209	0.3368	0.6632
8.6	0.0005	0.0003	0.0243	0.0197	0.3368	0.6632
8.7	0.0005	0.0002	0.0229	0.0185	0.3368	0.6632
8.8	0.0004	0.0002	0.0215	0.0174	0.3368	0.6632
8.9	0.0004	0.0002	0.0203	0.0164	0.3368	0.6632
9.0	0.0004	0.0002	0.0191	0.0155	0.3368	0.6632
9.1	0.0003	0.0002	0.0180	0.0145	0.3368	0.6632
9.2	0.0003	0.0001	0.0169	0.0137	0.3368	0.6632
9.3	0.0003	0.0001	0.0159	0.0129	0.3368	0.6632
9.4	0.0002	0.0001	0.0150	0.0121	0.3368	0.6632
9.5	0.0002	0.0001	0.0141	0.0114	0.3368	0.6632
9.6	0.0002	0.0001	0.0132	0.0107	0.3368	0.6632
9.7	0.0002	0.0001	0.0125	0.0101	0.3368	0.6632
9.8	0.0002	0.0001	0.0117	0.0095	0.3368	0.6632

Example 3
Solved with the implicit Euler method.

	consumer 1		consumer 2		prices	
time	x1	x2	x1	x2	p1	p2
0.0	3.0000	1.5000	1.5000	3.0000	0.6000	0.4000
0.1	2.6060	1.7051	1.3973	2.8946	0.6199	0.3801
0.2	2.2960	1.7964	1.2489	2.8191	0.7337	0.2663
0.3	2.0249	1.7892	1.0756	2.7547	0.8323	0.1677
0.4	1.7903	1.7087	0.9557	2.6502	0.7656	0.2344
0.5	1.5867	1.5949	0.8662	2.5249	0.7323	0.2677
0.6	1.4033	1.4654	0.7820	2.3953	0.7526	0.2474
0.7	1.2376	1.3307	0.7040	2.2614	0.7705	0.2295
0.8	1.0880	1.1973	0.6325	2.1242	0.7867	0.2133
0.9	0.9534	1.0694	0.5672	1.9848	0.8016	0.1984
1.0	0.8326	0.9495	0.5079	1.8445	0.8155	0.1845
1.1	0.7248	0.8390	0.4541	1.7049	0.8284	0.1716
1.2	0.6289	0.7382	0.4054	1.5675	0.8404	0.1596
1.3	0.5440	0.6473	0.3615	1.4336	0.8515	0.1485
1.4	0.4693	0.5659	0.3218	1.3046	0.8616	0.1384
1.5	0.4038	0.4934	0.2860	1.1815	0.8708	0.1292
1.6	0.3466	0.4294	0.2539	1.0651	0.8792	0.1208
1.7	0.2969	0.3729	0.2250	0.9561	0.8868	0.1132
1.8	0.2538	0.3233	0.1990	0.8549	0.8945	0.1055
1.9	0.2165	0.2799	0.1755	0.7615	0.9024	0.0976
2.0	0.1843	0.2419	0.1544	0.6760	0.9102	0.0898
2.1	0.1566	0.2089	0.1353	0.5981	0.9179	0.0821
2.2	0.1329	0.1801	0.1182	0.5276	0.9252	0.0748
2.3	0.1125	0.1552	0.1029	0.4640	0.9322	0.0678
2.4	0.0951	0.1335	0.0893	0.4071	0.9388	0.0612
2.5	0.0803	0.1148	0.0772	0.3562	0.9449	0.0551
2.6	0.0677	0.0986	0.0666	0.3110	0.9500	0.0500
2.7	0.0571	0.0847	0.0575	0.2711	0.9500	0.0500
2.8	0.0481	0.0727	0.0497	0.2359	0.9500	0.0500
2.9	0.0406	0.0624	0.0431	0.2051	0.9500	0.0500
3.0	0.0342	0.0535	0.0373	0.1781	0.9500	0.0500
3.1	0.0288	0.0459	0.0323	0.1545	0.9500	0.0500
3.2	0.0243	0.0393	0.0280	0.1339	0.9500	0.0500
3.3	0.0205	0.0337	0.0243	0.1160	0.9500	0.0500
3.4	0.0173	0.0289	0.0211	0.1004	0.9500	0.0500
3.5	0.0146	0.0247	0.0183	0.0869	0.9500	0.0500
3.6	0.0123	0.0212	0.0159	0.0752	0.9500	0.0500
3.7	0.0104	0.0181	0.0138	0.0651	0.9500	0.0500
3.8	0.0087	0.0155	0.0120	0.0563	0.9500	0.0500
3.9	0.0074	0.0132	0.0104	0.0487	0.9500	0.0500
4.0	0.0062	0.0113	0.0090	0.0421	0.9500	0.0500
4.1	0.0052	0.0097	0.0078	0.0364	0.9500	0.0500
4.2	0.0044	0.0083	0.0068	0.0314	0.9500	0.0500
4.3	0.0037	0.0071	0.0059	0.0272	0.9500	0.0500
4.4	0.0031	0.0060	0.0051	0.0235	0.9500	0.0500
4.5	0.0027	0.0052	0.0044	0.0203	0.9500	0.0500
4.6	0.0022	0.0044	0.0039	0.0176	0.9500	0.0500

4.7	0.0019	0.0038	0.0033	0.0152	0.9500	0.0500
4.8	0.0016	0.0032	0.0029	0.0131	0.9500	0.0500
4.9	0.0013	0.0027	0.0025	0.0113	0.9500	0.0500
5.0	0.0011	0.0023	0.0022	0.0098	0.9500	0.0500
5.1	0.0010	0.0020	0.0019	0.0085	0.9500	0.0500
5.2	0.0008	0.0017	0.0016	0.0073	0.9500	0.0500
5.3	0.0007	0.0015	0.0014	0.0063	0.9500	0.0500
5.4	0.0006	0.0012	0.0012	0.0055	0.9500	0.0500
5.5	0.0005	0.0011	0.0011	0.0047	0.9500	0.0500
5.6	0.0004	0.0009	0.0009	0.0041	0.9500	0.0500
5.7	0.0003	0.0008	0.0008	0.0036	0.9500	0.0500
5.8	0.0003	0.0007	0.0007	0.0031	0.9500	0.0500
5.9	0.0002	0.0006	0.0006	0.0027	0.9500	0.0500
6.0	0.0002	0.0005	0.0005	0.0023	0.9500	0.0500
6.1	0.0002	0.0004	0.0005	0.0020	0.9500	0.0500
6.2	0.0001	0.0003	0.0004	0.0017	0.9500	0.0500
6.3	0.0001	0.0003	0.0003	0.0015	0.9500	0.0500
6.4	0.0001	0.0002	0.0003	0.0013	0.9500	0.0500
6.5	0.0001	0.0002	0.0003	0.0011	0.9500	0.0500
6.6	0.0001	0.0002	0.0002	0.0010	0.9500	0.0500
6.7	0.0001	0.0002	0.0002	0.0008	0.9500	0.0500
6.8	0.0001	0.0001	0.0002	0.0007	0.9500	0.0500
6.9	0.0000	0.0001	0.0001	0.0006	0.9500	0.0500
7.0	0.0000	0.0001	0.0001	0.0005	0.9500	0.0500
7.1	0.0000	0.0001	0.0001	0.0005	0.9500	0.0500
7.2	0.0000	0.0001	0.0001	0.0004	0.9500	0.0500
7.3	0.0000	0.0001	0.0001	0.0004	0.9500	0.0500
7.4	0.0000	0.0000	0.0001	0.0003	0.9500	0.0500
7.5	0.0000	0.0000	0.0001	0.0003	0.9500	0.0500
7.6	0.0000	0.0000	0.0001	0.0002	0.9500	0.0500
7.7	0.0000	0.0000	0.0000	0.0002	0.9500	0.0500
7.8	0.0000	0.0000	0.0000	0.0002	0.9500	0.0500
7.9	0.0000	0.0000	0.0000	0.0001	0.9500	0.0500
8.0	0.0000	0.0000	0.0000	0.0001	0.9500	0.0500
8.1	0.0000	0.0000	0.0000	0.0001	0.9500	0.0500
8.2	0.0000	0.0000	0.0000	0.0001	0.9500	0.0500
8.3	0.0000	0.0000	0.0000	0.0001	0.9500	0.0500
8.4	0.0000	0.0000	0.0000	0.0001	0.9500	0.0500
8.5	0.0000	0.0000	0.0000	0.0001	0.9500	0.0500
8.6	0.0000	0.0000	0.0000	0.0001	0.9500	0.0500
8.7	0.0000	0.0000	0.0000	0.0000	0.9500	0.0500
8.8	0.0000	0.0000	0.0000	0.0000	0.9500	0.0500
8.9	0.0000	0.0000	0.0000	0.0000	0.9500	0.0500
9.0	0.0000	0.0000	0.0000	0.0000	0.9500	0.0500
9.1	0.0000	0.0000	0.0000	0.0000	0.9500	0.0500
9.2	0.0000	0.0000	0.0000	0.0000	0.9500	0.0500
9.3	0.0000	0.0000	0.0000	0.0000	0.9500	0.0500
9.4	0.0000	0.0000	0.0000	0.0000	0.9500	0.0500
9.5	0.0000	0.0000	0.0000	0.0000	0.9500	0.0500
9.6	0.0000	0.0000	0.0000	0.0000	0.9500	0.0500
9.7	0.0000	0.0000	0.0000	0.0000	0.9500	0.0500
9.8	0.0000	0.0000	0.0000	0.0000	0.9500	0.0500

5. Appendix 1

In this appendix we summarize some of the properties of topological degree theory.

This appendix was extracted from the notes of a course in nonlinear analysis by Rabinowitz [14] (another good reference is Schwartz [22]).

Let Ω be an open bounded subset of $\mathbf{R}^n$. We will denote by $C^k(\Omega,\mathbf{R}^m)$ the set of functions from Ω into $\mathbf{R}^m$ which are k times continuously differentiable, and by $C^k(\overline{\Omega},\mathbf{R}^m)$ the subspace of $C^k(\Omega,\mathbf{R}^m)$ consisting of all functions, that together with their derivatives up to order k coincide with the restriction of continuous functions on $\overline{\Omega}$. $C^0(\Omega,\mathbf{R}^m)$ and $C^0(\overline{\Omega},\mathbf{R}^m)$ will be denoted by $C(\Omega,\mathbf{R}^m)$ and $C(\overline{\Omega},\mathbf{R}^m)$ respectively.

We endow $C^k(\overline{\Omega},\mathbf{R}^m)$ with the norm

$$\|\varphi\|_{C^k(\overline{\Omega},\mathbf{R}^m)} := \max_{1\le\alpha\le k} \sup_{x\in\overline{\Omega}} \|\varphi^{(\alpha)}(x)\|_{\mathbf{R}^m}.$$

If $m = n$, $\varphi\in C^1(\Omega,\mathbf{R}^m)$, and $x\in\Omega$, $J_\varphi(x)$ will denote the Jacobian determinant of φ at x.

For the rest of this appendix Ω will be an open bounded set of $\mathbf{R}^m$, i.e we shall consider functions from $\Omega\subset\mathbf{R}^m$ to $\mathbf{R}^m$.

5.1 Definition

Let $\varphi \in C^1(\overline{\Omega},\mathbf{R}^m)$ and define:

$$S := \left\{ x \in \Omega \mid J_\varphi(x) = 0 \right\}.$$

Then for $b \in \mathbf{R}^m \setminus (\varphi(\partial\Omega) \cup \varphi(S))$ we define the Brouwer degree of φ with respect to Ω at b as:

$$\deg(\varphi,\Omega,b) := \sum_{x \in \varphi^{-1}(b)} \operatorname{sgn} J_\varphi(x).$$

By the implicit function theorem, the set $\varphi^{-1}(b)$ is discrete and since $\overline{\Omega}$ is compact, this set is finite.

5.2 Sard's theorem

If Ω, φ and S are defined as above then $\varphi(S)$ has measure 0.

5.3 Lemma

Let $\varphi \in C^2(\overline{\Omega},\mathbf{R}^m)$ and $b \notin \varphi(\partial\Omega) \cup \varphi(S)$. Then there is a neighborhood U of φ in $C^1(\overline{\Omega},\mathbf{R}^m)$ such that for all $\psi \in U \cap C^2(\Omega,\mathbf{R}^m)$ one has:

$$\begin{cases} b \notin \psi(\partial\Omega) \\ x \in \psi^{-1}(b) \implies J_\psi(x) \neq 0 \\ deg(\psi,\Omega,b) = deg(\varphi,\Omega,b). \end{cases}$$

5.4 Lemma

Let $\varphi \in C^2(\overline{\Omega}, \mathbf{R}^m)$ *and* $b, \beta \not\in \varphi(\partial\Omega) \cup \varphi(S)$. *If* b *and* β *belong to the same component of* $\mathbf{R}^m \setminus \varphi(\partial\Omega)$, *then* $deg(\varphi, \Omega, b) = deg(\varphi, \Omega, \beta)$.

These two lemmas together with Sard's theorem allow us to extend the definition of degree to every function in $C(\overline{\Omega}, \mathbf{R}^m)$.

If $b \in (\mathbf{R}^m \setminus \varphi(\partial\Omega)) \cup \varphi(S)$ $\deg(\varphi, \Omega, b)$ is not defined. However, since measure $(\varphi(S)) = 0$, we can choose $\tilde{b}$ as close as we please to b, such that $\tilde{b} \not\in \varphi(S)$. But $\mathbf{R}^m \setminus \varphi(\partial\Omega)$ is open; therefore we can choose $\tilde{b}$ to lie in the same component of $\mathbf{R}^m \setminus \varphi(\partial\Omega)$ as b. Then we can define:

$$\deg(\varphi, \Omega, b) := \deg(\varphi, \Omega, \tilde{b}).$$

By the previous lemma, this definition does not depend upon the choice of $\tilde{b}$. Also, this definition extends the last lemma to all $b, \beta \in \mathbf{R}^m \setminus \varphi(\partial\Omega)$.

5.5 Corollary

Let $\varphi \in C^2(\overline{\Omega}, \mathbf{R}^m)$ *and* $b \in \mathbf{R}^m \setminus \varphi(\partial\Omega)$. *Then there is a neighborhood* U *of* φ *in* $C^1(\overline{\Omega}, \mathbf{R}^m)$ *such that for all* $\psi \in U \cap C^2(\Omega, \mathbf{R}^m)$ *one has:*

$$\begin{cases} b \not\in \psi(\partial\Omega) \\ deg(\psi, \Omega, b) = deg(\varphi, \Omega, b). \end{cases}$$

5.6 Corollary

Let $H \in C^1([0,1]\times\overline{\Omega},\mathbb{R}^m)$ *be such that* $H(t,.) \in C^2(\overline{\Omega},\mathbb{R}^m)$ *for all* $t \in [0,1]$, *and* $b \notin H([0,1]\times\partial\Omega)$. *Then* $deg(H(t,.),\Omega,b)$ *is independent of* t.

5.7 Definition

Let $\varphi \in C(\overline{\Omega},\mathbb{R}^m)$, $b \notin \varphi(\partial\Omega)$ and $r := d(b,\varphi(\partial\Omega))$. Take any $\psi \in C^2(\overline{\Omega},\mathbb{R}^m)$ such that $||\varphi - \psi||_{C(\overline{\Omega},\mathbb{R}^m)} < \frac{r}{2}$ and define:

$$\deg(\varphi,\Omega,b) := \deg(\psi,\Omega,b).$$

In order to show that this definition does not depend upon the choice of ψ let us use the last corollary.

If $\psi_1,\psi_2 \in C^2(\overline{\Omega},\mathbb{R}^m)$ are such that $||\varphi - \psi_i||_{C(\overline{\Omega},\mathbb{R}^m)} < \frac{r}{2}$ define:

$$H(t,x) := t\psi_1(x) + (1-t)\psi_2(x) \quad x\in\Omega \quad t\in[0,1].$$

Then H satisfies all the conditions of the last corollary. Since $\psi_1(x),\psi_2(x) \in \varphi(x)+\frac{r}{2}B$ for all $x\in\Omega$, $H(t,x) \in \varphi(x)+\frac{r}{2}B$ for all $x\in\Omega$. Thus $b \notin H([0,1]\times\partial\Omega)$ and $\deg(\psi_1,\Omega,b) = \deg(\psi_2,\Omega,b)$.

5.8 Theorem

Let $\varphi \in C(\overline{\Omega},\mathbb{R}^m)$ *and* $b \in \mathbb{R}^m \setminus \varphi(\partial\Omega)$. *Then* $deg(\varphi,\Omega,b)$ *is defined and possesses the following properties:*

i) Normalization:

$$deg(id,\Omega,b) = \begin{cases} 1 & \text{if } b \in \Omega \\ 0 & \text{if } b \notin \Omega \end{cases}$$

where $id: \mathbf{R}^m \to \mathbf{R}^m$ *is the identity, i.e.* $id(x) = x$ *for all* $x \in \mathbf{R}^m$.

ii) Continuity with respect to φ*:*

There is a neighborhood V *of* φ *in* $C(\Omega,\mathbf{R}^m)$ *such that* $b \notin \psi(\partial\Omega)$ *and* $deg(\psi,\Omega,b) = deg(\varphi,\Omega,b)$ *for all* $\psi \in V$.

iii) Continuity with respect to b*:*

If β *belongs to the same component of* $\mathbf{R}^m \setminus \varphi(\partial\Omega)$ *as* b*, then* $deg(\varphi,\Omega,b) = deg(\varphi,\Omega,\beta)$, *i.e. deg is constant on components of* $\mathbf{R}^m \setminus \varphi(\partial\Omega)$.

iv) Homotopy invariance:

Let $H \in C([0,1]\times\overline{\Omega},\mathbf{R}^m)$ *and* $b \notin H([0,1]\times\partial\Omega)$. *Then* $deg(H(t,.),\Omega,b)$ *is constant on* $[0,1]$.

v) Additivity:

Suppose $\Omega = \Omega_1 \cup \Omega_2$ *where* Ω_1 *and* Ω_2 *are two disjoint open sets. If* $b \notin \varphi(\partial\Omega_1) \cup \varphi(\partial\Omega_2)$ *then*

$$deg(\varphi,\Omega,b) = deg(\varphi,\Omega_1,b) + deg(\varphi,\Omega_2,b).$$

vi) Excision:

If $K \subset \Omega$ *is closed and* $b \notin \varphi(K)$ *then*

$$deg(\varphi,\Omega,b) = deg(\varphi,\Omega\setminus K,b).$$

vii) Restriction:

Let $\psi \in C(\overline{\Omega}, \mathbb{R}^n \times \{0\})$, *where* $n < m$, *and* $\varphi(x) := x - \psi(x)$. *Let* $P : \mathbb{R}^m \to \mathbb{R}^n$ *denote the orthogonal projection of* $\mathbb{R}^m$ *into its first n components, i.e. if* $x = (x_1, .., x_m) \in \mathbb{R}^m$ *then* $P(x) = (x_1, .., x_n)$. *If* $b = (\beta, 0) \in \mathbb{R}^n \times \{0\} \setminus \varphi(\partial\Omega)$ *then*

$deg(\varphi, \Omega, b) = deg(\gamma, \Gamma, \beta)$ *where*

$\Gamma = P(\Omega \cap (\mathbb{R}^n \times \{0\}))$ and $\gamma(x) = P(\varphi(x, 0)) \quad \forall x \in \Gamma.$

viii) Cartesian product:

Let $\varphi_i \in C(\overline{\Omega}_i, \mathbb{R}^{m_i})$, *where* $\Omega_i \subset \mathbb{R}^{m_i}$ *is bounded and open, and* $b_i \in \mathbb{R}^{m_i} \setminus \varphi_i(\partial\Omega_i)$ *for i=1,2. Then*

$deg((\varphi_1, \varphi_2), \Omega_1 \times \Omega_2, (b_1, b_2)) = deg(\varphi_1, \Omega_1, b_1)\, deg(\varphi_2, \Omega_2, b_2).$

As a corollary, it can be shown that if $b \notin \varphi(\overline{\Omega})$ then $\deg(\varphi, \Omega, b) = 0$. Conversely if $\deg(\varphi, \Omega, b) \neq 0$ there exists $x \in \Omega$ such that $\varphi(x) = b$.

6. Appendix 2

Here we provide the listings of the computer programs used to construct the approximate solutions to the problem (P') of chapter 2.

We first give the complete listing of the program that uses the explicit Euler method (problem (P1) of chapter 4) at each step. The subroutines GRADD, DEMAND and GRADF are provided by the user. In the listing below, these three subroutines are constructed to solve example 1 of chapter 4.

Next we provide only part of the listing of the program that uses the implicit Euler method (problem (P2) of chapter 4) at each iteration. The first four subroutines of this program, SIMPLEX, ADJUST, PIVOT and DISPLAY are the same four subroutines listed in the first program and we do not list them again.

In the second program, the subroutines GRADD and GRADF are provided by the user. In the listing below, these correspond to the example 3 of chapter 4.

The programs were written in C, the "official" language of the UNIX system.

```
/*

        T H E   E X P L I C I T   E U L E R   M E T H O D

*/
#include <stdio.h>
#define poseps 0.0001       /* positive tolerance                         */
#define negeps -0.0001      /* negative tolerance                         */
#define NCONS  5            /* maximum number of consumers                */
#define MGOOD  10           /* maximum number of goods                    */
#define MGOOD2 20           /* 2 times MGOOD                              */
#define NM     50           /* NCONS times MGOOD                          */
#define NCOL   60           /* maximum number of columns in the tableau */
#define NROW   70           /* maximum number of rows in the tableau      */
#define RMAX   1000000.0 /* numerical infinite                            */

/* The first 3 subroutines are the SIMPLEX method */

SIMPLEX(R,I,J,N,M,TYPE,phase,endcond,opt)

/* This subroutine constructs a basis and inverts it. It also checks
   if there are redundant constraints and whether the problem is
   infeasible.
   All the columns associated to slack variables are put in the
   first basis. For each equality constraint we choose a column
   associated to a (structural) variable to put in the basis.
   If after inverting the basis and modifying the tableau the right
   hand side has negative components we add an artificial variable,
   put its column in the basis and solve an auxiliary problem to
   find a feasible basis for the original problem (phase 1).
   After finding a feasible basis we solve the problem (phase 2).

   R is the tableau. It contains the (modified) matrix associated to
   the non basic variables, the right hand side (in R[.][0]) and
   the objective function.

   I is the set of indices of basic variables.

   J is the set of indices of non basic variables.

   N is the number of variables of the problem (the tableau has N+1
   columns).

   M+1 is the number of rows in the tableau.

   TYPE[i] = -1    row i represents the objective function
              0    row i represents an equality constraint
              1    row i represents an inequality constraint (it must be <=).
*/
float R[][NCOL];
int *I,*J,*TYPE;
float *opt;
int N,*M,phase,*endcond;
```

```
{
int iobj,iobj1,iart,jpivot,i,j,k,n;
float pivot,aux;

*endcond=0; *opt= 0.0;

/* A basis is constructed and inverted */

for(j=1;j<=N;j++) J[j]=j-1;
n=N; i=0; iobj= *M+1;
while(i <= *M && *endcond == 0) {
    if(TYPE[i] == 0) {
        pivot=poseps;
        for(j=1;j <= N;j++) {
            aux=R[i][j];
            if(aux < 0.0) aux= -aux;
            if(aux > pivot) {
                pivot=aux;
                jpivot=j;
            }
        }

        if(pivot > poseps) {                /* It is possible to put a column  */
                                            /* in the basis.                   */
            ADJUST(R,N,*M,i,jpivot);
            I[i]=J[jpivot];
            if(jpivot < N) {
                J[jpivot]=J[N];
                for(k=0;k <= *M;k++) R[k][jpivot]=R[k][N];
            }
            N--;
        } else {
            aux=R[i][0];
            if(aux < 0.0) aux= -aux;

            if(aux > poseps) *endcond= 2; /* The problem is infeasible.       */

            else {                          /* The constraint is redundant and */
                                            /* we delete it from the tableau.  */
                if(i < *M) {
                    I[i]=I[*M];
                    for(j=0;j <= N;j++) R[i][j]=R[*M][j];
                    i--;
                }
                *M--;
            }
        }
    } else if(TYPE[i] == 1) {I[i]=n; n++;}
           else  iobj=i;
    i++;
}

I[iobj]=n+1;
```

```
iart=n; /* if we need it, the index of the artificial variable
will be equal to the number of (structural) variables + the number
of slack variables */

if(*endcond == 0) {
    iobj1= *M+1; pivot= negeps;
    for(i=0;i <= *M;i++)
    if(i!=iobj && R[i][0] < pivot) {
        iobj1=i;
        pivot=R[i][0];
    }

    if(iobj1 <= *M) {                       /* Phase 1 is required.          */

        for(j=0;j <= N;j++) {               /* Put in the basis the artificial */
            pivot=R[iobj1][j];              /* variable taking out the most   */
            R[iobj1][j]=0.0;                /* negative.                      */
            for(i=0;i <= *M;i++) if(i!=iobj) R[i][j]=R[i][j]-pivot;
        }
        n=N+1;
        for(i=0;i<=*M;i++) R[i][n]= -1.0; /* This is the objective function  */
                                            /* for phase 1.                   */
        R[iobj][n]=0.0;
        J[n]=I[iobj1]; I[iobj1]=iart;
        PIVOT(R,I,J,n,*M,iobj1,iobj,iart,1,endcond);

        if(I[iobj1] == iart) {              /* The artificial variable is     */
                                            /* still basic.                   */
            if(R[iobj1][0] <= poseps) {     /* The artificial variable is 0.  */

                jpivot=0; j=1;
                while(jpivot == 0) {
                    if(R[iobj1][j] < negeps) jpivot=j;
                    j++;
                }
                ADJUST(R,n,*M,iobj1,jpivot);
                I[iobj1]=J[jpivot];
                J[jpivot]=J[n];
                if(jpivot != n) for(i=0;i<=*M;i++) R[i][jpivot]=R[i][n];

            } else {                        /* The problem is infeasible.     */

                *endcond=2;
                *opt= R[iobj1][0];
            }
        } else {
            j=1; while(J[j]!= iart) j++;
            if(j!=n) {
                J[j]=J[n];
                for(i=0;i<=*M;i++) R[i][j]=R[i][n];
            }
        }
    }
```

```
    if(*endcond <= 1 && phase ==2) {        /* If the problem is feasible and  */
                                            /* we want an optimal solution     */
                                            /* solve phase 2.                  */
        *endcond=0;
        PIVOT(R,I,J,N,*M,iobj,iobj,iart,phase,endcond);
        if(*endcond == 1) *opt=R[iobj][0];
    }
}
}

ADJUST(R,N,M,ipivot,jpivot)

/* This subroutine adjust the tableau R. The pivot is
R[ipivot][jpivot] and the tableau has M+1 rows and N+1 columns. */

float R[][NCOL];
int N,M,ipivot,jpivot;
{
int i,j;
float pivot;

pivot= R[ipivot][jpivot];
R[ipivot][jpivot]= 1.0;
for(j=0;j <= N;j++) R[ipivot][j] /= pivot;
for(i=0;i <= M;i++)
if(i != ipivot) {
    pivot=R[i][jpivot];
    R[i][jpivot]= 0.0;
    for(j=0;j <= N;j++) R[i][j] -= pivot*R[ipivot][j];
}
}


PIVOT(R,I,J,N,M,iobj,iobj2,iart,phase,endcond)

/* This subroutine iterates with the simplex method. It assumes that
   a feasible basis is given and that it has been inverted so only
   the portion associated to the non basic variables is kept in the
   matrix R (the tableau is in the compact form).

   R is the tableau.

   I is the set of indices of basic variables.

   J is the set of indices of nonbasic variables.

   N is the number of variables of the problem (the tableau has N+1 columns).

   M is one less than the number of rows in the tableau. If one row represents
   the objective function, then M is the number of constraints of the linear
   problem.

   iobj is the row that represents the objective function for this phase.
```

```
   iobj2 is the row that represents the objective function for phase2. If we
   we dont want any row to be treated differently then iobj2 must be set to
   a number greater than M.

   iart is the index of the artificial variable (in case of phase 1).

   phase = 1 means we are seeking a feasible point
         = 2 means we are seeking an optimal point

   endcond = 1  optimal solution
           = 2  infeasable problem
           = 3  unbounded problem
*/
float R[][NCOL];
int *I,*J;
int N,M,iobj,iobj2,iart,phase,*endcond;
{
int ipivot,jpivot,i,j;
float aux,pivot,ratio;

while(*endcond == 0) {
    j=1;
    if(phase!=1) while(j <= N && R[iobj][j] >= negeps) j++;
    else         while(j <= N && R[iobj][j] <= poseps) j++;
    if(j <= N) {                               /* there is a negative cost */
        jpivot=j; ipivot=M+1; pivot=RMAX;
        for(i=0; i <= M; i++)
        if(i!=iobj2) {
            aux=R[i][jpivot];
            if(aux > poseps ) {
                ratio=R[i][0]/aux;
                if((ratio<pivot-poseps)||
                   (ratio < pivot+poseps && I[i] < I[ipivot])) {
                    pivot=ratio;
                    ipivot=i;
                }
            }
        }
        if(ipivot <= M) {                      /* the iteration is bounded */
            ADJUST(R,N,M,ipivot,jpivot);
            i=I[ipivot];
            if(i == iart) *endcond=1;
            I[ipivot]=J[jpivot];
            J[jpivot]=i;
        } else *endcond=3;                     /* the problem is unbounded */
    } else *endcond=1;                         /* the problem is solved    */
}
}

DISPLAY(X,P,N,M,iter1,iter2,h,out)

/* This subroutine prints the level of consumption for each consumer and the
   price vector after a period of time of iter*h.
```

```
    X contains the current level of consumption of each consumer.

    P is the current price vector.

    N is the number of consumers.

    M is the number of goods.

    iter1 is the number of iterations of the Euler method that have been
    performed.

    iter2 is the number of iterations of the Robinson's algorithm
    performed in this iteration of the Euler method.

    h is the stepsize.

    out indicates the form in which the output will be given. If out
    = 0 the output is given with some comments and additional
    information. If out != 0 the output is given in the compact form;
    only the consumtion levels and the prices for each iteration are
    printed in as many columns as required.
*/

float *X,*P;
float h;
int N,M,iter1,iter2,out;
{
int ii,i,k,nm;
nm = N*M;

if(out == 0) {
    printf("0teration %d   time = %.2f",iter1,iter1*h);
    printf("0umber of minor iterations performed = %d",iter2);
    ii=0;
    for(k=0;k<N;k++) {
        printf("0onsumer %d0,k);
        for(i=0;i<M;i++) {
            printf("%.5f  ",X[ii]);
            ii++;
        }
    }
    printf("0rices0);
    for(i=0;i<M;i++) printf("%.5f  ",P[i]);
    printf("0);
} else {
    for(i=0;i<nm;i++) printf("%8.4f",X[i]);
    for(i=0;i<M;i++) printf("%8.4f",P[i]);
    printf("0);
}
}

    /* The next two subroutines are constructed for the case where
    each consumer has a Cobb-Douglas demand function with 2 goods.
```

```
    */

GRADD(G,X,P,i,N,M)

/* This subroutine receives the level of consumption of each
   consumer and the current price, and returns the matrix G:

   G[k][j] = derivative of k-th component of the demand function of consumer
             i with respect to the price of the j-th good.

   X contains the level of consumption of each consumer.

   P contains the current price vector.

   i is index of the consumer.

   N is the number of consumers.

   M is the number of goods.

*/
float G[MGOOD][MGOOD];
float *X, *P;
int   i, N, M;
{
    int   j, k, point;
    float z[MGOOD];
    float alpha,beta,gamma;

    point = i*M;
    for(j=0; j<M; j++) z[j] = X[point+j];

    if(i == 0) {
        alpha = 0.3;
        beta  = 0.7;
    } else {
        alpha = 0.6;
        beta  = 0.4;
    }

    G[0][0] = -alpha*P[1]*z[1]/(P[0]*P[0]);
    G[0][1] = alpha*z[1]/P[0];
    G[1][0] = beta*z[0]/P[1];
    G[1][1] = -beta*P[0]*z[0]/(P[1]*P[1]);

}

DEMAND(D,X,P,i,N,M)

/* This subroutine receives the level of consumption of each
   consumer and the current price, and returns the vector D:

   D demand of consumer i.
```

```
   X contains the level of consumption of each consumer.

   P contains the current price vector.

   i is index of the consumer.

   N is the number of consumers.

   M is the number of goods.

*/
float *D, *X, *P;
int   i, N, M;
{
    int   j, k, point;
    float z[MGOOD];
    float alpha,beta,gamma,sum;

    point = i*M;
    for(j=0; j<M; j++) z[j] = X[point+j];

    if(i == 0) {
        alpha = 0.3;
        beta  = 0.7;
    } else {
        alpha = 0.6;
        beta  = 0.4;
    }

    sum=0.0;
    for(j=0;j<2;j++) sum += P[j]*z[j];

    D[0] = alpha*sum/P[0] - z[0];
    D[1] = beta*sum/P[1] - z[1];

}

/* The next subroutine describes a circular feasible region by the
inequality:
            (x1+1)**2 + (x2+1)**2 <= 100.
*/

GRADF(G,z)

/*  This subroutine receives a point z and returns the value and
    derivatives of the functions defining the feasible set, at the
    point z.

    G[i][0]  is the value of the i-th function.

    G[i][j]  is the derivative of the i-th function with respect to
             the j-th variable.
*/
```

```
float G[MGOOD][MGOOD];
float *z;
{

int j,k;

    G[0][0] = (z[0] + 1.0)*(z[0] + 1.0) + (z[1] + 1.0)*(z[1] + 1.0)-100.0;
    G[0][1] = 2.0*(z[0] + 1.0);
    G[0][2] = 2.0*(z[1] + 1.0);

}

CUADRO(R,X,RO,TYPE,h,pmin,N,M,M0,endcond)

/* This subroutine set up the tableau for the SIMPLEX method.

   R is the tableau

   X contains the current level of consumtion for each consumer

   RO contains the approximation to the new price vector

   TYPE characterizes each row of the tableau as an equality
   constraint, inequality constraint or objective function

   h is the stepsize for the implicit Euler method

   pmin is the lower bound for each price

   N is the number of consumers

   M is the number of goods

   M0 is the number of constraints characterizing the set M of
   available commodities

   endcond will be set to 1, in case the simplex method is not
   necessary for this iteration, and to 2 otherwise
*/

float R[][NCOL];
float *X,*RO;
int *TYPE;
float h,pmin;
int N,M,M0,*endcond;
{
int i,j,k,l,ii,point,m1,m10,m20,m30;
float G[MGOOD][MGOOD],SUM[MGOOD][MGOOD],Z[MGOOD],D[MGOOD];
float sum;

m1 = M+1; m10 = m1+M0; m20 = m10+M; m30 = m20+M;

for(i=0;i<=m30;i++)
```

```
for(j=0;j<=m1;j++) R[i][j] = 0.0;

for(i=0;i<M;i++)
for(j=0;j<M;j++) SUM[i][j] = 0.0;

point = M;
for(j=0;j<M;j++) Z[j] = X[j];
for(i=1;i<N;i++) {
    for(j=0;j<M;j++) Z[j] += X[point+j];
    point += M;
}

TYPE[0] = -1;
TYPE[1] = 0;
for(i=2;i<=m30;i++) TYPE[i] = 1;

R[0][m1] = 1.0;

for(j=0;j<m1;j++) R[1][j] = 1.0;

ii = 1;
for(i=1;i<m1;i++) {
    ii++;
    R[ii][0] = -pmin;
    R[ii][i] = -1.0;
}

for(i=0;i<N;i++) {
    DEMAND(D,X,RO,i,N,M);
    GRADD(G,X,RO,i,N,M);
    for(k=0;k<M;k++) {
        for(j=0;j<M;j++) SUM[k][j] += G[k][j];
        Z[k] += h*D[k];
    }
}

point = M0;
GRADF(G,Z);
for(i=0;i<M0;i++) {
    ii++;
    R[ii][0] = -G[i][0];
    if(R[ii][0] <= negeps) point = i;
    for(j=0;j<M;j++) {
        sum = 0.0;
        for(k=0;k<M;k++)
            sum += G[i][k+1]*SUM[k][j];
        R[ii][j+1] = h*sum;
        R[ii][0] += h*sum*RO[j];
    }
}
if(point < M0) *endcond = 2; else *endcond = 1;

for(i=0;i<M;i++) {
```

```
    ii++;
    R[ii][0] = RO[i];
    R[ii][i+1] = 1.0;
    R[ii][m1] = -1.0;
}

for(i=0;i<M;i++) {
    ii++;
    R[ii][0] = -RO[i];
    R[ii][i+1] = -1.0;
    R[ii][m1] = -1.0;
}

}

main( )

/* n     number of consumers
   m     number of goods
   m0    number of constraints defining the production set
   iter1 maximum number of major iterations
   iter2 maximum number of minor iterations
   h     stepsize
   pmin  lower bound for each price
   epsi  precision required for minor iterations
*/

{

float R[NROW][NCOL];
float X[NM],P[MGOOD],RO[MGOOD];
int I[NROW],J[NCOL],TYPE[NROW];
float h,pmin,epsi;
int n,m,m0,iter1,iter2,out;
float norm,opt,aux;
int i,j,k,l,ii,kiter1,kiter2,endcond,nm,N,M;

scanf("%d%d%d%d%d%d%f%f%f",&n,&m,&m0,&iter1,&iter2,&out,&h,&pmin,&epsi);
nm=n*m; N=m+1;

/* X[k*m+i] = amount of good i consumed by consumer k */
for(i=0;i<nm;i++) scanf("%f",&X[i]);

/* P[i] = price of good i */
for(i=0;i<m;i++) scanf("%f",&P[i]);

printf("The stepsize is h = %8.5f0,h);
DISPLAY(X,P,n,m,0,0,h,out);

endcond= 1; kiter1= 0;
while(kiter1 < iter1 && endcond < 2) {
    for(i=0;i<m;i++) RO[i]=P[i];
    kiter2=0; norm=RMAX;
```

```
    while(kiter2 < iter2 && norm > epsi && endcond < 2) {
        CUADRO(R,X,RO,TYPE,h,pmin,n,m,m0,&endcond);
        if(endcond == 2) {
            M= 3*m+1+m0;
            SIMPLEX(R,I,J,N,&M,TYPE,2,&endcond,&opt);
            if(endcond < 2) {
                norm = -R[0][0];
                for(i=0;i<m;i++) RO[i]=0.0;
                for(i=1;i<=M;i++) {
                    ii=I[i];
                    if(ii<m) RO[ii]=R[i][0];
                }
                kiter2++;
            }
        } else {
            norm = 0.0;
        }
    }
    if(endcond = 1) {
        if(norm <= epsi) {
            ii = 0;
            for(i=0;i<m;i++)  P[i]= RO[i];
            for(i=0;i<n;i++) {
                DEMAND(RO,X,P,i,n,m);
                for(j=0;j<m;j++) X[ii+j] += h*RO[j];
                ii += m;
            }
            kiter1++;
            DISPLAY(X,P,n,m,kiter1,kiter2,h,out);

        } else {
            endcond= 3;
            printf("0** error *** %d minor iterations done without
            converging0,kiter2);
        }
    } else printf("0** error *** the minor iteration is infeasible");
}
printf("0);
}
```

```
/*

     T H E   I M P L I C I T   E U L E R   M E T H O D

*/
#include <stdio.h>
#define poseps 0.00001     /* positive tolerance                            */
#define negeps -0.00001    /* negative tolerance                            */
#define NCONS  5           /* maximum number of consumers                   */
#define MGOOD  10          /* maximum number of goods                       */
#define MGOOD2 20          /* 2 times MGOOD                                 */
#define NM     50          /* NCONS times MGOOD                             */
#define NCOL   60          /* maximum number of columns in the tableau */
#define NROW   70          /* maximum number of rows in the tableau         */
#define RMAX   1000000.0   /* numerical infinite                            */

GRADD(G,X,P,i,N,M)

/* This subroutine receives the level of consumption of each
   consumer and the current price, and returns the matrix G:

   G[.][0] = demand of consumer i.

   G[k][j] = derivative of k-th component of the demand function of consumer
             i with respect to its j-th variable.

   X contains the level of consumption of each consumer.

   P contains the current price vector.

   i is index of the consumer.

   N is the number of consumers.

   M is the number of goods.

*/
float G[MGOOD][MGOOD2];
float *X, *P;
int   i, N, M;
{
    int   j, k, point;
    float z[MGOOD];

    point = i*M;
    for(j=0; j<M; j++) z[j] = X[point+j];

    if(i == 0) {
        G[0][1] = P[0]*(-2.0+0.6*z[0]) + P[1]*(-1.0-2.4*z[0]+z[1]);
        G[0][2] = P[1]*(1.0+z[0]-0.4*z[1]);
        G[0][3] = -2.0*z[0]+0.3*z[0]*z[0];
        G[0][4] = -z[0]+z[1]-1.2*z[0]*z[0]+z[0]*z[1]-0.2*z[1]*z[1];
        G[1][1] = P[0]*(0.5+2.0*z[0]+0.5*z[1]);
```

```
        G[1][2] = P[0]*(-2.0+0.5*z[0]-2.0*z[1]) + P[1]*(-1.0+0.4*z[1]);
        G[1][3] = 0.5*z[0]-2.0*z[1]+z[0]*z[0]+0.5*z[0]*z[1]-z[1]*z[1];
        G[1][4] = -z[1]+0.2*z[1]*z[1];
    } else {
        G[0][1] = P[0]*(-2.0+0.8*z[0]-0.3*z[1])+P[1]*(-1.5-0.2*z[0]+0.5*z[1]);
        G[0][2] = P[0]*(-0.3*z[0]) + P[1]*(2.0+0.5*z[0]-0.8*z[1]);
        G[0][3] = -2.0*z[0]+0.4*z[0]*z[0]-0.3*z[0]*z[1];
        G[0][4] = -1.5*z[0]+2.0*z[1]+0.1*z[0]*z[0]+0.5*z[0]*z[1]-0.4*z[1]*z[1];
        G[1][1] = P[0]*(1.8-0.72*z[0]+0.3*z[1]);
        G[1][2] = P[0]*(-2.0+0.3*z[0]+0.8*z[1]) + P[1]*(-1.0-0.2*z[1]);
        G[1][3] = 1.8*z[0]-2.0*z[1]-0.36*z[0]*z[0]+0.3*z[0]*z[1]+0.4*z[1]*z[1];
        G[1][4] = -z[1]-0.1*z[1]*z[1];
    }
    G[0][0] = G[0][3]*P[0] + G[0][4]*P[1];
    G[1][0] = G[1][3]*P[0] + G[1][4]*P[1];

}

GRADF(G,z)

/*  This subroutine receives a point z and returns the value and
    derivatives of the functions defining the set of available
    commodities, at the point z.

    G[i][0]  is the value of the i-th function.

    G[i][j]  is the derivative of the i-th function with respect to
             the j-th variable.
*/

float G[MGOOD][MGOOD2];
float *z;
{

    G[0][0] = z[1] + 0.1*z[0] - 5.0;
    G[0][1] = 0.1;
    G[0][2] = 1.0;
    G[1][0] = z[0] + 0.1*z[1] - 5.0;
    G[1][1] = 1.0;
    G[1][2] = 0.1;

}

CUADRO(R,G,X,PSI,RO,TYPE,h,pmin,N,M,M0)

/* This subroutine set up the tableau for the SIMPLEX method.

   R is the tableau

   X contains the current level of consumtion for each consumer

   PSI contains the approximation to the new level of consumption
   for each consumer
```

```
   RO contains the approximation to the new price vector

   TYPE characterizes each row of the tableau as an equality
   constraint, inequality constraint or objective function

   h is the stepsize for the implicit Euler method

   pmin is the lower bound for each price

   N is the number of consumers

   M is the number of goods

   M0 is the number of constraints characterizing the set M of
   available commodities
*/
float R[][NCOL], G[][MGOOD2];
float *X,*PSI,*RO;
int *TYPE;
float h,pmin;
int N,M,M0;
{
int i,j,k,ii,point,nm,nm1,nm2,n1m,n1m1,n2m,n2m1,n3m,n3m1,n4m,n5m;
float z[MGOOD];
float sum;

nm=N*M; nm1=nm+1; nm2=nm1+1;
n1m= nm+M; n1m1= n1m+1;
n2m=nm1+n1m; n2m1=n2m+1;
n3m=n2m+n1m; n3m1=n3m+1;
n4m=n3m+M0; n5m=n4m+M;

for(i=0;i<=n5m;i++)
for(j=0;j<=n1m1;j++) R[i][j]= 0.0;

TYPE[0]= -1;
for(i=1;i<=nm1;i++) TYPE[i]=0;
for(i=nm2;i<=n5m;i++) TYPE[i]=1;

R[0][n1m1]=1.0; ii=1; point=0;
for(i=0;i<N;i++) {
    GRADD(G,PSI,RO,i,N,M);
    for(k=0;k<M;k++) {
        sum=G[k][0];
        for(j=1;j<=M;j++) {
            sum -= (G[k][j]*PSI[point+j-1]+G[k][j+M]*RO[j-1]);
            R[ii][j+point] = -h*G[k][j];
            R[ii][nm+j] = -h*G[k][j+M];
        }
        R[ii][ii] += 1.0;
        R[ii][0] = X[ii-1]+h*sum;
        ii++;
    }
```

```
      point += M;
  }

  R[nm1][0] = 1.0;  for(j=nm1;j<=n1m;j++) R[nm1][j] = 1.0;

  for(i=0;i<nm;i++) {
      ii++;
      R[ii][0] = PSI[i];
      R[ii][i+1] = 1.0;
      R[ii][n1m1] = -1.0;
  }
  for(i=0;i<M;i++) {
      ii++;
      R[ii][0] = RO[i];
      R[ii][i+nm1] = 1.0;
      R[ii][n1m1] = -1.0;
  }
  for(i=0;i<nm;i++) {
      ii++;
      R[ii][0] = -PSI[i];
      R[ii][i+1] = -1.0;
      R[ii][n1m1] = -1.0;
  }
  for(i=0;i<M;i++) {
      ii++;
      R[ii][0] = -RO[i];
      R[ii][i+nm1] = -1.0;
      R[ii][n1m1] = -1.0;
  }
  point=0; for(j=0; j<M; j++) z[j] = 0.0;
  for(k=0; k<N; k++) {
     for(j=0; j<M; j++) z[j] += PSI[point+j];
     point += M;
  }
  GRADF(G,z);
  for(k=0;k<M0;k++)
      for(j=1;j<=M;j++) G[k][0] -= G[k][j]*z[j-1];
  for(i=0;i<M0;i++) {
      ii++; R[ii][0] = -G[i][0]; point = 0;
      for(k=0;k<N;k++) {
          for(j=1;j<=M;j++) R[ii][point+j] = G[i][j];
          point += M;
      }
  }

  for(i=0;i<M;i++) {
      ii++;
      R[ii][0] = -pmin;
      R[ii][i+nm1] = -1.0;
  }

  }
```

```
main( )

/* n      number of consumers
   m      number of goods
   m0     number of constraints defining the production set
   iter1 maximum number of major iterations
   iter2 maximum number of minor iterations
   h      stepsize
   pmin  lower bound for each price
   epsi  precision required for minor iterations
*/
{
float R[NROW][NCOL], G[MGOOD][MGOOD2];
float X[NM],PSI[NM],P[MGOOD],RO[MGOOD];
int I[NROW],J[NCOL],TYPE[NROW];
float h,pmin,epsi;
int n,m,m0,iter1,iter2,out;
float norm,opt,aux;
int i,j,k,l,ii,kiter1,kiter2,endcond,nm,n1m,N,M;

scanf("%d%d%d%d%d%d%f%f%f",&n,&m,&m0,&iter1,&iter2,&out,&h,&pmin,&epsi);
nm=n*m; n1m=nm+m; N= n1m+1;

/* X[k*m+i] = amount of good i consumed by consumer k */
for(i=0;i<nm;i++) scanf("%f",&X[i]);

/* P[i] = price of good i */
for(i=0;i<m;i++) scanf("%f",&P[i]);

printf("The stepsize is h = %8.5f0,h);
DISPLAY(X,P,n,m,0,0,h,out);

k=0;
for(i=0;i<n;i++) {
    GRADD(G,X,P,i,n,m);
    for(j=0;j<m;j++) PSI[k+j] = h*G[j][0];
    k += m;
}
endcond= 1; kiter1= 0;
while(kiter1 < iter1 && endcond < 2) {
    for(i=0;i<nm;i++) PSI[i] += X[i];
    for(i=0;i<m;i++) RO[i]=P[i];

    kiter2=0; norm=RMAX;
    while(kiter2 < iter2 && norm > epsi && endcond < 2) {
        CUADRO(R,G,X,PSI,RO,TYPE,h,pmin,n,m,m0);
        M= 3*n1m+1+m0;
        SIMPLEX(R,I,J,N,&M,TYPE,2,&endcond,&opt);
        if(endcond < 2) {
            norm = -R[0][0];
            for(i=0;i<nm;i++) PSI[i]=0.0;
            for(i=0;i<m;i++) RO[i]=0.0;
            for(i=1;i<=M;i++) {
```

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```
                    ii=I[i];
                    if(ii<nm) PSI[ii]=R[i][0];
                    else if(ii<n1m) RO[ii-nm]=R[i][0];
                }
                kiter2++;
            }
        }
        if(endcond = 1) {
            if(norm <= epsi) {
                for(i=0;i<nm;i++) {
                    aux = X[i];
                    X[i]= PSI[i];
                    PSI[i] -= aux;
                }
                for(i=0;i<m;i++)  P[i]= RO[i];
                kiter1++;
                DISPLAY(X,P,n,m,kiter1,kiter2,h,out);
            } else {
                endcond= 3;
                printf("0** error *** %d minor iterations done without converging0,iter2);
            }
        } else printf("0** error *** the minor iteration is infeasible");
    }
    printf("0);
}
```

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