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GENERALIZED EQUATIONS

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ABSTRACT

The term "generalized equation" has recently been used to describe certain kinds of inclusions that involve multivalued functions, particularly normal-cone operators. Such problems include static generalized equations, which extend ordinary nonlinear equations, as well as generalized differential equations, which extend ordinary differential equations to the situation in which the defining relation contains multivalued functions (again, particularly normal-cone operators).

This survey gives an overview of what is known about these problems as of late 1982. Because of space limitations, it has not been practical to present an encyclopedic description of all recent work. Rather, this paper tries to exhibit samples of recent results in several different areas, and to lead the reader to works in the literature that explain those results in more detail and that contain references to other work not mentioned here. The choice of samples to be presented was based on the author's particular knowledge and interests.

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1. What are generalized equations? The term "generalized equation" has recently come to be used to describe certain types of relations that are similar to equations except that one side of the relation is multivalued; more specifically, the multivalued expression often involves a normal-cone operator. In this section we will define the terms needed to describe generalized equations and we will give some examples.

First, suppose that C is a closed convex set in $\mathbb{R}^n$. The normal cone to C at a point $x \in \mathbb{R}^n$ is defined to be:

$$\partial\psi_C(x) := \begin{cases} \phi, & \text{if } x \notin C \\ \{y \mid \langle y, c-x \rangle \leqq 0 \text{ for each } c \in C\}, & \text{if } x \in C. \end{cases}$$

It is easy to see that $\partial\psi_C(x)$ is a closed convex cone; geometrically it is the cone of all outward normals to C at x_0.

The type of generalized equation with which we shall deal in the first part of this paper is of the form

$$0 \in F(x) + \partial\psi_C(x), \tag{1.1}$$

where F is a function from an open set $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^n$. In cases where the generalized equation formalism seems particularly useful, F is often a fairly smooth function, and the expression (1.1) is useful in separating the "smooth" part of the problem at hand from the part involving "corners". We shall see some examples of this later on.

To see what (1.1) means in terms of the set C, note that if (1.1) holds then the sum on the right is nonempty (it contains 0), so $\partial\psi_C(x)$

is nonempty, and this means that $x \in C$. Also, $-F(x)$ must belong to $\partial\psi_C(x)$, so for each $c \in C$,

$$\langle -F(x), c-x \rangle \leqq 0.$$

Thus we can see that (1.1) will hold if and only if x satisfies the so-called variational inequality:

$$x \in C, \text{ and for each } c \in C \ \langle F(x), c-x \rangle \geqq 0, \tag{1.2}$$

and this means geometrically that $F(x)$ is an inward normal to C at x.

One might then ask why we are interested in (1.1), since (1.2) is equivalent to it. The answer is that the form of (1.1) acts as an aid to analyzing problems. It recalls immediately the analogy with ordinary equations (which can be regarded as the special cases of (1.1) in which $C = \mathbb{R}^n$, since then $\partial\psi_C(x) \equiv \{0\}$, and (1.1) holds if and only if $F(x) = 0$). This analogy turns out to be quite helpful in developing results for generalized equations that are extensions of those already known for ordinary equations, such as implicit-function theorems and computational algorithms (e.g. Newton-type methods). We shall see a number of examples of this kind of extension later in the paper.

In the second part of the paper we extend our consideration from static problems to dynamic ones, in which we have some kind of evolution which, if the variable $x(t)$ were unrestricted, would be described by a relation like

$$0 = \mathring{x}(t) + F[x(t)] \tag{1.3}$$

(i.e., $-\mathring{x}(t) = F[x(t)]$, where $\mathring{x}(t)$ denotes $\frac{d}{dt}x(t)$). However, if we suppose that $x(t)$ is to be confined to some closed convex set C, then

(1.3) might not be appropriate, since if $x(t)$ were on the boundary of C it might be impossible to satisfy (1.3) while ensuring that $x(t)$ remained in C. In such a situation we can modify (1.3) by requiring that $\dot{x}(t)$ be the projection of $-F[x(t)]$ on the tangent cone to C at $x(t)$. Under appropriate conditions this can be stated equivalently as the requirement that $\dot{x}(t)$ be the smallest element of $-F[x(t)] - \partial\psi_C[x(t)]$; that is,

$$-\dot{x}(t) = \{F[x(t)] + \partial\psi_C[x(t)]\}^{\dagger}, \tag{1.4}$$

where $A^{\dagger}$ denotes the smallest element of the closed convex set A. In fact, by making appropriate extensions of the ideas of tangent and normal cones, we can extend this reasoning to sets C which may not be convex.

Note that if we ask for an "equilibrium" situation in 1.4 (i.e., one in which $\dot{x}(t) = 0$), then we are led to the problem of finding points x which satisfy (1.1). Thus it is appropriate to regard (1.4) as the dynamic extension of (1.1).

We shall return to the dynamic case in Sections 6 of this paper. In the meantime, we consider in more detail the static problem (1.1), beginning in the next section with some examples of how (1.1) can be used to express familiar problems in optimization and in mathematical economics.

2. <u>How are generalized equations useful?</u> We will exhibit in this section some ways in which the "static" generalized equation (1.1) can be used as a unifying device to model relationships found in a number of applications in the areas of optimization, complementarity, and mathematical economics. In keeping with the interpretation of (1.1) as the equilibrium case of (1.4), we shall see that these relationships are typically of the "static equilibrium" type: they express the conditions for optimality in a mathematical programming problem or for some type of equilibrium in a problem from complementarity or from economics.

Let us first consider the mathematical programming problem

$$\begin{aligned} &\text{minimize} && f(y) \\ &\text{subject to} && g(y) \in K^{\circ} \\ & && y \in L, \end{aligned} \tag{2.1}$$

where f and g are functions from an open subset U of $\mathbb{R}^p$ to $\mathbb{R}$ and $\mathbb{R}^m$ respectively, L is a closed convex set in $\mathbb{R}^p$, K is a closed convex cone in $\mathbb{R}^m$, and K° denotes the <u>polar cone</u> of K, defined by

$$K^{\circ} := \{y \in \mathbb{R}^m \mid \langle y,k\rangle \leqq 0 \text{ for each } k \in K\}.$$

The formulation in (2.1) is general enough to express a great many of the specific optimization problems found in practice. For example, if the constraints of the problem one is dealing with are of the form

$$g_1(y) \leqq 0,\ldots,g_k(y) \leqq 0;\ g_{k+1}(y) = 0,\ldots,g_{k+\ell}(y) = 0,$$

then one has only to set $K = \mathbb{R}^k_+ \times \mathbb{R}^{\ell}$ (so that $K^{\circ} = \mathbb{R}^k_- \times \{0\}^{\ell}$), and $L = \mathbb{R}^p$.

If we define the standard Lagrangian associated with (2.1) to be $L(y,u) := f(y) + \langle u, g(y) \rangle$ for $y \in U$ and $u \in \mathbb{R}^m$, then if y is a local optimizer of (2.1) at which an appropriate constraint qualification holds, there will exist $u \in \mathbb{R}^m$ which, with y, satisfies the necessary optimality conditions:

$$\begin{aligned} 0 &\in \frac{\partial}{\partial y} L(y,u) + \partial\psi_L(y) \\ 0 &\in -\frac{\partial}{\partial u} L(y,u) + \partial\psi_K(u). \end{aligned} \tag{2.2}$$

See Robinson (1976a) for details of the derivation and of the kinds of constraint qualifications under which (2.2) can be expected to hold.

To express (2.2) as a generalized equation we need only set $x = (y,u)$, $F(x) = [\frac{\partial}{\partial y} L(y,u), -\frac{\partial}{\partial u} L(y,u)]$, and $C = L \times K$. Noting that $\partial\psi_{L\times K} = \partial\psi_L \times \partial\psi_K$, we then see that (2.2) is equivalent to

$$0 \in F(x) + \partial\psi_C(x). \tag{2.3}$$

It may help in understanding the structure of (2.2) to see what the generalized equation (2.3) looks like in the particular case of (generalized) quadratic programming, in which f and g in (2.1) have the special form

$$f(y) = \tfrac{1}{2}\langle y, Qy \rangle + \langle p, y \rangle$$

$$g(y) = Ay - a,$$

where Q and A are linear transformations from $\mathbb{R}^p$ to $\mathbb{R}^p$ and $\mathbb{R}^m$ respectively, $p \in \mathbb{R}^p$ and $a \in \mathbb{R}^m$. Note that here the variables y may be subject to implicit constraints, such as upper and lower bounds, which are enforced through the set L. In this case (2.3) takes the special form

$$0 \in \begin{bmatrix} Q & A^T \\ -A & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} + \begin{bmatrix} p \\ a \end{bmatrix} + \partial\psi_{L\times K} \begin{bmatrix} y \\ u \end{bmatrix}, \tag{2.4}$$

and except for the presence of the normal-cone operator, (2.4) looks like a system of linear equations (to which, indeed, it would reduce if $L = \mathbb{R}^p$ and $K = \mathbb{R}^m$: that is, if the variables y were unconstrained and the explicit constraints were linear equations). To specialize (2.4) to the case of linear programming, we just set $Q = 0$, and then the matrix in (2.4) becomes a skew matrix.

Given a linear transformation M from $\mathbb{R}^n$ to itself and a point $m \in \mathbb{R}^n$, if we set $F(x) = Mx + m$ then (1.1) becomes

$$0 \in Mx + m + \partial\psi_C(x), \tag{2.5}$$

and we shall call this a <u>linear generalized equation</u>. As we have just seen, linear generalized equations arise naturally from the optimality conditions for quadratic programming. They also occur in a number of other contexts; in fact, as we shall see in Section 4 below, linear generalized equations play a rôle with respect to nonlinear generalized equations that is analogous to the rôle played by linear equations with respect to nonlinear equations.

Another common problem that gives rise to generalized equations is that of <u>complementarity</u>. Given a function F from an open subset of $\mathbb{R}^n$ to itself, and a closed convex cone $K \subset \mathbb{R}^n$, the <u>generalized</u>

complementarity problem for F and K is that of finding a point x such that

$$x \in K,\ F(x) \in K^*,\ \langle x, F(x) \rangle = 0, \tag{2.6}$$

where $K^* := -K^\circ$ is the dual cone of K. However, if we recall that because K is a cone,

$$\partial\psi_K(x) = \begin{cases} \phi & \text{if } x \notin K \\ \{y \in K^\circ \mid \langle y, x \rangle = 0\} & \text{if } x \in K, \end{cases}$$

then we can see at once that (2.6) is equivalent to

$$0 \in F(x) + \partial\psi_K(x). \tag{2.7}$$

The usual nonlinear complementarity problem found in the literature is the special case of (2.6) in which $K = \mathbb{R}^n_+$, and the linear complementarity problem is the special case of the nonlinear problem in which $F(x) = Mx + m$. Thus, linear complementarity problems give rise to linear generalized equations of a special type; namely, those in which the set whose normal-cone operator appears in (2.5) is actually a cone. For more information about linear and nonlinear complementarity problems, see the papers by Lemke (1970), Eaves (1971), Cottle (1976), Cottle and Dantzig (1968), and Karamardian (1969a, 1969b, 1972), among many others.

A number of models from mathematical economics can be expressed as generalized equations, and we shall examine two of these here. The first is the model proposed by Hansen and Koopmans (1972) of a capital stock invariant under optimization. In this model, Hansen and Koopman consider an economic growth problem in which the technology is linear (i.e., has constant returns to scale), involving goods of three types: capital goods, resources, and consumption goods. The problem is to find a (technologically) feasible operating path over time which maximizes a sum of discounted utilities involving the consumption goods. More particularly, the authors ask whether such a path can be found in which the stock of capital goods is invariant over time, and they prove that the problem of finding such an invariant capital stock is substantially equivalent to a certain single-period problem. It is this single-period problem with which we shall be concerned here. By incorporating the consumption goods in the utility function, one can reduce the variables of the problem to the following classes, denoted by the letters shown opposite each class:

Capital goods:	$z \in \mathbb{R}^L$
Resources:	$w \in \mathbb{R}^M$
Activity levels:	$x \in \mathbb{R}^I$.

These goods are related by the following inequalities, in which A, B, and C denote linear transformations on the appropriate spaces:

$$\begin{aligned} Ax \leqq z &\leqq Bx \\ Cx &\leqq w \\ x &\geqq 0. \end{aligned} \tag{2.8}$$

If one now poses the problem of maximizing, for a fixed $z \geqq 0$, a concave differentiable function $v(x)$ subject to (2.8), the necessary optimality conditions will associate dual variables with the inequality constraints and will prescribe relations that must be satisfied by these dual variables. If we denote by q_A the dual variable associated with the inequality $Ax \leqq z$, by q_B that associated with $z \leqq Bx$, and by r that associated with $Cx \leqq w$, then the optimality conditions prescribe, in addition to (2.8), the following relationships:

$$\begin{aligned} v'(x) - q_A A + q_B B - rC &\leqq 0 \\ \langle v'(x),x\rangle - \langle q_A,z\rangle + \langle q_B,z\rangle - \langle r,w\rangle &= 0 \\ q_A,\ q_B,\ r &\geqq 0. \end{aligned} \tag{2.9}$$

The problem considered by Hansen and Koopmans is, given w, to solve (2.8) and (2.9) for x, z, q_A, q_B and r in such a way that $z \geqq 0$ and $q_B = \alpha q_A$, where α is a prescribed discount factor in the interval $(0,1)$. It is this particular requirement on the Lagrange multipliers that forces the one-period optimization problem to yield an invariant capital stock z.

In order to formulate this problem as a generalized equation, let us first consider the problem of maximizing $v(x)$ over all x <u>and</u> z satisfying (2.8) and the additional requirement that $z \geqq 0$. Keeping the same notation for the Lagrange multipliers, we obtain the optimality conditions

$$\begin{aligned} v'(x) - q_A A + q_B B - rC &\leqq 0 \\ q_A - q_B &\leqq 0 \\ \langle v'(x) - q_A A + q_B B - rC,\ x\rangle &= 0 \\ \langle q_A - q_B,\ z\rangle &= 0 \\ \langle q_A,\ Ax - z\rangle &= 0 \\ \langle q_B,\ z - Bx\rangle &= 0 \\ \langle r,\ Cx - w\rangle &= 0 \\ q_A,\ q_B,\ r &\geqq 0, \end{aligned}$$

together with (2.8) and the condition that $z \geqq 0$. We can organize all of this rather extensive set of conditions into a single generalized equation by writing

$$0 \in \begin{bmatrix} 0 & 0 & A^T & -B^T & C^T \\ 0 & 0 & -I & I & 0 \\ -A & I & 0 & 0 & 0 \\ B & -I & 0 & 0 & 0 \\ -C & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ q_A \\ q_B \\ r \end{bmatrix} + \begin{bmatrix} -v'(x) \\ 0 \\ 0 \\ 0 \\ w \end{bmatrix} + \partial\psi_C \begin{bmatrix} x \\ z \\ q_A \\ q_B \\ r \end{bmatrix}, \tag{2.11}$$

where C is the product $\mathbb{R}^I_+ \times \mathbb{R}^L_+ \times \mathbb{R}^L_+ \times \mathbb{R}^L_+ \times \mathbb{R}^M_+$. Note that the matrix in (2.11) is skew; indeed, this skewness serves as a guide in translating (2.8) and (2.10) into (2.11).

However, (2.11) does not express exactly the conditions required by Hansen and Koopmans, since (2.11) contains the complementary system

$$-q_A + q_B \geqq 0, \; z \geqq 0, \; \langle -q_A + q_B, z \rangle = 0 \tag{2.12}$$

(i.e., $0 \in -q_A + q_B + \partial\psi_{\mathbb{R}^L_+}(z)$), instead of the relations $-\alpha q_A + q_B = 0$, $z \geqq 0$ demanded by Hansen and Koopmans. At this point, we note that in the Hansen-Koopmans formulation the matrix A is required to have non-negative elements, so that the inequality $-Ax + z \geqq 0$, already present in the model, together with $x \geqq 0$, will guarantee that $z \geqq 0$ even if z is not explicitly constrained. Let us therefore replace (2.12) by the complementary system $-\alpha q_A + q_B = 0$, $z \in \mathbb{R}^L$. This system is equivalent to

$$0 \in -\alpha q_A + q_B + \partial\psi_{\mathbb{R}^L}(z),$$

and we can make this change in (2.11) simply by replacing $-I$ in the (2,3) position of the matrix by $-\alpha I$ and the first copy of $\mathbb{R}^L_+$ in C by $\mathbb{R}^L$. With these changes, and with $C' := \mathbb{R}^I_+ \times \mathbb{R}^L \times \mathbb{R}^L_+ \times \mathbb{R}^L_+ \times \mathbb{R}^M_+$, the new generalized equation is

$$0 \in \begin{bmatrix} 0 & 0 & A^T & -B^T & C^T \\ 0 & 0 & -\alpha I & I & 0 \\ -A & I & 0 & 0 & 0 \\ B & -I & 0 & 0 & 0 \\ -C & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ q_A \\ q_B \\ r \end{bmatrix} + \begin{bmatrix} -v'(x) \\ 0 \\ 0 \\ 0 \\ w \end{bmatrix} + \partial\psi_{C'} \begin{bmatrix} x \\ z \\ q_A \\ q_B \\ r \end{bmatrix} . \tag{2.13}$$

Although (2.13) now expresses precisely the conditions demanded by Hansen and Koopmans, we note that its matrix is no longer skew, reflecting the fact that it no longer corresponds exactly to the optimality conditions for a linearly-constrained optimization problem; indeed, Hansen and Koopmans had to use a fixed-point algorithm to solve their problem. This illustrates the fact that generalized equations can be used effectively to model equilibrium type relations even when these do not correspond to optimization problems. Our next example is also of this type.

For the second example of modeling a problem from mathematical economics, we shall examine the structure of a model of energy equilibrium. This is a simplified model that retains the conceptual structure of the Project Independence Evaluation System model discussed by Hogan (1975). The model consists of two sectors; a production (and transportation) sector using a linear technology to produce a prescribed (vector) quantity q of different forms of energy at minimum cost, and a consumption sector which demands varying amounts of each energy form depending upon the prices, p, of all available forms. The key requirement is to find a pair (p,q) such that (i) q is the vector of energy forms demanded by consumers when the (given) prices p are in effect, and (ii) p is the dual (price) variable associated, in the suppliers' linear programming problem, with the constraint that the (given) amounts q of energy forms must be produced. Thus the pair (p,q) has to appear in both sides of the production-consumption system, and this is what makes the problem one of equilibrium rather than of optimization.

To formulate the problem in more precise terms, let us suppose that the energy production system has been modeled as a linear programming problem

$$\begin{aligned} \text{minimize} \quad & \langle c,x \rangle \\ \text{subject to} \quad & Ax = q \\ & Bx = b \\ & x \geqq 0, \end{aligned} \tag{2.14}$$

where x is a vector of n non-negative activity levels, c is a vector of costs associated with the activities, and the ℓ constraints $Bx = b$ represent material balance constraints, upper and lower bounds, and other structural properties of the production system. Of course, these constraints may include inequalities as well as equations, but we assume that the inequalities have already been transformed to equations by the use of appropriate slack variables. The k constraints $Ax = q$ in (2.14) express the relation between the activity levels x and the final output q of k different forms of energy.

The consumption of energy, on the other hand, is assumed to be modeled by a demand function $q_D(p)$ giving consumer demands for energy as functions of the prices in effect. This function might, for example, be estimated by econometric methods. We can now express in a different way the requirements placed on the pair (p,q) by saying that we want to find a solution of the linear programming problem (2.14) with the element q of the right-hand side equal to $q_D(p)$, in which p is an optimal dual variable corresponding to the first constraint. Note that this is a similar situation to that developed earlier in the Hansen-Koopmans model; we begin with an optimization model, then alter it by placing an additional requirement on the Lagrange multipliers.

If we have an optimal solution x of (2.14) with the specified right-hand side, then we must have

$$\begin{aligned} Ax &= q_D(p) \\ Bx &= b \\ x &\geqq 0, \end{aligned} \tag{2.15}$$

and the dual variables p and r corresponding to the two constraints must satisfy

$$\begin{aligned} c + pA + rB &\geqq 0 \\ \langle c + pA + rB,\ x \rangle &= 0 \end{aligned} \tag{2.16}$$

We can model the relations (2.15) and (2.16) as a generalized equation by writing

$$0 \in \begin{bmatrix} 0 & A^T & B^T \\ -A & 0 & 0 \\ -B & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ p \\ r \end{bmatrix} + \begin{bmatrix} c \\ q_D(p) \\ b \end{bmatrix} + \partial\psi_{\mathbb{R}^n_+ \times \mathbb{R}^k \times \mathbb{R}^\ell} \begin{bmatrix} x \\ p \\ r \end{bmatrix} . \qquad (2.17)$$

Note that this is almost like the linear generalized equation that would result from writing the optimality conditions for the linear programming problem (2.14); the difference is that the "constant term" is now no longer constant since it contains the function $q_D(p)$.

More details about this way of formulating such equilibrium models can be found in Josephy (1979c, 1979d). There the formulation is carried out in terms of the inverse function, $p_D(q)$, corresponding to $q_D(p)$. However, for conceptual purposes these formulations are equivalent.

We have shown in this section how generalized equations can be used to formulate optimality conditions, complementarity problems, and economic equilibrium problems in a conceptually simple, economical and unified way. In the next two sections we turn from the question of formulation to questions of analysis and numerical solution. We shall ask when solutions of generalized equations exist, whether they are stable when they exist, and how we can compute them. We begin in the next section with some results that hold when monotonicity is present.

3. Existence and Stability: The monotone case. This section treats some results that are available to us when the expression $F(x) + \partial\psi_C(x)$ in (1.1) is a monotone operator in the variable x. We begin by reviewing the definition, and some properties, of monotone operators. For a multifunction (multivalued function) A we shall write $(x,y) \in A$ to mean $y \in A(x)$ (i.e., (x,y) belongs to the graph of A).

DEFINITION 3.1: A multifunction $A: \mathbb{R}^n \to \mathbb{R}^n$ is a monotone operator if for each (x_1,y_1) and (x_2,y_2) in A, $\langle x_1-x_2, y_1-y_2 \rangle \geq 0$. A is a maximal monotone operator if its graph is not properly contained in that of any other monotone operator.

Examples of maximal monotone operators that occur naturally in connection with generalized equations include:

a. The normal cone operator $\partial\psi_C$ associated with a closed convex set C. In fact, the subdifferential mapping associated with any closed proper convex function is maximal monotone. For proofs and more details see Brezis (1973).

b. The linear operator represented by

$$\begin{pmatrix} Q & A^T \\ -A & 0 \end{pmatrix}$$

(cf. (2.4)), whenever Q is positive semidefinite. This will be the case when Q is the Hessian of a convex function. More generally, any positive semidefinite matrix represents a monotone operator, and in particular any skew matrix does.

Since our earlier examples of generalized equations involved sums of operators (e.g., a linear or nonlinear operator plus a normal-cone operator) it is of interest to determine when such sums will be monotone if their components are monotone. The following theorem gives a convenient criterion for such monotonicity. It specializes to $\mathbb{R}^n$ a result of Rockafellar (1970). We write dom F for $\{x|F(x)\neq\phi\}$, and ri for relative interior (interior relative to the affine hull).

THEOREM 3.2: Let F and G be maximal monotone operators from $\mathbb{R}^n$ to $\mathbb{R}^n$. If ri dom $F \cap$ ri dom $G \neq \phi$, then $F + G$ is maximal monotone.

This result shows that under rather mild assumptions, if the function F appearing in (1.1) is monotone then (1.1) itself will be a problem of finding a zero of a monotone operator. Therefore, it will be of interest to us to review some known facts about existence and stability of such zeros. One of the simplest such facts applies in case the operator involved is strongly monotone, and so we turn next to the definition of strong monotonicity.

DEFINITION 3.3: A multifunction F from $\mathbb{R}^n$ to itself is strongly monotone if there exists a constant $\gamma > 0$ such that for each (x_1, y_1) and (x_2, y_2) in F,

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geqq \gamma \| x_1 - x_2 \|^2 .$$

Now suppose F is a maximal monotone operator which is strongly monotone with modulus γ. A fundamental result about monotone operators says that F is maximal monotone if and only if for any $\lambda > 0$, $(I+\lambda F)^{-1}$ is a contraction that is nonempty on the entire space. That is, for any z_1 and z_2 there exist (x_1,y_1) and (x_2,y_2) in F with $x_1 + \lambda y_1 = z_1$, $x_2 + \lambda y_2 = z_2$, and $\|x_1-x_2\| \leqq \|z_1-z_2\|$. This result is discussed in Brezis (1973). It is clear that $(I+\lambda F)^{-1}$ is then single-valued (take $z_1 = z_2$). However, if F happens to be strongly monotone, we can deduce even more; suppose we form inner products as follows:

$$\begin{aligned} \langle x_1-x_2, z_1-z_2 \rangle &= \langle x_1-x_2, (x_1+\lambda y_1) - (x_2+\lambda y_2) \rangle \\ &= \|x_1-x_2\|^2 + \lambda \langle x_1-x_2, y_1-y_2 \rangle \geqq (1+\lambda\gamma)\|x_1-x_2\|^2, \end{aligned} \tag{3.1}$$

where the inequality comes from strong monotonicity. Recalling that $\langle x_1-x_2, z_1-z_2 \rangle \leqq \|x_1-x_2\| \, \|z_1-z_2\|$, we can rearrange (3.1) to yield

$$\|x_1-x_2\| \leqq (1+\lambda\gamma)^{-1} \|z_1-z_2\|,$$

which shows that $(I+\lambda F)^{-1}$ is actually a strong contraction since $(1+\lambda\gamma)^{-1} < 1$. Such an operator has a unique fixed point by the contraction mapping theorem, and it is easy to see that the fixed points of $(I+\lambda F)^{-1}$ are exactly the zeros of F. Hence if we are dealing with a strongly monotone operator we will have a unique zero.

However, it is very often the case that the operator with which we have to deal is not strongly monotone. In such a case, another existence result can often be helpful: this result is local, rather then global, in nature. To state it we need another definition.

DEFINITION 3.4: A multifunction A is said to be locally bounded at a point x_0 if there exists a neighborhood U of x_0 such that $A(U)(=\{y|y\in Ax \text{ for some } x\in U\})$ is a bounded set.

The following theorem translates to $\mathbb{R}^n$ a result of Rockafellar (1969).

THEOREM 3.5: Let G be a maximal monotone operator from $\mathbb{R}^n$ to itself. Then G is locally bounded at y_0 if and only if y_0 is not a boundary point of dom G.

Now consider a case in which the maximal monotone operator G is known to be upper semicontinuous at $y_0 \in$ dom G. If $G(y_0)$ is bounded, then so will be $G(U)$ for some neighborhood U of y_0 (by upper semicontinuity). But then G is locally bounded at y_0, and by the theorem we conclude that y_0 is not a boundary point of dom G. But as $y_0 \in$ dom G by assumption, we must have $y_0 \in$ int dom G.

This reasoning can be particularly helpful if we take G to be the inverse of a maximal monotone operator F (and hence itself maximal monotone), with $y_0 = 0$. Then we conclude that if:

a. F^{-1} is upper semicontinuous at 0,

and

b. $F^{-1}(0)$ is bounded and nonempty

then the inclusion $y \in F(x)$ is solvable for all y in some neighborhood of the origin; moreover, if Q is any open set containing $F^{-1}(0)$, then for some neighborhood V of 0 we have $F^{-1}(V) \subset Q$.

This observation can be applied to yield, for example, a rather complete stability theory for linear generalized equations involving positive semidefinite matrices and polyhedral convex sets. These include problems of linear programming and of convex quadratic programming (for which the matrix has the special form shown in (2.4) above), as well as linear complementarity problems whose matrices are positive semidefinite (though not necessarily symmetric). This is discussed by Robinson (1979), where the following theorem is proved. In the statement of the theorem, B denotes the Euclidean unit ball.

THEOREM 3.6: *Let* A *be a positive semidefinite* $n \times n$ *matrix,* C *be a nonempty polyhedral convex set in* $\mathbb{R}^n$, *and* a *be a point of* $\mathbb{R}^n$. *Then the following are equivalent:*

a. *The solution set of the linear generalized equation*

$$0 \in Ax + a + \partial\psi_C(x) \tag{3.2}$$

is nonempty and bounded.

b. *There exists* $\varepsilon > 0$ *such that for each* $n \times n$ *matrix* A' *and each* $a' \in \mathbb{R}^n$ *with*

$$\varepsilon' := \max\{\|A'-A\|, \|a'-a\|\} < \varepsilon, \tag{3.3}$$

the set

$$S(A',a') := \{x \mid 0 \in Ax+a+\partial\psi_C(x)\}$$

is nonempty.

Further, if these conditions hold, then for each open bounded set $Q \supset S(A,a)$ there are positive numbers η and μ such that for each (A',a') with ε' (defined by (3.3)) $< \eta$, one has

$$\phi \neq S(A',a') \cap Q \subset S(A,a) + \mu\varepsilon' B. \tag{3.4}$$

Finally, if (A',a') are restricted to values for which $S(A',a')$ is known to be connected (for example, if A' is restricted to be positive semidefinite), then Q can be replaced by $\mathbb{R}^n$.

The inclusion (3.4) means that for each solution, say x_1, of

$$0 \in A'x + a' + \partial\psi_C(x)$$

in Q, there is a solution of (3.2), say x_0, with $\|x_1 - x_0\| \leqq \mu \max\{\|A'-A\|, \|a'-a\|\}$. Thus the solution sets obey a set-valued analogue of Lipschitz continuity, called "upper Lipschitz continuity" [see Robinson (1979, 1981)].

We may remark that the condition that A' be positive semidefinite (under which the "isolating" set Q is not required in Theorem 3.6) will hold automatically in a large class of practical applications of the theorem. Of course this will be true if A is actually positive definite, but it may well be true even if A is only semidefinite, because of the structure of A. For example, consider the linear programming problem

$$\begin{aligned} &\text{minimize} \quad \langle c,x \rangle \\ &\text{subject to} \quad Dx = d \\ &\qquad\qquad\qquad x \geqq 0, \end{aligned} \tag{3.5}$$

where $D: \mathbb{R}^n \to \mathbb{R}^m$, $d \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. We can formulate this problem as the generalized equation

$$0 \in \begin{bmatrix} 0 & D^T \\ -D & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} -c \\ d \end{bmatrix} + \partial\psi_{\mathbb{R}^n_+ \times \mathbb{R}^m} \begin{bmatrix} x \\ u \end{bmatrix},$$

and we now observe that if (D',d',c') represent perturbed data close to (D,d,c) then the perturbed generalized equation is

$$0 \in \begin{bmatrix} 0 & (D')^T \\ -D' & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} -c' \\ d' \end{bmatrix} + \partial\psi_{\mathbb{R}^n_+ \times \mathbb{R}^m} \begin{bmatrix} x \\ u \end{bmatrix},$$

whose matrix is positive semidefinite regardless of what D' is. Thus, in this problem we will always have a positive semidefinite matrix because of the particular structure imposed on the generalized equation by the optimality conditions of (3.5). Another such example, involving quadratic programming, is given in Robinson (1979).

4. Existence and Stability: the case of continuity or differentiability.
We now turn to the question of existence and stability of solutions for generalized equations in which we do not have monotonicity. Here we shall use a topological tool (the Brouwer fixed-point theorem) to establish some general existence results; then we shall investigate stability questions by employing local analytical methods analogous to the implicit-function theorem. In fact, we shall establish an implicit-function theorem for generalized equations and derive several related results.

Our first existence theorem is a simple but useful fact that has been noted by several authors [e.g., Hartman and Stampacchia (1966), Eaves (1971), Karamardian (1972)].

THEOREM 4.1: Let C be a compact convex set in $\mathbb{R}^n$ and let $F: C \to \mathbb{R}^n$ be a continuous function. Then the generalized equation

$$0 \in F(x) + \partial\psi_C(x) \qquad (1.1)$$

has a solution (in C).

For a very easy proof of this theorem (Eaves (1971)), define a continuous self-map Φ of C by letting $\Phi(c)$ be the projection of $c - F(c)$ on C. As projections are nonexpansive, Φ is continuous, so by the Brouwer theorem it must have at least one fixed point. But its fixed points are precisely the solutions of (1.1).

We can extend this result to one involving unbounded sets C in a number of ways. Typically, one assumes some kind of condition on F at large elements of C, then uses Theorem 4.1. Several such results are discussed in

Moré (1974a, 1974b) and elsewhere; we exhibit one as a sample of the sorts of conditions that may be imposed on F (see Moré (1974a), Theorem 2.4):

THEOREM 4.2: Let F be a continuous function from the closed convex set $C \subset \mathbb{R}^n$ to $\mathbb{R}^n$. Suppose that there is a positive number μ such that for each $x \in C$ with $\|x\| = \mu$, there is some $u \in C$ with $\|u\| < \mu$ and $\langle x-u, F(x) \rangle \geqq 0$. Then (1.1) has a solution x with $\|x\| \leqq \mu$.

Of course, the hypothesis of Theorem 4.2 will be satisfied if one can show that the inequality $\langle x-x_0, F(x) \rangle \geqq 0$ holds for some $x_0 \in C$ and all x with sufficiently large norm. As an illustration of how this may be applied, consider a linear generalized equation

$$0 \in Mx + m + \partial\psi_K(x) \tag{4.1}$$

where K is a cone and the matrix M is strictly K-copositive: i.e., if $x \in K\backslash\{0\}$ then $\langle x,Mx \rangle > 0$. Then for some $\sigma > 0$, and all $x \in K$, $\langle x,Mx \rangle \geqq \sigma\|x\|^2$. Given any $m \in \mathbb{R}^n$, take $\mu = \sigma^{-1}\|m\|$; then if $x \in K$ with $\|x\| = \mu$ we have, with $u = 0$,

$$\begin{aligned} \langle x-0, Mx+m \rangle &= \langle x,Mx \rangle + \langle x,m \rangle \\ &\geqq \sigma\|x\|^2 - \|x\|\,\|m\| = 0 . \end{aligned}$$

Thus, by Theorem 4.2 there is a solution x of (4.1) with $\|x\| \leqq \sigma^{-1}\|m\|$. For example, if $K = \mathbb{R}^n_+$ and M is a non-negative matrix with positive diagonal, we can take σ to be the minimum diagonal element.

The above results are based on the Brouwer fixed-point theorem, but they can also be established by degree arguments (since Brouwer's theorem can be proved by such methods). We remark that Reinoza (1979) has developed a definition of degree for multivalued functions of the type appearing in (1.1), and Kojima (1980) made extensive use of degree arguments in his study of "strongly stable" solutions of nonlinear programming problems. These latter results are closely related to a class of stability results for generalized equations, which we shall discuss next.

We now shift our attention from results promising existence of a solution to results describing the stability of an existing solution when the problem is slightly perturbed. Here we will need to use differentiability properties of F, whereas in the first two theorems of this section we needed only continuity.

The generalized equation problem with which we shall deal is that of finding x so that

$$0 \in F(p,x) + \partial\psi_C(x), \tag{4.2}$$

which differs from (1.1) in that the parameter p has been added. Its function is to introduce perturbations into F (but not C) so that we may study the dependence of the solution(s) on such perturbations. We shall ask whether, if x_0 solves (4.2) for a given value $p = p_0$, and if we then allow p to vary near p_0, there is some function $x(p)$ yielding a solution x of (4.2) for each p near p_0. If such a function exists, we should like to gain information about its behavior. In other words, we are seeking the type of information about (4.2) that could be provided by an implicit-function theorem.

We shall now show that an implicit-function theorem can indeed be established for (4.2). In order to do this, we have to introduce some definitions. The first idea is that of the linearization of (1.1): if we suppose that Ω is an open subset of $\mathbb{R}^n$ and that $F: \Omega \to \mathbb{R}^n$ is Fréchet differentiable at x_0, then the <u>linearization of</u> $F(x) + \partial\psi_C(x)$ <u>about</u> x_0 is defined to be $F(x_0) + F'(x_0)(x-x_0) + \partial\psi_C(x)$: in other words, we just linearize the function appearing in (1.1) but leave the normal-cone operator alone.

At this point we might return briefly to the questions, considered earlier in the paper, of why the generalized-equation formalism can be helpful in dealing with problems. We shall see that the linearization defined here works, in the sense that good properties of this linearization guarantee (locally) good properties of the original nonlinear generalized equation. Thus, it seems to be an appropriate tool to use, and indeed it seems obvious when we look at (1.1) that this is the way in which we should linearize it. However, if C is a cone and if we then write (1.1) in the equivalent form of a complementary system

$$F(x) \in C^*, \quad x \in C, \quad \langle x, F(x) \rangle = 0, \tag{4.3}$$

then in looking at (4.3) one might be tempted to linearize not only the first inclusion, but also the complementarity equation on the right. In fact, the linearization obtained in this way does not work well, and so at least in this case the use of the generalized-equation symbolism (4.2) leads one naturally to the correct method of analysis.

Having introduced the idea of linearization, we next define a regular solution of (1.1).

DEFINITION 4.3: Suppose x_0 is a point at which F is Fréchet differentiable, and that x_0 solves (1.1): i.e., $0 \in F(x_0) + \partial\psi_C(x_0)$. Define an operator T by $T(x) := F(x_0) + F'(x_0)(x-x_0) + \partial\psi_C(x)$. Then x_0 is a regular solution of (1.1) if there exist neighborhoods U of 0 and V of x_0 such that $(T^{-1} \cap V)|U$ is single-valued and Lipschitzian: in other words, the function that associates to each $u \in U$ the set of $v \in V$ such that $u \in T(v)$ is Lipschitzian on U.

This property was originally called "strong regularity" in Robinson (1980), because a weaker property had been analyzed in Robinson (1976b) under the name of "regularity." However, the present property has proven to be much more useful in a variety of situations, so we shall use the term "regularity" to refer to it.

One of the consequences of regularity is the following implicit-function theorem for (4.2). It is taken from Robinson (1980).

THEOREM 4.4: Let P be an open subset of a normed linear space and Ω be an open subset of $\mathbb{R}^n$, with $F: P \times \Omega \to \mathbb{R}^n$. Write $F_x(p,x)$ for $\frac{\partial}{\partial x} F(p,x)$, and suppose that:

a. F and F_x are continuous on $P \times \Omega$.

b. For each $x \in \Omega$, $F(\cdot,x)$ is Lipschitzian on P with Lipschitz modulus ν (independent of $x \in \Omega$).

c. x_0 is a regular solution of (4.2) (for $p=p_0$) with associated Lipschitz modulus λ.

Then for any $\varepsilon > 0$ there exist neighborhoods N_ε of p_0 and W_ε of x_0, and a single-valued function $x: N_\varepsilon \to W_\varepsilon$ with Lipschitz modulus $\nu(\lambda+\varepsilon)$, such that for any $p \in N_\varepsilon$, $x(p)$ is the unique solution of (4.2) in W_ε.

This theorem says in effect that if the linearization has, locally, a Lipschitzian inverse then so does the original generalized equation. Moreover, as we shall see in the next theorem, the linearization can be used to approximate solutions of the nonlinear problem.

THEOREM 4.5: Assume the notation and hypotheses of Theorem 4.4. For each $\varepsilon > 0$ and for each $p \in N_\varepsilon$ let $\xi(p)$ be the (unique) solution in W_ε of the linear generalized equation

$$0 \in F(x_0,p) + F_x(x_0,p_0)(\xi-x_0) + \partial\psi_C(\xi) .$$

Then there exists a function $\alpha_\varepsilon: N_\varepsilon \to \mathbb{R}$ such that

$$\lim_{p\to p_0} \alpha_\varepsilon(p) = 0$$

and for any $p \in N_\varepsilon$,

$$\|x(p)-\xi(p)\| \leqq \alpha_\varepsilon(p)\|p-p_0\| .$$

As Theorem 4.4 shows that a generalized equation with a regular solution remains solvable if slightly perturbed, we might wonder whether the regularity property is preserved for these perturbed solutions. The next theorem shows that the answer to this question is yes.

THEOREM 4.6: *Let* A *be a linear transformation from* $\mathbb{R}^n$ *to itself, let* $a \in \mathbb{R}^n$ *and let* C *be a closed convex set in* $\mathbb{R}^n$. *Suppose that* x_0 *is a regular solution of*

$$0 \in Ax + a + \partial\psi_C(x)$$

with associated neighborhoods U *of* 0 *and* V *of* x_0 *and Lipschitz modulus* λ. *Then there exist neighborhoods* M *of* 0 *and* N *of* x_0, *and a positive number* ε, *such that for any* A' *and* a' *with*

$$\max \{||A'-A||, ||a'-a||\} < \varepsilon,$$

if $T'(x) := A'x + a' + \partial\psi_C(x)$, *then* $[(T')^{-1} \cap N]|M$ *is a single-valued function with Lipschitz modulus* $\lambda' := \lambda(1-\lambda||A'-A||)^{-1}$.

We remark that, by Theorem 4.4, for each pair (A',a') near (A,a), the generalized equation

$$0 \in A'x + a' + \partial\psi_C(x) \tag{4.4}$$

has a unique solution near x'. What Theorem 4.6 says is that this solution is in fact a regular solution of (4.4), and that its associated Lipschitz modulus is not much greater than λ. Moreover, the neighborhoods involved in the definition of regularity can be taken to be the same for all nearby versions of (4.4). It therefore shows that regularity is an "open" property, and it provides an analogue for generalized equations of the Banach lemma for linear operators (see, e.g., Kantorovich and Akilov (1964), Theorem 3(2.V)).

For the particular case in which $C = \mathbb{R}^r \times \mathbb{R}^s_+$, which is very frequently seen in applications, a characterization of regularity for a linear generalized equation is given by Robinson (1980) (see Theorem 3.1 of that paper). Given a solution x_0, one removes the "inactive" variables (i.e., elements of x_0 that are non-negatively constrained but are in fact strictly positive), as well as those non-negatively constrained variables that must remain equal to zero because the corresponding function values are strictly positive. Regularity then holds if and only if the square matrix corresponding to the remaining "reduced" problem satisfies the property that its principal submatrix corresponding to the unconstrained variables is nonsingular and the Schur complement of that submatrix has positive principal minors. Here the Schur complement of the nonsingular principal submatrix A_{11} in the square matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

is defined to be

$$(A/A_{11}) := A_{22} - A_{21}A_{11}^{-1}A_{12}.$$

See Cottle (1974) for further information on Schur complements.

Among other results dealing with local existence and stability of solutions to generalized equations, we mention the "strong positivity conditions" of Reinoza (1981) and the work of Kummer (1982) on solvability of very general multivalued inclusions, as well as work of Spingarn (1977) on "cyrtohedra" and on perturbed optimization problems. In the next section, we shall see how the idea of linearization and some of the above stability results may be applied to develop efficient computational methods.

5. Computing solutions: The Newton method and some variants. Given a generalized equation such as (1.1), we often want to compute a solution to it in order to solve some practical problem. In this section we describe methods of Newton type for computing such solutions. Although Newton methods were proposed by Robinson (1976b) and by Eaves (1978), the results that we give here were obtained by Josephy (1979a, 1979b, 1979c, 1979d). They extend to generalized equations the well known theorem of Kantorovich (see Kantorovich and Akilov (1964), Theorem 6 (1.XVIII)) for conventional equations.

Let us consider solving the generalized equation

$$0 \in F(x) + \partial\psi_C(x) \tag{1.1}$$

by repeated linearization. That is, starting at some given point x_0, for each $k \geqq 0$ we construct x_{k+1} from x_k by solving the linear generalized equation

$$0 \in F(x_k) + F'(x_k)\ (x-x_k) + \partial\psi_C(x). \qquad (5.1)_k$$

Obvious questions arise: can we be sure that the problems $(5.1)_k$ will be solvable? If they are solvable, will they have (at least locally) unique solutions? If so, will the sequence $\{x_k\}$ converge to a solution of (1.1)?

In general, one clearly could not expect positive answers to such questions. However, what Josephy showed was that if the first linearized problem $(5.1)_0$ had a regular solution x_1, and if certain inequalities held, then all of the problems $(5.1)_k$ would have (regular) solutions, and the sequence of solutions would converge R-quadratically to a solution of (1.1). The main result along these lines is Theorem 2 of Josephy (1979a), which we restate here as Theorem 5.1. In that theorem, we denote by $B(x,\rho)$ the closed ball of radius ρ about x.

THEOREM 5.1 [Josephy (1979a]: Let F, C and Ω be as previously defined, and suppose further that Ω is convex and that F has a Fréchet derivative that is Lipschitz continuous on Ω with Lipschitz modulus L. Let $x_0 \in \Omega$, and suppose that the generalized equation

$$0 \in F(x_0) + F'(x_0)(x-x_0) + \partial\psi_C(x) \qquad (5.1)_0$$

has a regular solution x_1 with associated Lipschitz modulus λ. Choose $r > 0$, $R > 0$ and $\rho > 0$ so that [Theorem 4.6] for any $x \in B(x_0,\rho)$ the operator $[F(x) + F'(x)[(\cdot) - x] + \partial\psi_C(\cdot)]^{-1} \cap B(x_1,r)$, restricted to $B(0,R)$, is single valued and Lipschitzian, with the Lipschitz modulus $\lambda[1-\lambda\|F'(x)-F'(x_1)\|]^{-1}$.

Define $\eta := \|x_1-x_0\|$, and let $h := \lambda L\eta$. Assume that

a. $0 < h \leqq \frac{1}{2}$,

and

b. $L\eta^2 \leqq 2R$.

Define

$$t^* := \left[\frac{1-(1-2h)^{\frac{1}{2}}}{h}\right]\eta ,$$

and assume that

c. $B(x_0,t^*) \subset \Omega \cap B(x_0,\rho)$.

Then the sequence $\{x_k\}$ *defined by letting* x_{k+1} *be the solution of* $(5.1)_k$ *in* $B(x_1,r)$ *is well defined and converges to* $x^* \in B(x_0,t^*)$ *with* $0 \in F(x^*) + \partial\psi_C(x^*)$. *Further, for each* $k \geqq 1$ *one has*

$$\|x^*-x_k\| \leqq (2^n\lambda L)^{-1}(2h)^{(2^k)}.$$

Thus, Josephy's result infers the existence of a solution from the regularity condition and from the bounds expressed by assumptions (a), (b), and (c) of Theorem 5.1; it also establishes R-quadratic convergence in the sense of Ortega and Rheinboldt (1970) provided that $h < \frac{1}{2}$. Of course, if we are willing to assume the existence of a regular solution x^* of (1.1), we can obtain from Theorem 5.1 a "point of attraction" result to the effect that for any starting point x_0 close enough to x^*, the sequence $\{x_k\}$ defined by $(5.1)_k$ will be well defined and will converge quadratically to x^*. Theorem 1 of Josephy (1979a) establishes this result with the additional conclusion that $\{x_k\}$ converges Q-quadratically, as well as R-quadratically, to x^*.

Josephy tested his Newton method on a number of problems, among them a version of the Hansen-Koopmans capital stock model; the results are reported in Josephy (1979a). In all cases Lemke's method (see Cottle and Dantzig (1968)) was used to solve the subproblems, which in these cases were linear complementarity problems. He also tested the Newton algorithm on an example of an energy-equilibrium problem of PIES type, given by Hogan (1975). The tests are reported, and some properties of the model are analyzed in Josephy (1979c, 1979d).

It should be pointed out that if one applied Josephy's method to the nonlinear generalized equation resulting from the optimality conditions for a nonlinear programming problem (see Section 2), then the linearized problems will be the linear generalized equations arising from certain quadratic programming problems. These quadratic programming problems are precisely the approximating problems proposed by Wilson (1963) in his algorithm for solving nonlinear programming problems. Thus, in the case of nonlinear programming Josephy's work provides a proof of the implementability and convergence of Wilson's method under less stringent hypotheses than those previously known. In particular, it was previously shown by Robinson (1974) that Wilson's method would converge locally to a solution of a nonlinear programming problem satisfying the second-order sufficient condition, linear independence of the gradients of the binding constraints, and strict complementary slackness. However, in Robinson (1980) it is shown that the generalized equation associated with the optimality conditions of a nonlinear programming problem will have a regular solution if the corresponding solution of the nonlinear programming problem satisfies a strengthened second-order sufficient condition and linear independence of the gradients of the binding constraints (without strict complementary slackness). Therefore, Josephy's result shows that Wilson's method will converge quadratically to such solutions too.

In addition to his work on Newton's method, Josephy considered quasi-Newton methods in which $(5.1)_k$ is replaced by

$$0 \in F(x_k) + A_k(x-x_k) + \partial\psi_C(x), \qquad (5.2)_k$$

in which A_k is an approximation of some kind to $F'(x_k)$, chosen to reduce the computational labor of setting up $(5.1)_k$. He showed in Josephy (1979b) that two standard convergence theorems for quasi-Newton methods could be extended to generalized equations, again using the machinery of regularity. These theorems assert (1) local linear convergence to a regular solution, and (2) Q-superlinear convergence when the updates satisfy an appropriate limit condtion. The results for ordinary equations are given by Dennis and Moré (1977), Theorems 3.1 and 5.1. Thus, Josephy's work makes available for the solution of generalized equations the use of approximations to $F'(x_k)$ via updates, such as are commonly used for ordinary equations. Again, Josephy tested some of these methods; some results are reported in Josephy (1979b).

In this section we have dealt with the properties of local, Newton-like, methods for solving generalized equations. We have not treated the global methods which go by the names of "simplicial," or "path-following," algorithms, simply because there is already an enormous literature on these methods. Although algorithms of this type can be, and have been, used to solve complementarity problems and related multivalued problems, there is no point in our duplicating here the excellent descriptions that have appeared elsewhere. In particular, for a very complete survey of this field the reader may consult the paper of Allgower and Georg (1980) and the more than 200 references contained therein.

6. A brief look at generalized differential equations. In this concluding section we will survey very briefly some new results in the theory of generalized differential equations. Recall that in Section 1 we discussed problems such as

$$-\mathring{x}(t) = \{F[x(t)] + \partial\psi_C[x(t)]\}^{\dagger}, \tag{1.4}$$

where $A^{\dagger}$ denotes the smallest element of A. The results we discuss here apply to problems even more general than (1.4). They appear in papers of Cornet (1981a, 1981b), which contain many references to previous work in this area. Because of space limitations, we shall confine ourselves here to describing the main existence result of Cornet (1981b). We note, however, that particular cases of generalized differential equations, such as that in which the operators involved are monotone, have been studied for some time; see the references in Brézis (1973), for example.

Cornet's theorem deals with sets and generalized equations somewhat more general than those encountered earlier in this paper. For example, the sets involved need not be convex, and the functions involved may be multivalued. We therefore first quote a theorem of Cornet that characterizes the sets to which the existence theorem applies. We shall denote by $T_X(x_0)$ the Bouligand tangent cone to a subset X of $\mathbb{R}^n$ at a point $x_0 \in \operatorname{cl} X$, defined by

$$T_X(x_0) := \{y \in \mathbb{R}^n \mid \text{there exist } \{x_n\} \subset X,\ \{\lambda_n\} \subset (0,+\infty), \text{ with } x_n \to x_0 \text{ and } \lambda_n(x_n - x_0) \to y\}.$$

Having $T_X(x_0)$ we define the normal cone by

$$N_X(x_0) := T_X(x_0)^\circ.$$

If X happens to be convex then $N_X(x_0) = \partial\psi_X(x_0)$ as defined earlier. Finally, we need the Clarke tangent cone, defined by

$$TC_X(x_0) := \{y \in \mathbb{R}^n \mid \lim_{\substack{w \to x_0 \\ w \in X \\ \theta \to 0^+}} \theta^{-1} d[w+\theta y, X] = 0\},$$

where $d[\cdot, X]$ denotes the distance to X. Cornet first proves the following.

THEOREM 6.1 [Cornet (1981a), Th. I.3.1]: Let $x_0 \in X \subset \mathbb{R}^n$; suppose for some $\alpha > 0$, $X \cap B(x_0, \alpha)$ is compact.

a. The following are equivalent:

i) $N_X(\cdot)$ is closed at x_0.

ii) $T_X(\cdot)$ is lower semicontinuous at x_0.

iii) $T_X(x_0) = TC_X(x_0)$.

b. If the equivalent properties in (a) hold, then $T_X(x_0)$ is convex.

Having this characterization, Cornet next defines a set $X \subset \mathbb{R}^n$ to be tangentially regular if X is locally compact and the equivalent conditions in (a) of Theorem 6.1 hold at each $x_0 \in X$. This definition establishes a wide class of "nice" sets in $\mathbb{R}^n$, including in particular all locally compact convex sets. Cornet then extends to tangentially regular sets an existence theorem of Henry (1973) for generalized differential equations over convex sets [see Cornet (1981a), Th. II.4.1], and he proves the following key theorem:

THEOREM 6.2 [Cornet (1981b), Th. 3.1]: *Let* X *be a nonempty, tangentially regular subset of* $\mathbb{R}^n$, *and let* Φ *be a continuous multifunction from* X *to* $\mathbb{R}^n$ *with nonempty compact convex values.*

Then for each $x_0 \in X$ *there exist* $T > 0$ *and a Lipschitzian function* $x: [0,T] \to X$ *such that* $x(0) = x_0$ *and, for almost every* $t \in [0,T]$,

$$-\dot{x}(t) = \{\Phi[x(t)] + N_X[x(t)]\}^{\dagger}.$$

Theorem 6.2 thus establishes an existence result for a class of generalized differential equations even broader than that represented by (1.4). The application of these generalized differential equations to the modeling of dynamic systems, for example in economics, is the subject of current research efforts.

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