CHARACTERIZATION OF POSITIVE DEFINITE AND SEMIDEFINITE MATRICES VIA QUADRATIC PROGRAMMING DUALITY

by

S.-P. Han & O. L. Mangasarian

Computer Sciences Technical Report #473

June 1982

UNIVERSITY OF WISCONSIN-MADISON
COMPUTER SCIENCES DEPARTMENT

CHARACTERIZATION OF POSITIVE DEFINITE AND SEMIDEFINITE MATRICES VIA QUADRATIC PROGRAMMING DUALITY

S.-P. Han & O. L. Mangasarian

Technical Report #473
June 1982
Revised January 1983

ABSTRACT

Positive definite and semidefinite matrices induce well known duality results in quadratic programming. The converse is established here. Thus if certain duality results hold for a pair of dual quadratic programs, then the underlying matrix must be positive definite or semidefinite. For example if a strict local minimum of a quadratic program exceeds or equals a strict global maximum of the dual, then the underlying symmetric matrix Q is positive definite. If a quadratic program has a local minimum then the underlying matrix Q is positive semidefinite if and only if the primal minimum exceeds or equals the dual global maximum and $x^TQx = 0$ implies $Qx = 0$. A significant implication of these results is that the Wolfe dual may not be meaningful for nonconvex quadratic programs and for nonlinear programs without locally positive definite or semidefinite Hessians, even if the primal second order sufficient optimality conditions are satisfied.

AMS (MOS) Subject Classifications: 90C20, 15A63
Key Words: Positive definite matrices, quadratic programming, duality

---

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based on work supported by the National Science Foundation under Grants ENG-7903881, MCS-7901066 and MCS-8200632.

CHARACTERIZATION OF POSITIVE DEFINITE AND SEMIDEFINITE MATRICES VIA QUADRATIC PROGRAMMING DUALITY

S.-P. Han & O. L. Mangasarian

1. INTRODUCTION

It is well known [3,4,11,10] that the dual quadratic programs

$$\text{(1a)} \qquad \begin{array}{ll} \underset{x}{\text{Minimize}} & \frac{1}{2}x^TQx + p^Tx \\ \text{subject to} & Ax \leqq b \\ & Cx = d \end{array}$$

$$\text{(1b)} \qquad \begin{array}{ll} \underset{x,u,v}{\text{Maximize}} & -\frac{1}{2}x^TQx - b^Tu - d^Tv \\ \text{subject to} & Qx + A^Tu + C^Tv + p = 0 \\ & u \geqq 0 \end{array}$$

where Q, A and C are given real matrices of order $n\times n$, $m\times n$ and $k\times n$ respectively, with $Q = Q^T$, and p, b and d are given vectors in the real finite dimensional Euclidean spaces R^n, R^m and R^k respectively, possess many important relations when Q is positive semidefinite or positive definite. In this paper we are interested in the converse: What sort of duality relations between (1a) and (1b) induce positive definiteness or semidefinitness in Q? A key role in deriving these converse relations is played by the following conjugate cone characterization of positive definite and semidefinite matrices [8].

1.1 Theorem [8] Let K be a nonempty convex polyhedral cone in R^n. The $n\times n$ real matrix P is positive semidefinite if and only if P is positive semidefinite plus on the cone K and positive semidefinite on the conjugate cone K^P, that is

$$\text{(2)} \qquad \begin{array}{l} x \in K \Rightarrow x^TPx \geqq 0 \\ x^TPx = 0,\ x \in K \Rightarrow (P+P^T)x = 0 \end{array}$$

(3) $$y \in K^P := \{y \mid y(P+P^T)x \leqq 0,\ \forall x \in K\} \Rightarrow y^T P y \geqq 0$$

1.2 Theorem [8] Let K be a nonempty convex polyhedral cone in R^n. The $n \times n$ real matrix P is positive definite if and only if P is positive definite on K and K^A that is:

$$0 \neq x \in K \Rightarrow x^T P x > 0$$

$$0 \neq y \in K^P \Rightarrow y^T P y > 0$$

With the help of these characterization theorems and the second order optimality conditions of quadratic programming [6,9,2,1] we show for example in Theorem 3.5 that if a unique local minimum of a quadratic program exceeds or equals a strict global maximum of the dual, then the matrix Q must be positive definite. In Theorem 3.6 we show that if a quadratic program has a local minimum then Q is positive semidefinite if and only if the primal minimum exceeds or equals the dual global maximum, and $Qx = 0$ whenever $x^T Q x = 0$. In Corollary 3.7 we show that if the primal feasible and dual feasible sets are nonempty and if the weak duality relation holds, that is the primal objective exceeds or equals the dual objective over their respective feasible regions, and if $Qx = 0$ whenever $x^T Q x = 0$, then Q is positive semidefinite. In [7] positive-definiteness of the Hessian of the Lagrangian of nonlinear programs was established under more restrictive assumptions.

The import of these and our other results is that when certain simple and desirable duality results are satisfied by a pair of dual quadratic programs, then the underlying matrix must be positive definite or semidefinite. This leads to the conclusion that the Dennis-Dorn-Wolfe quadratic dual programs [3,4,11] are meaningful only if the underlying matrix is positive definite or semidefinite. For example even if the primal quadratic problem (1a) has a unique global minimum solution (and thus satisfying the second order sufficient optimality condition), and if the underlying matrix is not positive semidefinite then the dual quadratic problem (1b) may not have a solution. Thus the example

$$\text{minimize} \quad x_1^2 - x_2^2 \quad \text{subject to} \quad x_2 = 0$$

has the unique global solution $x_1 = x_2 = 0$ but its dual

$$\text{maximize} \quad x_1^2 - x_2^2 + vx_2 \quad \text{subject to} \quad x_1 = 0, \ -2x_2 + v = 0$$

is unbounded above. Similarly the Wolfe dual for nonlinear programs may not be meaningful unless the Hessian of the Lagrangian is locally positive definite or semidefinite in the neighborhood of a stationary point of the primal problem [7]. Thus even if the second order sufficient optimality conditions are satisfied but the Hessian of the Lagrangian is not positive definite or semidefinite in a neighborhood of a local minimum solution, the dual problem may not have a solution.

We shall need second order optimality conditions for the dual quadratic programs (1a) and (1b) which have local and strictly local solutions. These results can be found in [9,2,1] which we summarize here in a convenient form. The points $(\bar{x},\bar{u},\bar{v}) \in R^{n+m+k}$ and $(\bar{x},\bar{u},\bar{v},\bar{w}) \in R^{n+m+k+n}$ are Karush-Kuhn-Tucker points of (1a) and (1b) respectively if they satisfy the following respective conditions [10]

(4a)
$$\begin{aligned} Q\bar{x} + A^T\bar{u} + C^T\bar{v} + p &= 0 \\ A\bar{x} &\leqq b \\ C\bar{x} &= d \\ \bar{u} &\geqq 0 \\ \bar{u}^T(A\bar{x}-b) &= 0 \end{aligned}$$

(4b)
$$\begin{aligned} -Q\bar{x} + Q\bar{w} &= 0 \\ A\bar{w} - b &\leqq 0 \\ C\bar{w} - d &= 0 \\ Q\bar{x} + A^T\bar{u} + C^T\bar{v} + p &= 0 \\ \bar{u} &\geqq 0 \\ \bar{u}^T(A\bar{w}-b) &= 0 \end{aligned}$$

Note that if $(\bar{x},\bar{u},\bar{v})$ is a Karush-Kuhn-Tucker point of (1a) then $(\bar{x},\bar{u},\bar{v},\bar{x})$ is a Karush-Kuhn-Tucker point of (1b). To characterize local solutions we need to define the following index sets associated with a Karush-Kuhn-Tucker point $(\bar{x},\bar{u},\bar{v})$ of (1a):

$$J := \{i \mid A_i\bar{x} = b_i,\ \bar{u}_i > 0\}$$

$$K := \{i \mid A_i\bar{x} = b_i,\ \bar{u}_i = 0\}$$

$$I := \{i \mid A_i\bar{x} < b_i,\ \bar{u}_i = 0\}$$

The notation A_J will represent the rows A_i of A with $i \in J$. We can now state the following.

1.3 Theorem [2,1] (Characterization of local solutions of quadratic programs) A point $\bar{x} \in R^n$ is local minimum solution of the quadratic program (1a) if and only if $\bar{x}$ and some $(\bar{u},\bar{v}) \in R^{m+k}$ satisfy the Karush-Kuhn-Tucker conditions (4a) and

$$\text{(5a)} \qquad A_J x = 0,\ A_K x \leqq 0,\ Cx = 0 \Rightarrow x^T Q x \geqq 0$$

The Karush-Kuhn-Tucker point $(\bar{x},\bar{u},\bar{v})$ of (1a) is a local maximum solution of the dual quadratic program (1b) if and only if

$$\text{(5b)} \qquad Qx + A^T u + C^T v = 0,\ u_K \geqq 0,\ u_I = 0 \Rightarrow x^T Q x \geqq 0$$

1.4 Theorem [9,2,1] (Characterization of strict local solutions of quadratic programs) A point $\bar{x} \in R^n$ is a strict local minimum solution of the quadratic program (1a) if and only if $\bar{x}$ and some $(\bar{u},\bar{v}) \in R^{m+k}$ satisfy the Karush-Kuhn-Tucker conditions (4a) and

$$\text{(6a)} \qquad A_J x = 0,\ A_K x \leqq 0,\ Cx = 0,\ x \neq 0 \Rightarrow x^T Q x > 0$$

The Karush-Kuhn-Tucker point $(\bar{x},\bar{u},\bar{v})$ of (1a) is a strict local maximum solution of the dual quadratic program (1b) if and only if

$$\text{(6b)} \quad Qx + A^T u + C^T v = 0,\ u_K \geqq 0,\ u_I = 0,\ (x,u,v) \neq 0 \Rightarrow x^T Q x > 0.$$

In the next two sections we characterize positive definite and semidefinite problems in terms of equality-constrained quadratic programs (Section 2) and inequality-constrained quadratic programs (Section 3). This split into equality- and inequality-constrained problems permits the statement of somewhat sharper results for the former. For simplicity we confine the results of Section 3 to inequality constraints only. Problems with both equality and inequality constraints can be handled in a straightforward extension of the results of Section 3.

2. EQUALITY-CONSTRAINED QUADRATIC PROGRAMS

We specialize here the dual problems (1a) and (1b) to the following equality-constrained dual quadratic programs

(7a) $\underset{x}{\text{Minimize}}\ \frac{1}{2}x^TQx + p^Tx$ subject to $Cx = d$

(7b) $\underset{x,v}{\text{Maximize}}\ -\frac{1}{2}x^TQx - d^Tv$ subject to $Qx + C^Tv + p = 0$

We say that a problem is feasible, if the set of points satisfying its constraints is nonempty.

2.1 Theorem (Characterization of positive semidefinite and definite matrices) Let (7a) be feasible.

(i) Let (7b) be feasible. A necessary and sufficient condition for Q to be positive semidefinite is that (7a) has a local minimum, (7b) has a local maximum solution and

(8) $$x^TQx = 0,\ Cx = 0 \Rightarrow Qx = 0$$

(ii) A sufficient condition for Q to be positive definite is that (7a) has a strict local minimum solution and (7b) has a strict local maximum solution. This condition is also necessary if C has linearly independent rows.

Proof (i) Necessity follows from existence and duality theory of convex quadratic programming [5,10]. We establish sufficiency now by means of Theorem 1.1. Define

(9) $$K := \{x \mid Cx = 0\}$$

Then

(10) $$\begin{aligned} K^Q &:= \{y \mid y^TQx \leqq 0,\ \forall x \in K\} \\ &= \{y \mid y^TQx > 0,\ Cx = 0 \text{ has no solution } x\} \\ &= \{y \mid Qy + C^Tv = 0\} \end{aligned}$$

Since (7a) has a local minimum solution it follows by Theorem 1.3 (5a) and (9) that

(11) $$x^TQx \geqq 0 \quad \text{for} \quad x \in K$$

Since (7b) has a local maximum solution, it follows also by Theorem 1.3, (5b) and (10) that

(12) $$y^TQy \geqq 0 \quad \text{for} \quad y \in K^Q$$

Hence by (11), (8), (12) and Theorem 1.1, Q is positive semidefinite.

(ii) (Necessity) That both (7a) and (7b) have solutions follows from the feasibility of (7a) and the positive definiteness of Q. The uniqueness of solution for (7a) follows from the positive definitness of Q. The uniqueness of solution for (7b) follows from the positive definiteness of Q, the linear independence of the rows of C and Theorem 1.4 (6b).

(Sufficiency) We establish sufficiency by means of Theorem 1.2. Since (7a) has a strict local minimum solution it follows by Theorem 1.4 (6a) and (9) that

(13) $$x^TQx > 0 \quad \text{for} \quad 0 \neq x \in K$$

Since (7b) has a strict local maximum solution, it follows also by Theorem 1.4 (6a) that

$$x^TQx > 0 \quad \text{for} \quad Qx + C^Tv = 0, \ (x,v) \neq 0$$

and hence by (10)

(14) $$y^TQy > 0 \quad \text{for} \quad 0 \neq y \in K^Q$$

Hence by (13), (14) and Theorem 1.2, Q is positive definite. □

3. INEQUALITY-CONSTRAINED QUADRATIC PROGRAMS

We turn our attention now to the following inequality constrained dual quadratic programs

(15a) $\underset{x}{\text{Minimize}}\ \frac{1}{2}x^TQx + p^Tx$ subject to $Ax \leqq b$

(15b) $\underset{x,u}{\text{Maximize}}\ -\frac{1}{2}x^TQx - b^Tu$ subject to $Qx + A^Tu + p = 0$, $u \geqq 0$

3.1 Theorem (Characterization of positive semidefinite and definite matrices) Let (15a) be feasible.

(i) Let (15b) be feasible. A necessary and sufficient condition for Q to be positive semidefinite is that (15a) has a local minimum solution $\bar{x}$ with multiplier $\bar{u}$, that $(\bar{x},\bar{u})$ is a local maximum solution of (15b) and

(16) $$x^TQx = 0,\ A_Jx = 0,\ A_Kx \leqq 0 \Rightarrow Qx = 0$$

(ii) A sufficient condition for Q to be positive definite is that (15a) have a strict local minimum solution $\bar{x}$ with multiplier $\bar{u}$, and $(\bar{x},\bar{u})$ is a strict local maximum solution of (15b). If in addition the rows of A_J are linearly independent, $A_Jx = 0,\ A_Kx > 0$ has a solution, then this condition is also necessary.

Proof (i) Necessity follows from existence and duality theory of convex quadratic programs. We establish sufficiency now by means of Theorem 1.1. Define

(17) $$K := \{x \mid A_Jx = 0,\ A_Kx \leqq 0\}$$

Then

$$K^Q = \{y \mid y^TQx \leqq 0, \forall x \in K\} = \{y \mid y^TQx > 0, A_Jx = 0, A_Kx \leqq 0, \text{ has no solution } x\}$$
$$= \{y \mid Qy - A_J^Tu_J - A_K^Tu_K = 0,\ u_K \geqq 0\}$$

Therefore

(18) $$-K^Q = \{x \mid Qx + A^Tu = 0,\ u_K \geqq 0,\ u_I = 0\}$$

Since $\bar{x}$ is a local minimum solution of (15a) with multiplier $\bar{u}$ it follows from Theorem 1.3 (5a) and (17) that

(19) $$x^T Qx \geqq 0 \quad \text{for} \quad x \in K$$

Since $(\bar{x},\bar{u})$ is also a local maximum solution of (15b) it follows from Theorem 1.3 (5b) and (18) that $x^T Qx \geqq 0$ for for $x \in -K$ which is equivalent to

(20) $$x^T Qx \geqq 0 \quad \text{for} \quad x \in K^Q$$

Conditions (19), (16), (20) and Theorem 1.1 imply that Q is positive semidefinite.

(ii) (Necessity) That both (15a) and (15b) have solutions follows from the feasibility of (15a) and the positive definiteness of Q. The uniqueness of solution of (15a) follows from the positive definitness of Q. The uniqueness of solution of (15b) follows from the positive definiteness of Q, the linear independence of the rows of A_J, the existence of a solution to $A_J x = 0$, $A_K x > 0$ and Theorem 1.4 (6b).

(Sufficiency) We establish sufficiency by means of Theorem 1.2. Since (15a) has a strict local minimum solution $\bar{x}$ it follows by Theorem 1.4 (6a) and (17) that

(21) $$x^T Qx > 0 \quad \text{for} \quad 0 \neq x \in K$$

Since $(\bar{x},\bar{u})$ is a strict local maximum solution of (15b) it follows from Theorem 1.4 (6b) and (18) that

(22) $$x^T Qx > 0 \quad \text{for} \quad 0 \neq x \in K^Q$$

Hence by (21), (22) and Theorem 1.2, Q is positive definite. □

3.2 <u>Corollary</u> (Globalization of local dual solutions)

(i) Let $\bar{x}$ be a local minimum solution of (15a) with multiplier $\bar{u}$, let $(\bar{x},\bar{u})$ be a local maximum solution of (15b) and let (16) hold. Then Q is positive semidefinite and hence $\bar{x}$ is a global minimum solution of (15a) and $(\bar{x},\bar{u})$ is a global maximum solution of (15b).

(ii) Let $\bar{x}$ be a strict local minimum solution of (15a) with multiplier $\bar{u}$, let $(\bar{x},\bar{u})$ be a strict local maximum solution of (15b) then Q is positive definite and hence $\bar{x}$ is a unique global minimum solution of (15a) and $(\bar{x},\bar{u})$ is a global maximum solution of (15b).

We note that condition (16) of Theorem 3.1 cannot be dispensed as shown by the following pair of dual programs:

Minimize	x_1x_2	Maximize	$-x_1x_2$
subject to	$x_1 \geqq 0$	subject to	$x_1 - u_2 = 0$
	$x_2 \geqq 0$		$x_2 - u_1 = 0$
			$(u_1,u_2) \geqq 0$

Clearly $(x_1,x_2) = (0,0)$ is a global solution to the primal problem, $(x_1,x_2,u_1,u_2) = (0,0,0,0)$ is a Karush-Kuhn-Tucker point for the primal problem as well as a global solution to the dual problem. However the underlying matrix $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is not positive semidefinite because condition (16) is violated.

We establish now other duality results which induce positive definiteness or semidefiniteness. We begin by a few preliminary results.

3.3 <u>Lemma</u> Let $(\bar{x},\bar{u})$ satisfy the Karush-Kuhn-Tucker conditions of (15a). Then

$$\frac{1}{2}\bar{x}^TQ\bar{x} + p^T\bar{x} \geqq -\frac{1}{2}x^TQx - b^Tu \tag{23}$$

implies that

$$-\frac{1}{2}\bar{x}^TQ\bar{x} - b^T\bar{u} \geqq -\frac{1}{2}x^TQx - b^Tu \tag{24}$$

Proof From the Karush-Kuhn-Tucker conditions of (15a) we have that

$$-\bar{x}^T Q\bar{x} - p^T\bar{x} - b^T\bar{u} = 0$$

which when added to (23) yields (24). □

3.4 Lemma Let $(\bar{x},\bar{u})$ satisfy the Karush-Kuhn-Tucker conditions of (15a) such that for all (x,u) feasible for the dual quadratic program (15b)

$$\frac{1}{2}\bar{x}^T Q\bar{x} + p^T\bar{x} \geqq -\frac{1}{2}x^T Qx - b^T u \tag{25}$$

Then $(\bar{x},\bar{u})$ solves (15b).

Proof Since $(\bar{x},\bar{u})$ is feasible for the dual quadratic program (15b) and since by (25) and Lemma 3.3

$$-\frac{1}{2}\bar{x}^T Q\bar{x} - b^T\bar{u} \geqq -\frac{1}{2}x^T Qx - b^T u$$

for all dual feasible (x,u), it follows that $(\bar{x},\bar{u})$ solves (15b). □

3.5 Theorem (Sufficient condition for positive definiteness) If a strict local minimum of the quadratic program (15a) exceeds or equals a unique global maximum of the dual quadratic program (15b) then Q is positive definite.

Proof Let $\bar{u}$ be a multiplier associated with the strict local minimum solution of (15a). By Lemma 3.4, $(\bar{x},\bar{u})$ is a global maximum solution of (15b). By assumption this global maximum is unique Hence by Theorem 3.1 (ii) Q is positive definite. □

3.6 Theorem (Characterization of positive semidefinite matrices) Let $\bar{x}$ be a local minimum solution of (15a). The matrix Q is positive semidefinite if and only if (16) holds and for any dual feasible (x,u)

$$\frac{1}{2}\bar{x}^T Q\bar{x} + p^T\bar{x} \geqq -\frac{1}{2}x^T Qx - b^T u \tag{26}$$

Proof Necessity follows from the duality theory of quadratic programming and the fact that $x^TQx = 0$ implies $Qx = 0$ for any symmetric positive semidefinite matrix. To prove sufficiency we note that there exists a $\bar{u}$ such that $(\bar{x},\bar{u})$ is a Karush-Kuhn-Tucker point of (15a) and by (26) and Lemma 3.4, $(\bar{x},\bar{u})$ is a global maximum solution to (16b). Hence by Theorem 3.1 (i) Q is positive semidefinite.

A direct consequence of Theorem 3.6 is the following characterization of positive semidefinite matrices in terms of the weak duality [10] relation of quadratic programs.

3.7 Corollary (Positive semidefiniteness via weak duality) Let the quadratic programs (15a) and (15b) be feasible. The matrix Q is positive semidefinite if and only if for all primal feasible x and all dual feasible (y,u)

(27) $$\frac{1}{2}x^TQx + p^Tx \geqq -\frac{1}{2}y^TQy - b^Tu$$

and

(28) $$z^TQz = 0 \Rightarrow Qz = 0$$

REFERENCES

1. J. M. Borwein: "Necessary and sufficient conditions for quadratic minimality", Carnegie-Mellon University, Department of Mathematics, November, 1981.
2. L. Contesse: "Une caractérisation complète des minima locaux en programmation quadratique", Numerische Mathematik 34, 1980, 315-312.
3. J. B. Dennis: "Mathematical programming and electrical networks", Wiley, New York, 1959.
4. W. S. Dorn: "Duality in quadratic programming", Quarterly of Applied Mathematics 18, 1960, 155-162.
5. B. C. Eaves: "On quadratic programming", Management Science 17, 1971, 698-711.
6. A. Fiacco and G. P. McCormick: "Nonlinear programming: Sequential unconstrained minimization techniques", Wiley, New York, 1968.
7. O. Fujiwara, S.-P. Han and O. L. Mangasarian: "Local duality of nonlinear programs", University of Wisconsin, Madison, Mathematics Research Center, Technical Summary Report #2329, February 1982.
8. S.-P. Han and O. L. Mangasarian: "Conjugate cone characterization of positive and semidefinite matrices", University of Wisconsin, Madison, Computer Sciences Department, Technical Report #471, March 1982.
9. O. L. Mangasarian: "Locally unique solutions of quadratic programs, linear and nonlinear complementarity problems", Mathematical Programming 19, 1980, 200-212.
10. O. L. Mangasarian: "Nonlinear programming", McGraw-Hill, New York, 1969.
11. P. Wolfe: "A duality theorem for nonlinear programming", Quarterly of Applied Mathematics 19, 1961, 239-244.

Errata to MRC TSR #2386

Page	Line	Now	Should be
4	Equation (5a)	$A_E x \leqq 0$	$A_J x = 0,\ A_K x \leqq 0$
4	Equation (5b)	$(u_K, u_I) \geqq 0$	$u_K \geqq 0,\ u_I = 0$
6	16	(5a)	(7a)
6	17	(5b)	(7b)
7	3	(5a)	,(5b)
8	Equation (17)	$\subset \{x \mid A_E x \leqq 0\}$	Deleted

Errata to CS TR #473

Page	Line	Now	Should be
4	Equation (5a)	$A_E x \leqq 0$	$A_J x = 0,\ A_K x \leqq 0$
4	Equation (5b)	$(u_K, u_I) \geqq 0$	$u_K \geqq 0,\ u_I = 0$
6	16	(5a)	(7a)
6	17	(5b)	(7b)
7	3	(5a)	,(5b)
8	Equation (17)	$\subset \{x \mid A_E x \leqq 0\}$	Deleted