FOUNDATIONS OF MATHEMATICAL ANALYSIS

§1. Sets and Propositions

§2. Products, Relations, and Functions

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THE FOUNDATIONS OF MATHEMATICAL ANALYSIS

0. INTRODUCTION

The purpose of this monograph is to present a brief overview of the basic structures upon which broad areas of mathematical analysis are built. It is intended to carry the reader from the material covered in elementary courses on linear algebra ('vectors, matrices, and determinants') and analysis ('advanced calculus'), to the point at which he or she can understand the level of abstraction and the fundamental concepts of courses on topology, real and complex function theory, measure and integration theory, probability theory, and functional analysis.

No attempt will be made to explain all the detailed ramifications of the mathematical structures presented; but the reader wishing to move on to higher levels of specialization will have been led through an outline of the principal concepts, definitions, vocabulary, and properties, sufficient to give him or her an understanding of the sense and flavor of these essential topics underlying modern mathematical analysis.

1. SETS AND PROPOSITIONS

We begin with the undefined concepts of an object (or *element* or *point*) and of a set (or *class* or *collection* or *family*) of objects. If an object denoted by the symbol x is in a set denoted by the symbol A, we say that x *belongs* to (or is a *member* of) A, and we write

$$x \in A. \tag{1.1}$$

If x is *not* in A, we write $\quad x \,\overline{\in}\, A \quad$ or $\quad x \notin A. \qquad (1.2)$

Two symbols representing objects will be considered *equal* if and only if they represent the *same* object: if x and y represent the same object, we write $x = y$; if not, $x \neq y$.

Two expressions or symbols denoting sets will be considered equal (or *identical*) and will be said to refer to the same set when every object belonging to one of the sets belongs to the other and vice versa. Thus no considerations of *order*, *arrangement*, or *repetition* are relevant to identifying a set. If A and B denote the same set, we write

$$A = B; \quad \text{if not,} \quad A \neq B. \tag{1.3}$$

If every element of a set A is in a set B, we say that A is a *subset* of (or is *contained* in) B, and conversely, that B *contains* A, and we write

$$A \subseteq B \quad \text{or} \quad B \supseteq A. \tag{1.4}$$

We note that a set may itself be considered as an object and be a member of another set. A set may also be used to label a family of objects (possibly sets): when so used, it is called an *index set*: for example, if the family K consists of the sets S_α, S_β, S_γ, and S_δ, where the set whose elements are α, β, γ, and δ is J, then J will be referred to as the index set and we may say that K is made up of all sets S_λ such that $\lambda \in J$. Introducing a useful formal concept, we call the set to which no object belongs the null (or *empty*) set and denote it by $\emptyset$. Thus, for whatever object is symbolized by x,

$$x \,\overline{\in}\, \emptyset. \tag{1.5}$$

Objects and sets whose membership relations can be unambiguously determined are called *well-defined*.

A set may be *specified* by *enumeration* of its members: the elements are listed, separated by commas, between curly brackets. For example,

$$\left.\begin{array}{lrcl} & \text{Family} & = & \{\text{Pa, Ma, Dick, Jane, Spot}\}, \\ \text{or} & D & = & \{x, y, z, p, q\}. \end{array}\right\} \qquad (1.6)$$

The enumeration may be implicit, using formulae and/or continuation denoted by ellipsis (...): for example,

$$\text{Even Numbers} = \{2, 4, 6, 8, \ldots, 2n, \ldots\}. \qquad (1.7)$$

It may sometimes be useful to use semi-colons as separators: for example, the signed integers may be specified as

$$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = \{0; 1, 2, 3, \ldots; -1, -2, -3, \ldots\}, \qquad (1.8)$$

according to convenience. Alternatively, a set may be specified by a *common property* of all its members: the curly brackets contain an arbitrary symbol, followed by a colon (sometimes a vertical bar is used instead) and then by a statement about the object represented by the symbol, which is true of all members of the set and false of all other objects. For example,

$$\left.\begin{array}{rcl} \text{Even Numbers} & = & \{k\text{: } k \text{ is a positive integer divisible by 2}\}, \\ \text{Children of John} & = & \{\maltese\text{: John is the father of } \maltese\}, \\ \text{Unit Disk} & = & \{z \mid z = x + iy \text{ and } x^2 + y^2 \leqslant 1\}. \end{array}\right\} \quad (1.9)$$

We note that the symbols used in (1.9) may each be replaced by another (so long as distinct symbols remain distinct in any one specification) without changing the set specified: such symbols are called *dummy variables*. For example $\{x\text{: } x \text{ is red}\} = \{y \mid y \text{ is red}\}$, and the Unit Disk is $\{C \mid C = p + iq \text{ and } p^2 + q^2 \leqslant 1\}$. The colon or vertical bar may be read as "such that": the Even Numbers are "the set of k, *such that* k is a positive integer divisible by 2." Often, some property of the members of a set is mentioned before the "such that" symbol: for example,

$$\left.\begin{aligned} \text{Even Numbers} &= \{\text{positive integer } k \mid k \text{ is divisible by } 2\}, \\ \text{Unit Disk} &= \{z = x + iy\colon x^2 + y^2 \leqslant 1\}. \end{aligned}\right\} \qquad (1.10)$$

We carefully distinguish between objects and the sets of which they are members, even if they are the *only* members. Thus, $x \in \{x\} \in \{\{x\}\}$; but these three entities are entirely different: $\{x\}$ is a subset of $\{x, y, z\}$; while $\{\{x\}\}$ is a subset of $\{\{x\}, \{x, y, z\}, A, B\}$ and also of $\{x, \{x\}\}$; and x is not necessarily a set at all. $\emptyset$ is the empty set; but the set $\{\emptyset\}$ is *not* empty: it has the element $\emptyset$.

We assume the undefined concepts of **true** and **false**, as applied to **statements** (or *assertions*): loosely described as grammatically correct sentences in the indicative mood.) We say that a statement is *about* an object if its *meaning* explicitly depends on understanding the nature of the object (or if the object appears, or is *referred to*, in the statement.) A statement is called a **proposition** if its truth or falsehood can be unambiguously determined. We shall sometimes find it useful to associate a *truth-value* with a proposition: if the proposition is true, its truth-value is 1, and if it is false, its truth-value is 0. If two propositions, symbolized by ϕ and ψ, say, have the same truth-value, we say that they are **equivalent** and write

$$\phi \Leftrightarrow \psi. \qquad (1.11)$$

We observe that equivalence of propositions is an *equivalence relation* (this idea will be returned to later), having the properties (true for all propositions ϕ, ψ, and χ) that

$$\left.\begin{array}{c} \phi \Leftrightarrow \phi; \\ \text{if} \quad \phi \Leftrightarrow \psi, \quad \text{then} \quad \psi \Leftrightarrow \phi; \\ \text{if} \quad \phi \Leftrightarrow \psi \quad \text{and} \quad \psi \Leftrightarrow \chi, \quad \text{then} \quad \phi \Leftrightarrow \chi. \end{array}\right\} \qquad (1.12)$$

We further note that the assertion of a proposition is equivalent to the assertion that it is true:

$$\phi \Leftrightarrow (\phi \Leftrightarrow 1); \qquad (1.13)$$

where we use parentheses in the usual mathematical way.

We observe that (1.11) is itself a statement, "ϕ is equivalent to ψ", which is constructed from the propositions ϕ and ψ: the beginning of an algebra of propositions, which we shall proceed to develop. We now introduce the idea of a truth-table: a table, for an expression involving a number of propositions, listing its truth-values corresponding to every possible combination of the truth-values of the constituent propositions. If the expression is constructed from n distinct propositions, then the truth-table will have 2^n lines. To illustrate, the truth-table for (1.11) is

ϕ	ψ	$\phi \Leftrightarrow \psi$
1	1	1
1	0	0
0	1	0
0	0	1
		▲

or, more compactly,

ϕ	$\Leftrightarrow$	ψ
1	1	1
1	0	0
0	0	1
0	1	0
	▲	

(1.14)

When used as above, the truth-table *defines* the effect of the operator $\Leftrightarrow$. We may also use truth-tables to *prove* or *verify* identities (that is, propositions which are necessarily true.) For example, we may prove (1.13) as follows:

ϕ	$\Leftrightarrow$	$(\phi$	$\Leftrightarrow$	$1)$
1	1	1	1	1
0	1	0	0	1
	▲			

(1.15)

In this case, the table has two lines, because there is only one constituent proposition, ϕ. In the first line, ϕ is given the truth-value 1, and in the second line, the truth-value 0. Of course, the "1" has the constant value 1, which is entered in each line. Using the definition (1.14), we next enter in each line the truth-value of "$(\phi \Leftrightarrow 1)$" beneath the corresponding "$\Leftrightarrow$". Finally, beneath the other "$\Leftrightarrow$", we enter the truth-value of the equivalence of the "ϕ" and of the expression in parentheses, whose truth-values we have just entered. If the entries in this final column are all "1" (see the column marked "▲"), then the identity is proved, being always true.

Let us now define three operations on propositions, the *negation* of ϕ, denoted by not ϕ or $\sim\phi$, the *disjunction* of ϕ and ψ, denoted by ϕ or ψ or by $\phi \vee \psi$, and the *conjunction* of ϕ and ψ, denoted by ϕ and ψ or $\phi \wedge \psi$. These are defined by the truth-tables:

$\sim$	ϕ
0	1
1	0
▲	

ϕ	$\vee$	ψ
1	1	1
1	1	0
0	1	1
0	0	0
	▲	

ϕ	$\wedge$	ψ
1	1	1
1	0	0
0	0	1
0	0	0
	▲	

(1.16)

To avoid having to use excessively many parentheses to determine the order, we adopt the usual algebraic convention, that *operations* ($\sim$, $\vee$, $\wedge$) take precedence over *relations* ($\Leftrightarrow$, $\Rightarrow$, $\Leftarrow$). It then follows rather easily that:

$$\left.\begin{array}{c} \sim(\sim\phi) \Leftrightarrow \phi; \\ \phi \vee (\psi \vee \chi) \Leftrightarrow (\phi \vee \psi) \vee \chi; \quad \phi \wedge (\psi \wedge \chi) \Leftrightarrow (\phi \wedge \psi) \wedge \chi; \\ \sim(\phi \vee \psi) \Leftrightarrow (\sim\phi) \wedge (\sim\psi); \quad \sim(\phi \wedge \psi) \Leftrightarrow (\sim\phi) \vee (\sim\psi); \\ \phi \wedge (\psi \vee \chi) \Leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi); \quad \phi \vee (\psi \wedge \chi) \Leftrightarrow (\phi \vee \psi) \wedge (\phi \vee \chi). \end{array}\right\} \quad (1.17)$$

For example, we give the truth-table proofs of the fourth and seventh identities:

$\sim$	$(\phi$	$\vee$	$\psi)$	$\Leftrightarrow$	$(\sim$	$\phi)$	$\wedge$	$(\sim$	$\psi)$
0	1	1	1	1	0	1	0	0	1
0	1	1	0	1	0	1	0	1	0
0	0	1	1	1	1	0	0	0	1
1	0	0	0	1	1	0	1	1	0
				▲					

ϕ	$\vee$	$(\psi$	$\wedge$	$\chi)$	$\Leftrightarrow$	$(\phi$	$\vee$	$\psi)$	$\wedge$	$(\phi$	$\vee$	$\chi)$
1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	0	0	1	1	1	1	1	1	1	0
1	1	0	0	1	1	1	1	0	1	1	1	1
1	1	0	0	0	1	1	1	0	1	1	1	0
0	1	1	1	1	1	0	1	1	1	0	1	1
0	0	1	0	0	1	0	1	1	0	0	0	0
0	0	0	0	1	1	0	0	0	0	0	1	1
0	0	0	0	0	1	0	0	0	0	0	0	0
					▲							

The second and third identities are called *associative laws*; the sixth and seventh are called *distributive laws*.

We say that a proposition ϕ *implies* a proposition ψ (or ψ is *implied* by ϕ, or *if* ϕ *then* ψ, or ψ *if* ϕ, or ϕ *only if* ψ) when ψ is true whenever ϕ is true: we write

$$\phi \Rightarrow \psi \quad \text{or} \quad \psi \Leftarrow \phi. \qquad (1.18)$$

The corresponding truth-tables are

ϕ	$\Rightarrow$	ψ
1	1	1
1	0	0
0	1	1
0	1	0
	▲	

and

ϕ	$\Leftarrow$	ψ
1	1	1
1	1	0
0	0	1
0	1	0
	▲	

(1.19)

From this it follows immediately that

$$\left.\begin{aligned} &(\phi \Rightarrow \psi) \;\Leftrightarrow\; (\sim \phi) \vee \psi \\ \text{and that}\qquad &(\phi \Leftarrow \psi) \;\Leftrightarrow\; \phi \vee (\sim \psi)\ . \end{aligned}\right\} \tag{1.20}$$

We can easily verify the further identities

$$\left.\begin{aligned} (\phi \Rightarrow \psi) &\;\Leftrightarrow\; (\psi \Leftarrow \phi); \\ (\phi \Rightarrow \psi) &\;\Leftrightarrow\; ((\sim \phi) \Leftarrow (\sim \psi)); \\ (\phi \Leftrightarrow \psi) &\;\Leftrightarrow\; ((\phi \Rightarrow \psi) \wedge (\phi \Leftarrow \psi)); \\ \phi &\;\Rightarrow\; \phi; \\ \phi &\;\Leftarrow\; \phi; \\ ((\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \chi)) &\;\Rightarrow\; (\phi \Rightarrow \chi). \end{aligned}\right\} \tag{1.21}$$

The fourth of these identities demonstrates that the relation $\Rightarrow$ is *reflexive*, and the sixth, that $\Rightarrow$ is *transitive*: between them, they show that $\Rightarrow$ is an *order relation* for propositions (we shall return to this idea also, at a later stage.) It also follows that $\Leftarrow$ is an order relation; indeed, by the first identity in (1.21), $\Leftarrow$ is the *inverse* relation to $\Rightarrow$. The third identity accounts for the common way of reading $\Leftrightarrow$ as "if and only if" —— it is also sometimes written as iff.

If a proposition refers to one or more objects, this may be explicitly indicated by a notation such as $\phi(x)$ or $\psi(p, q, r)$. If the proposition $\phi(x)$ is true for all choices of the object symbolized by x, we write

$$(\forall x)\ \phi(x). \tag{1.22}$$

We call the expression $(\forall x)$ a *quantifier*, and we may read it as "for every choice of x" or "for all x". We note that, here also, the "x" is a dummy variable, replaceable with no effect by any other symbol. If there is at least one object for which a proposition $\phi(x)$ is true, we may write

$$(\exists x)\ \phi(x), \tag{1.23}$$

using the quantifier $(\exists x)$, which may be read as "for some x" or "there exists an x

such that". It follows at once that

$$\sim ((\forall x)\ \phi(x)) \quad \Leftrightarrow \quad (\exists x)\ (\sim \phi(x))\ , \tag{1.24}$$

which may be abbreviated to the quantifier identity,

$$\sim (\forall x) \quad \Leftrightarrow \quad (\exists x) \sim ; \tag{1.25}$$

and similarly, we have that

$$\sim (\exists x) \quad \Leftrightarrow \quad (\forall x) \sim . \tag{1.26}$$

A third quantifier is sometimes useful: $(\exists !x)$, which may be read "there is a *unique* x such that". We see that

$$(\forall x) \;\Rightarrow\; (\exists x) \quad \text{and} \quad (\exists !x) \;\Rightarrow\; (\exists x). \tag{1.27}$$

We now return to the consideration of sets. First, we observe that any well-formed mathematical statement is a proposition, and in particular, that (1.1) and (1.2) are propositions. In fact, we see that

$$(x \notin A) \quad \Leftrightarrow \quad \sim (x \in A)\ . \tag{1.28}$$

Further, we have the tautology,

$$(\forall A)\ A = \{x \mid x \in A\}. \tag{1.29}$$

In fact, given any proposition $\phi(x)$, we may formally define a corresponding set by writing

$$A = \{x \mid \phi(x)\}. \tag{1.30}$$

Owing to certain problems and paradoxes, it is necessary to limit the objects considered to members of a given universe of discourse (or *global set*) W: thus, to define a valid set, we must modify (1.30) to the form

$$A = \{x \mid x \in W \wedge \phi(x)\} \quad \text{or} \quad A = \{x \in W\colon \phi(x)\}. \tag{1.31}$$

In other words, we may only define sets as subsets of a universe of discourse —— which may itself be any well-defined set. (Nevertheless, the reference to W is often omitted; though it must be understood.)

Applying the algebra of propositions to statements of the form (1.1), we may now construct an algebra of sets. (Both of these algebras are given the name *Boolean algebra*.) Corresponding to the equivalence of propositions, we have the *equality* of sets:

$$(A = B) \Leftrightarrow ((x \in A) \Leftrightarrow (x \in B)). \tag{1.32}$$

Clearly, equality is also an equivalence relation, with the three characteristic properties exhibited in (1.12): for all sets A, B, and C,

$$\left.\begin{aligned} A &= A; \\ (A = B) &\Rightarrow (B = A); \\ (A = B) \wedge (B = C) &\Rightarrow (A = C). \end{aligned}\right\} \tag{1.33}$$

We now define set *operations* (^{c}, $\cup$, $\cap$) and *relations* ($=$, $\subseteq$, $\supseteq$, $\in$), again giving the former precedence over the latter, in the absence of parentheses. Corresponding to the negation of a proposition, we have the complement of a set:

$$A^{c} = \{x\colon \sim (x \in A)\}. \tag{1.34}$$

Corresponding to the disjunction of two propositions, we have the union of two sets:

$$A \cup B = \{x\colon (x \in A) \vee (x \in B)\}. \tag{1.35}$$

Corresponding to the conjunction of two propositions, we have the intersection of two sets:

$$AB = A \cap B = \{x\colon (x \in A) \wedge (x \in B)\}. \tag{1.36}$$

The latter notation is formally preferable (compare (1.36) with (1.35)), but the former has the irresistible advantage of brevity. Seven identities analogous to those in (1.17) follow immediately: for all sets A, B, and C,

$$\left.\begin{aligned} (A^{c})^{c} &= A; & & \\ A \cup (B \cup C) &= (A \cup B) \cup C; & A \cap (B \cap C) &= (A \cap B) \cap C; \\ (A \cup B)^{c} &= (A^{c}) \cap (B^{c}); & (A \cap B)^{c} &= (A^{c}) \cup (B^{c}); \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C); & A \cup (B \cap C) &= (A \cup B) \cap (A \cup C). \end{aligned}\right\} \tag{1.37}$$

(The method of proof of these is discussed in Exercises ⟨1.9⟩ and ⟨1.10⟩.)

Corresponding to implication between propositions, we have *containment* between sets:

$$\left.\begin{array}{ll} & (A \subseteq B) \Leftrightarrow (\forall x)\,((x \in A) \Rightarrow (x \in B)), \\ \text{or equivalently,} & (A \supseteq B) \Leftrightarrow (\forall x)\,((x \in A) \Leftarrow (x \in B)). \end{array}\right\} \tag{1.38}$$

The analog of (1.20) is now the assertion

$$(A \subseteq B) \Leftrightarrow (W = (A^{c}) \cup B) \quad \text{or} \quad (A \supseteq B) \Leftrightarrow (W = A \cup (B^{c})); \tag{1.39}$$

or using a quantifier,

$$(A \subseteq B) \Leftrightarrow (\forall x)\ ((x \notin A) \vee (x \in B)), \tag{1.40}$$

where we recall that "$(\forall x)$" is an abbreviation for "$(\forall x \in W)$". The six additional identities in (1.21) give us that

$$\left.\begin{array}{l} (A \subseteq B) \Leftrightarrow (B \supseteq A); \quad (A \subseteq B) \Leftrightarrow ((A^{c}) \supseteq (B^{c})); \\ (A = B) \Leftrightarrow ((A \subseteq B) \wedge (A \supseteq B)); \\ A \subseteq A\ ; \qquad A \supseteq A\ ; \\ (A \subseteq B) \wedge (B \subseteq C) \Rightarrow (A \subseteq C). \end{array}\right\} \tag{1.41}$$

Thus we see that $\subseteq$ and $\supseteq$ are mutually inverse order relations on sets.

Using quantifiers and index sets, we may extend the concepts of union and intersection of sets to arbitrary families of sets. Let J be the index set and let

$$\mathsf{F} = \{E_\alpha \mid \alpha \in J\}. \tag{1.42}$$

Then we define the union of F as

$$\cup\, \mathsf{F} = \cup_{\alpha \in J} E_\alpha = \{x \mid (\exists \alpha \in J)\ x \in E_\alpha\}, \tag{1.43}$$

and the intersection of F as

$$\cap\, \mathsf{F} = \cap_{\alpha \in J} E_\alpha = \{x \mid (\forall \alpha \in J)\ x \in E_\alpha\}. \tag{1.44}$$

Sometimes, the family F is finite (say, with $J = \{1, 2, 3, \ldots, n\}$) or countably infinite (this idea will be discussed later) (say $J = \{1, 2, 3, \ldots\}$). Then we write

$$\left.\begin{array}{ll} & \cup\, \mathsf{F} = \cup_{\alpha=1}^{n} E_\alpha \quad \text{and} \quad \cap\, \mathsf{F} = \cap_{\alpha=1}^{n} E_\alpha, \\ \text{or} & \cup\, \mathsf{F} = \cup_{\alpha=1}^{\infty} E_\alpha \quad \text{and} \quad \cap\, \mathsf{F} = \cap_{\alpha=1}^{\infty} E_\alpha, \text{ respectively,} \end{array}\right\} \tag{1.45}$$

in close analogy with the mathematical notation for sums (Σ) and products (Π).

A few more identities and relations will round off our discussion. First, we note that, formally,

$$\emptyset = \{x\colon 0\} \quad \text{and} \quad W = \{x\colon 1\}. \tag{1.46}$$

We also see that

$$\sim \phi \Rightarrow (\phi \Rightarrow \psi). \tag{1.47}$$

(This is sometimes expressed by saying that "a false proposition implies every Proposition.") Pairing off identities for propositions and sets, we have:

$$\left.\begin{array}{llll} \emptyset^c = W, & \sim 0 \Leftrightarrow 1; & W^c = \emptyset, & \sim 1 \Leftrightarrow 0; \\ A \cup A = A = A \cap A, & \phi \vee \phi \Leftrightarrow \phi \Leftrightarrow \phi \wedge \phi; & & \\ A \cup \emptyset = A, & \phi \vee 0 \Leftrightarrow \phi; & A \cup W = W, & \phi \vee 1 \Leftrightarrow 1; \\ A \cap \emptyset = \emptyset, & \phi \wedge 0 \Leftrightarrow 0; & A \cap W = A, & \phi \wedge 1 \Leftrightarrow \phi; \\ A \cap B \subseteq A \subseteq A \cup B. & & & \end{array}\right\} \tag{1.48}$$

Some of our results may be generalized for arbitrary families of sets: for instance, the last relation above gives

$$(\forall \zeta \in J)\ \cap_{\alpha\in J} E_\alpha \subseteq E_\zeta \subseteq \cup_{\alpha\in J} E_\alpha. \tag{1.49}$$

Similarly, we can verify that

$$\left.\begin{array}{ll} (\cup_{\alpha\in J} E_\alpha)^c = \cap_{\alpha\in J} (E_\alpha)^c; & (\cap_{\alpha\in J} E_\alpha)^c = \cup_{\alpha\in J} (E_\alpha)^c; \\ A \cap (\cup_{\alpha\in J} E_\alpha) = \cup_{\alpha\in J} (A \cap E_\alpha); & A \cup (\cap_{\alpha\in J} E_\alpha) = \cap_{\alpha\in J} (A \cup E_\alpha). \end{array}\right\} \tag{1.50}$$

If $AB = \emptyset$, we say that the sets A and B are **disjoint** (the intersection used to be called the *join*; and the union was the *meet* — though it seems as if it should be the other way around! The same kind of paradox of nomenclature occurs in the fact that, if proposition $\phi(x)$ implies proposition $\psi(x)$, we say that $\phi(x)$ *contains* $\psi(x)$; yet it is $\{x\colon \psi(x)\}$ which contains $\{x\colon \phi(x)\}$.) More or less conversely, if $A^cB \neq \emptyset$ and $A \subseteq B$, we say that A is a *proper subset* of B and write $A \subset B$ or $B \supset A$. Finally, we shall find it convenient to refer to *disjoint unions* of sets; that is, to unions of the form (1.43), in which every $E_\alpha E_\beta = \emptyset$ (for $\alpha \neq \beta$): this will be indicated by the notation $\uplus_{\alpha\in J} E_\alpha$.

One last fact will be mentioned, for its intrinsic beauty and its importance to computer design. We may define a new operation between propositions, denoted by ϕ nor ψ or $\phi \downarrow \psi$ and defined by the truth-table and formula:

ϕ	$\downarrow$	ψ
1	0	1
1	0	0
0	0	1
0	1	0
	▲	

and $$\phi \downarrow \psi \Leftrightarrow \sim(\phi \vee \psi). \tag{1.51}$$

It is then easily verified that the three previously defined operations may be expressed in terms of $\downarrow$ (and so can the relations), as follows:

$$\left.\begin{aligned} \sim\phi &\Leftrightarrow \phi \downarrow \phi; \\ \phi \vee \psi &\Leftrightarrow (\phi \downarrow \psi) \downarrow (\phi \downarrow \psi); \\ \phi \wedge \psi &\Leftrightarrow (\phi \downarrow \phi) \downarrow (\psi \downarrow \psi). \end{aligned}\right\} \tag{1.52}$$

Finally, to reduce the use of parentheses even further, we follow common algebraic practice by adopting a hierarchy of *precedence* for operations and relations occurring in expressions involving sets and propositions:

[1] The *interior* of a matched pair of *delimiters* (such as $(\cdots)$, $(\cdots)$, $[\cdots]$, $[\![\cdots]\!]$, $\{\cdots\}$, $\langle\cdots\rangle$, $|\cdots|$, $|\cdots|$, $\|\cdots\|$, &c.) is evaluated before the *exterior*.

[2] Operations and relations between *sets* are evaluated before operations and relations between *propositions* (naturally, since relations between sets are propositions, but operations and relations between propositions yield propositions, not sets.)

[3] Both for sets and for propositions, *operations* are evaluated before *relations*. (This rule has already been formulated for both cases.)

[4]

Operations on sets:	^{c} before $\cap$ before $\cup$.
Relations between sets:	$=$, $\subseteq$, $\supseteq$, $\in$, all together.
Operations on propositions:	$\sim$ before $\wedge$ before $\vee$ before $\downarrow$.
Relations between propositions:	$\Rightarrow$ and $\Leftarrow$ (together) before $\Leftrightarrow$.

[5] *Quantifiers* affect all subsequent expressions until a *binary propositional operator* ($\wedge$, $\vee$, $\downarrow$) not enclosed in parenthetic delimiters is encountered, or until the end of any parenthesis containing the quantifier in question.

Examples of the simplification afforded by these rules follow:

(1.17), fourth & seventh: $\sim(\phi \vee \psi) \Leftrightarrow \sim\phi \wedge \sim\psi, \quad \phi \vee \psi \wedge \chi \Leftrightarrow (\phi \vee \psi) \wedge (\phi \vee \chi);$

(1.33), third: $A = B \wedge B = C \Rightarrow A = C;$

(1.37), fifth & sixth: $(AB)^{c} = A^{c} \cup B^{c}, \quad A \cap (B \cup C) = AB \cup AC;$

(1.39), second: $A \supseteq B \Leftrightarrow W = A \cup B^{c};$

(1.41), third & sixth: $A = B \Leftrightarrow A \subseteq B \wedge A \supseteq B, \quad A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C.$

EXERCISES 1

It cannot be emphasized too strongly that the material in the exercises is an integral part of the text, and that the results quoted therein is important and will, in many cases, be used later. To this end, results given in exercises are numbered for reference.

In the first place, the reader is urged to verify in detail all the identities and other results stated without proof in the text.

⟨1.1⟩ If $\mathsf{F} = \{E_j : j = 1, 2, 3, \ldots\}$ is a countably infinite family of sets E_j, then we write

$$\cup\, \mathsf{F} = \cup_{j=1}^{\infty} E_j = \sup_{j\to\infty} E_j \quad \text{and} \quad \cap\, \mathsf{F} = \cap_{j=1}^{\infty} E_j = \inf_{j\to\infty} E_j \tag{1.53}$$

(compare (1.45).) Show that (compare (1.49))

$$\inf_{j\to\infty} E_j \subseteq \sup_{j\to\infty} E_j. \tag{1.54}$$

⟨1.2⟩ For the same countably infinite family F, we write

$$\liminf_{j\to\infty} E_j = \cup_{j=1}^{\infty} \cap_{i=j}^{\infty} E_i \quad \text{and} \quad \limsup_{j\to\infty} E_j = \cap_{j=1}^{\infty} \cup_{i=j}^{\infty} E_i. \tag{1.55}$$

Show that

$$\left.\begin{array}{l} x \in \liminf_{j\to\infty} E_j \ \text{ iff } \ x \text{ is in all but a finite number of the } E_j; \\ x \in \limsup_{j\to\infty} E_j \ \text{ iff } \ x \text{ is in infinitely many of the } E_j. \end{array}\right\} \tag{1.56}$$

and that

Hence show that

$$\liminf_{j\to\infty} E_j \subseteq \limsup_{j\to\infty} E_j. \tag{1.57}$$

⟨1.3⟩ The class of *all subsets* of a given set A is called the **power class** of A and is denoted by $\mathsf{P}(A)$:

$$\mathsf{P}(A) = \{X \mid X \subseteq A\}. \tag{1.58}$$

Show that, if A has just n elements, then $\mathsf{P}(A)$ has exactly 2^n members.

⟨1.4⟩ Prove the following results:

$$A \subseteq \emptyset \Rightarrow A = \emptyset; \quad A \subseteq B \Rightarrow A \cup B = B \wedge A \cap B = A; \quad A = AB \uplus AB^c. \tag{1.59}$$

⟨1.5⟩ Prove that $x \in S \Leftrightarrow \{x\} \subseteq S$ and $\bigcup_{x \in S} \{x\} = S$. (1.60)

⟨1.6⟩ Prove that $\left(\bigcup_{\alpha \in A} E_\alpha\right) \cap \left(\bigcup_{\beta \in B} F_\beta\right) = \bigcup_{\alpha \in A} \bigcup_{\beta \in B} (E_\alpha \cap F_\beta)$. (1.61)

and that $\left(\bigcap_{\alpha \in A} E_\alpha\right) \cup \left(\bigcap_{\beta \in B} F_\beta\right) = \bigcap_{\alpha \in A} \bigcap_{\beta \in B} (E_\alpha \cup F_\beta)$. (1.62)

⟨1.7⟩ Prove that

$$A \subseteq P \wedge B \subseteq Q \Rightarrow A \cup B \subseteq P \cup Q \wedge A \cap B \subseteq P \cap Q. \tag{1.63}$$

⟨1.8⟩ Prove that, if $(\forall \alpha \in A)\ E_\alpha \subseteq F_\alpha$, then

$$\left(\bigcup_{\alpha \in A} E_\alpha\right) \subseteq \left(\bigcup_{\alpha \in A} F_\alpha\right) \quad \text{and} \quad \left(\bigcap_{\alpha \in A} E_\alpha\right) \subseteq \left(\bigcap_{\alpha \in A} F_\alpha\right). \tag{1.64}$$

⟨1.9⟩ Prove that the definitions (1.34), (1.35), and (1.36) are respectively equivalent to stating that, for all $x \in W$ and all sets A and B contained in W,

$$(x \in A^{\mathrm{c}}) \Leftrightarrow \sim(x \in A); \tag{1.65}$$

$$(x \in (A \cup B)) \Leftrightarrow (x \in A) \vee (x \in B); \tag{1.66}$$

$$(x \in (A \cap B)) \Leftrightarrow (x \in A) \wedge (x \in B). \tag{1.67}$$

⟨1.10⟩ Use the forms (1.65), (1.66), and (1.67) to prove the identities in (1.37). [For example, to prove the fourth identity; we note that (by the fourth identity in (1.17))

$$(x \in (A \cup B)^{\mathrm{c}}) \Leftrightarrow \sim(x \in (A \cup B)) \Leftrightarrow \sim((x \in A) \vee (x \in B)) \Leftrightarrow (\sim(x \in A)) \wedge (\sim(x \in B))$$

$$\Leftrightarrow (x \in A^{\mathrm{c}}) \wedge (x \in B^{\mathrm{c}}) \Leftrightarrow (x \in ((A^{\mathrm{c}}) \cap (B^{\mathrm{c}}))), \quad \text{as required.]}$$

⟨1.11⟩ Prove that $A \cup B = AB^{\mathrm{c}} \uplus AB \uplus A^{\mathrm{c}}B$. (1.68)

2. PRODUCTS, RELATIONS, AND FUNCTIONS

A *non-empty* set S is said to be a *singleton* (or to have just one element) iff

$$x \in S \wedge y \in S \Rightarrow x = y; \tag{2.1}$$

or, equivalently, iff

$$x \in S \Rightarrow S\ \{x\}^c = \emptyset. \tag{2.2}$$

A *non-empty* set T is said to be a *pair* (or to have just two elements) iff

$$x \in T \Rightarrow T\ \{x\}^c \text{ is a singleton.} \tag{2.3}$$

Given any pair $T = \{x, y\}$, we may construct a new set, denoted by

$$[x, y] = \{\{x\}, \{x, y\}\}; \tag{2.4}$$

and, similarly, another new set denoted by

$$[y, x] = \{\{y\}, \{x, y\}\}. \tag{2.5}$$

The choice of (2.4) or (2.5) determines an *ordering* of T, and either of these new sets is called an ordered pair. (More extensive consideration of *number* and of *order* will be given later.)

We now define the Cartesian product of two arbitrary sets A and B as

$$A \times B = \{[a, b]: a \in A \wedge b \in B\}. \tag{2.6}$$

The properties of the algebra of sets established in §1 lead us readily to see that, in general, $A \times B$ and $B \times A$ will be quite different, and that

$$\left.\begin{array}{ll} \emptyset \times A = A \times \emptyset = \emptyset; & A \subseteq P \wedge B \subseteq Q \Leftrightarrow (A \times B) \subseteq (P \times Q); \\ (A \cup B) \times C = (A \times C) \cup (B \times C); & (A \cap B) \times C = (A \times C) \cap (B \times C); \\ A \times (B \cup C) = (A \times B) \cup (A \times C); & A \times (B \cap C) = (A \times B) \cap (A \times C). \end{array}\right\} \tag{2.7}$$

These concepts readily generalize to an arbitrary family of sets indexed as in (1.42). We define the general family of *indexed objects* denoted by

$$[x_\alpha]_{\alpha \in J} = \{[\alpha, x_\alpha]: \alpha \in J\} \tag{2.8}$$

[the purpose of this construction being to firmly label each x_α with its index α, even when some of the x_α are identical: we shall see, later, that we are, in fact, defining a *function* x mapping the index set J into the global set W, the image of α being x_α];

and then we may define the Cartesian product of the family F in (1.42) as

$$\mathsf{X}\,\mathsf{F} = \mathsf{X}_{\alpha\in J}\, E_\alpha = \{[x_\alpha]_{\alpha\in J}\colon (\forall\alpha \in J)\ x_\alpha \in E_\alpha\}. \tag{2.9}$$

We generally do not make a strong distinction between ordered and unordered Cartesian products: an ordered Cartesian product arises when the index set is itself ordered. Thus, the product defined in (2.6) is seen as the case of (2.9) arising when T is the set $\{1, 2\}$, which has the usual ordering $[1, 2]$; with $E_1 = A$ and $E_2 = B$. The relations in (2.7) naturally extend to (2.9): for example,

$$\left.\begin{aligned} &((\exists\alpha)\ E_\alpha = \emptyset) \ \Rightarrow\ \mathsf{X}_{\alpha\in J}\, E_\alpha = \emptyset \\ \text{and}\qquad &((\forall\alpha)\ E_\alpha \subseteq F_\alpha) \ \Leftrightarrow\ \mathsf{X}_{\alpha\in J}\, E_\alpha \subseteq \mathsf{X}_{\alpha\in J}\, F_\alpha. \end{aligned}\right\} \tag{2.10}$$

Given the product $A \times B$, any subset of the product is called a relation. In other words, a relation is an arbitrary set of ordered pairs, with the first element of the pair in a given set A and the second element of the pair in a given set B. Several notations are available: if

$$\left.\begin{aligned} &[a, b] \in R \subseteq (A \times B), \\ \text{then also}\qquad &R(a, b) \Leftrightarrow 1 \quad\text{and}\quad a\,R\,b. \end{aligned}\right\} \tag{2.11}$$

We also write $\quad R\colon A \to B \quad$ for $\quad R \subseteq (A \times B)$ $\hfill$ (2.12)

$$\left.\begin{aligned} \text{and}\qquad & b \in R(a) \quad\text{for}\quad [a, b] \in R, \\ \text{where}\qquad & R(a) = \{b\colon [a, b] \in R\}. \end{aligned}\right\} \tag{2.13}$$

Similarly, write $\quad R^{-1}(b) = \{a\colon [a, b] \in R\}.$ $\hfill$ (2.14)

We call $R(a)$ the *image* of a in B, and $R^{-1}(b)$, the *inverse image* of b in A. More generally, for any $H \subseteq A$ and $K \subseteq B$, define the image of H in B as

$$R(H) = \{b\colon (\exists a \in H)\ [a, b] \in R\}, \tag{2.15}$$

and the inverse image of K in A as

$$R^{-1}(K) = \{a\colon (\exists b \in K)\ [a, b] \in R\}. \tag{2.16}$$

If R is a relation from A into B, we may define the inverse relation by

$$R^{-1} = \{[b, a]\colon [a, b] \in R\} \subseteq (B \times A). \tag{2.17}$$

We then see that *the inverse image of a set or point* (as defined in (2.14) and (2.16)) *under a relation R is just the image of the inverse relation R^{-1}*; and that

$$(R^{-1})^{-1} = R. \tag{2.18}$$

We now consider special properties of relations. First, if

$$(\forall a \in A)\ R(a) \text{ is a singleton} \tag{2.19}$$

(that is, if the image of any point in A is a single point in B), then we call R a function or mapping *from A into B*. We do not usually write $a\ R\ b$ when R is a function (though there is no valid reason why we should not do so), but we do write

$$b = R(a) \quad \text{for} \quad \{b\} = R(a), \tag{2.20}$$

by a time-honored and excusable abbreviation of notation, and sometimes

$$b = R\,a \quad \text{or} \quad b = R_a, \tag{2.21}$$

according to choice and convenience.

For any relation of the form (2.11), we call A the domain of R and B the *codomain* of R. The image of A, $R(A)$, is called the range of R, and the inverse image of B, $R^{-1}(B)$, is called the *total support* of R. [The *support* of a real-valued function usually denotes the subset of the domain, on which the function takes non-zero values.] Clearly,

$$R(A) \subseteq B \quad \text{and} \quad R^{-1}(B) \subseteq A. \tag{2.22}$$

It follows from (2.19) that, if R is a function,

$$R^{-1}(B) = A. \tag{2.23}$$

If R is a function and

$$R(A) = B, \tag{2.24}$$

we say that R is *surjective* (or that R maps A *onto* B.) If R is a function and

$$(\forall b \in R(A))\ R^{-1}(b) \text{ is a singleton} \tag{2.25}$$

(that is, if every point in the range of R is the image of just one point in A), then we say that R is *injective* (or that R is a *one-to-one mapping of A into B*.) Finally, if R is both surjective and injective (that is, both one-to-one and onto); so that both R and R^{-1} are functions (from A onto B and from B onto A, respectively); then we call R a *bijection* from A to B.

If R is a function from A into B and $H \subseteq A$, then we write

$$R_{[H]} = R \cap (H \times B) \tag{2.26}$$

for the restriction of R to the domain H (often, we abbreviate notation by omitting the subscript $[H]$ and identifying R with its restriction.) Clearly, $R_{[H]}$ is a relation of H to B, and by (2.1), $R_{[H]}$ is a function from H to B. We see that

$$R_{[H]}(H) = R(H) \subseteq R(A). \tag{2.27}$$

We note that, *for functions, set operations and relations are preserved by the inverse-image relation*; that is, if R is a function from A into B, and E and F are subsets of B, then

$$\left.\begin{array}{l} R^{-1}(\emptyset) = \emptyset; \quad R^{-1}(B) = A; \quad R^{-1}(BE^{c}) = A\ (R^{-1}(E))^{c}; \\ R^{-1}(E \cup F) = R^{-1}(E) \cup R^{-1}(F); \quad R^{-1}(E \cap F) = R^{-1}(E) \cap R^{-1}(F); \\ E \subseteq F \ \Rightarrow\ R^{-1}(E) \subseteq R^{-1}(F); \quad EF = \emptyset \ \Rightarrow\ R^{-1}(E)\ R^{-1}(F) = \emptyset. \end{array}\right\} \tag{2.28}$$

⟦*NOTE: Proofs will be placed between double square brackets.* The inverse image is defined in (2.16): this yields the first identity immediately. The second identity is just (2.23), true for any function. Since a function maps each point into only one point, the inverse images of disjoint sets must be disjoint (that is, there is no x, such that $R(x)$ is in both E and F, if $EF = \emptyset$): this proves the seventh relation. Now, $y = R(x) \Leftrightarrow x \in R^{-1}(BE^{c}) \Leftrightarrow y \in BE^{c} \Leftrightarrow y \notin E \Leftrightarrow x \notin R^{-1}(E) \Leftrightarrow x \in A\ (R^{-1}(E))^{c}$, the third identity above. Iff $y = R(x)$ and $y \in E \cup F$, then $y \in E$ or $y \in F$; so $x \in R^{-1}(E)$ or $x \in R^{-1}(F)$; whence the fourth identity follows. Replacing "$\cup$" by "$\cap$" and "or" by "and" in this argument, we similarly obtain the fifth identity. Finally, if $E \subseteq F$, then any x for which $R(x) \in E$ is an x for which $R(x) \in F$: the sixth relation follows.⟧

Now suppose that a function f maps a set X into a set Y, and that another function g maps the set Y into a set Z. Then we may define the composition of the two functions as the function $g \circ f$, mapping X into Z in such a way that

$$(\forall x \in X)\ (g \circ f)(x) = g(f(x)). \tag{2.29}$$

It follows from this definition that, if a further function h maps Z into a set T, then

$$h \circ (g \circ f) = (h \circ g) \circ f. \tag{2.30}$$

Here, the concept of *equality of functions* derives directly from the equality of sets, by way of the definition (2.11), since functions are relations and relations are sets of ordered pairs. This may be expressed in the form

$$e = f \Leftrightarrow e \subseteq (X \times Y) \wedge f \subseteq (X \times Y) \wedge (\forall x \in X)\ e(x) = f(x). \tag{2.31}$$

It is clear from the definition that the Cartesian product $X \times X$ necessarily has the subset

$$1_X = \{[x, x]: x \in X\}. \tag{2.32}$$

This relation is evidently a function; and indeed, it is a bijection which is its own inverse. It is called the identity or *unit function* of X, and the subscript X will often be omitted, when the meaning of 1 is clear. It follows at once that, for any function $f: X \to Y$,

$$f = f \circ 1_X = 1_Y \circ f. \tag{2.33}$$

Given the function $f: X \to Y$, there may or may not exist a *left-inverse* f_L of f; that is, a function $f_L: Y \to X$ such that

$$f_L \circ f = 1_X. \tag{2.34}$$

If $[x, y] \in f$, then $[y, x]$ will belong to the inverse relation f^{-1}; but, for any $y = f(x)$, there may be more than one $z \in f^{-1}(y)$ (always including $z = x$, of course.) If this is the case, then no left-inverse can occur (since $f_L(y)$ needs to take two or more values); and if f^{-1} is a function from $f(X)$ onto X (that is, if f is injective), then f_L is a left-inverse iff its restriction to $f(X)$ is f^{-1}:

$$f_{L[f(X)]} = f^{-1}. \tag{2.35}$$

Similarly, there may or may not exist a *right-inverse* function of f; that is, $f_R: Y \to X$, such that

$$f \circ f_R = 1_Y. \tag{2.36}$$

If any y exists in Y which is not in $f(X)$, then such a y cannot be mapped into itself by any $f \circ f_R$ whatsoever; while if $f(X) = Y$ (that is, if f is surjective), then f_R is a right-inverse iff

$$(\forall y \in Y)\ f_R(y) \in f^{-1}(y). \tag{2.37}$$

We have thus shown that

$$\left.\begin{array}{l} f \text{ has a left-inverse iff it is injective,} \\ \text{and} \quad f \text{ has a right-inverse iff it is surjective.} \end{array}\right\} \tag{2.38}$$

Iff f is both injective and surjective, it is a bijection, and therefore has both a left and a right inverse. Since f is a surjection, $f(X) = Y$, so that, by (2.35), $f_L = f^{-1}$; and since f is an injection, f^{-1} is a function, so that, by (2.37), $f_R = f^{-1}$. Thus,

$$f_L = f_R = f^{-1} \text{ iff } f \text{ is a bijection.} \tag{2.39}$$

We now turn to general relations R, whose domain and codomain coincide:

$$R \subseteq (A \times A). \tag{2.40}$$

If

$$1_A \subseteq R \quad \text{or} \quad (\forall a \in A)\ a\ R\ a, \tag{2.41}$$

we say that R is **reflexive**. If

$$(\forall a, b \in A)\ a\ R\ b \Rightarrow b\ R\ a, \tag{2.42}$$

we say that R is **symmetric**. If

$$(\forall a, b, c \in A)\ (a\ R\ b \wedge b\ R\ c) \Rightarrow a\ R\ c, \tag{2.43}$$

we say that R is **transitive**. A relation which is both reflexive and transitive is called an **order relation** (or *ordering*) on its domain. A relation which is symmetric, as well as reflexive and transitive, is called an **equivalence relation**. The inverse of an ordering is also an ordering (the *inverse ordering*), and any equivalence relation is a self-inverse order relation. Indeed, given any ordering R, the relation

$$R^0 = R \cap R^{-1} = \{[a, b]: a\ R\ b \wedge b\ R\ a\} \qquad (2.44)$$

will be an equivalence relation: we shall call it the core of the ordering R.

An order relation R whose core is the identity of its domain, so that

$$R^0 = 1_A \quad \text{or} \quad a\ R\ b \wedge b\ R\ a \Leftrightarrow a = b, \qquad (2.45)$$

will be called a *proper ordering* of A; and since any ordering is reflexive (see (2.41)), we see that, always, $R^0 \supseteq 1_A$, with equality for a proper ordering. A set A with a proper ordering R will be called a (partially) ordered set (or a poset); and if

$$R \cup R^{-1} = (A \times A) \quad \text{or} \quad (\forall a, b \in A)\ (a\ R\ b \vee b\ R\ a), \qquad (2.46)$$

then we call A a totally ordered set. Finally, if a totally ordered set is such that

$$(\forall a, b \in A)\ (\exists c \in A)\ (a\ R\ c \wedge b\ R\ c), \qquad (2.47)$$

we call A a directed set (or a net.) Of course, we might well call A a directed set if (2.47) were to hold with R replaced by R^{-1}; but we shall consider the directing order to be R when (2.47) holds, R^{-1} when the inverse property holds.

For example, it is easily seen that = is an equivalence relation for W, and also for $\mathsf{P}(W)$ (see (1.58) and (1.33)), and that $\Leftrightarrow$ is an equivalence relation for all propositions (see (1.12).) Similarly, we see that $\subseteq$ (and its inverse $\supseteq$) is an ordering for $\mathsf{P}(W)$, whose core is = (see (1.41)), making $\mathsf{P}(W)$ a poset; and $\Rightarrow$ (and its inverse $\Leftarrow$) is an ordering for all propositions, whose core is $\Leftrightarrow$ (see (1.21)): whether we consider this ordering proper depends on whether we consider the equivalence $\Leftrightarrow$ of propositions to be identity —— this is a moot point. Of course, the paradigm of all order relations is the relation $\leqslant$ between real numbers, with inverse ordering $\geqslant$.

Our definitions guarantee that, if X is a poset with respect to the ordering R and A is a subset of X, then A is also partially ordered by R. Any element x of X, such that

$$(\forall a \in A)\ a\ R\ x, \qquad (2.48)$$

is called a *majorant* of A, and we may define the (possibly empty) set

$$\ulcorner A \;=\; \{x \in X\colon (\forall a \in A)\; a\,R\,x\}. \tag{2.49}$$

Similarly, any $y \in X$ such that

$$(\forall a \in A)\; y\,R\,a \tag{2.50}$$

is called a *minorant* of A, and we may define the set

$$\llcorner A \;=\; \{y \in X\colon (\forall a \in A)\; y\,R\,a\}. \tag{2.51}$$

(The terminology is rooted in concepts of *size* and *height*, since the concept of order originated in the relation $\leqslant$: thus, a majorant is often also called an *upper bound*, and a minorant a *lower bound*.) If there exists a member x of A, such that (2.48) holds, then the property (2.45) (called antisymmetry) of the partial ordering R guarantees that there is at most one such element: it is then called the maximum element of A and is denoted by max A. Similarly, if there is a $y \in A$ such that (2.50) holds, it must be unique also, is called the minimum element of A and is denoted by min A. Again, if min $\ulcorner A$ exists, it is called the *least upper bound* (or lub or supremum) of A and is denoted by sup A: its uniqueness is guaranteed. Similarly, if max $\llcorner A$ exists, it must be unique, and it is called the *greatest lower bound* (or glb or infimum) of A and is denoted by inf A. Clearly,

$$\text{if } \max A \text{ exists then } \quad \sup A = \max A; \quad \text{if } \min A \text{ exists then } \quad \inf A = \min A. \tag{2.52}$$

A totally ordered set is sometimes called a *linearly ordered* set or a *chain*. If A is totally ordered by R, and if a, b, and c are in A, then every pair of points is related (by R or R^{-1}). The possibilities are (i) $a\,R\,b \wedge b\,R\,c \wedge c\,R\,a$, or $b\,R\,a \wedge c\,R\,b \wedge a\,R\,c$; or (ii) $a\,R\,b \wedge b\,R\,c \wedge a\,R\,c$, or $a\,R\,c \wedge c\,R\,b \wedge a\,R\,b$, or $b\,R\,c \wedge c\,R\,a \wedge b\,R\,a$, or $b\,R\,a \wedge a\,R\,c \wedge b\,R\,c$, or $c\,R\,a \wedge a\,R\,b \wedge c\,R\,b$, or $c\,R\,b \wedge b\,R\,a \wedge c\,R\,a$ (this exhausts all possibilities, since there are three pairs with two possible relations for each.) However, the two cases in (i) (so-called *cyclic order*) are only possible if a, b, and c are identical objects. ⟦By (2.43), $a\,R\,b \wedge b\,R\,c \Rightarrow a\,R\,c$; and with $c\,R\,a$, by (2.45), we get that $a = c$, whence also $a = b$. The second case yields the same result.⟧ The six remaining cases in (ii) can unambiguously be described by expressions of the form

$a\ R\ b\ R\ c$	and we say that	b	is *between*	a	and	c,
$a\ R\ c\ R\ b$	" "	c	"	a	"	b,
$b\ R\ c\ R\ a$	" "	c	"	b	"	a,
$b\ R\ a\ R\ c$	" "	a	"	b	"	c,
$c\ R\ a\ R\ b$	" "	a	"	c	"	b,
$c\ R\ b\ R\ a$	" "	b	"	c	"	a.

We note that "$a\ R\ b\ R\ c$" means not only that $a\ R\ b$ and $b\ R\ c$, but also that $a\ R\ c$. We also observe that, if one of the three elements being considered is the maximum of the totally ordered set, then it must necessarily appear on the right end of the string of characters (in the position occupied by "c" in "$a\ R\ b\ R\ c$"); and if one of the three elements is the minimum of A, it must appear on the left end of the string (in the position occupied by "a".)

We turn now to equivalence relations. If R is an equivalence relation for A, we may, for any $a \in A$, define its equivalence set as

$$E_a = \{b \in A:\ a\ R\ b\}. \tag{2.53}$$

It follows from (2.41), (2.42), and (2.43) that

$$a \in E_a; \quad (\forall b,\ c \in E_a)\ b\ R\ c; \quad b \in E_a \Rightarrow a \in E_b. \tag{2.54}$$

By (1.60), we have $\{a\} \subseteq E_a$, and by (2.53), $E_a \subseteq A$. Thus, by (1.64), on taking the union over all a in A, we get that

$$A = \cup_{a\in A}\ \{a\} \subseteq \cup_{a\in A}\ E_a \subseteq \cup_{a\in A}\ A = A. \tag{2.55}$$

This process of determining that a set lies between two others, with respect to the ordering $\subseteq$, and then showing that the two sets between which it lies are equal; so that all three are necessarily equal (by the third identity of (1.41)); is a very useful and powerful tool of analysis, which we call **bracketing**. Thus we have shown that

$$\cup_{a\in A}\ E_a = A. \tag{2.56}$$

Now consider the intersection E_aE_b: if $x \in E_aE_b$ then $x \in E_a$ (whence, if $y \in E_a$ then $y\ R\ a\ R\ x$; that is, $y\ R\ x$) and $x \in E_b$ (whence, similarly, if $z \in E_b$ then $z\ R\ x$); thus, either $E_aE_b = \emptyset$ or $E_a = E_b$. With (2.56), this shows that A is *partitioned* into a number of sets E_k whose disjoint union is A:

$$(\exists K \subseteq A)\ \big(((\forall j,\ k \in K)\ j \neq k \Rightarrow E_j\ E_k = \emptyset) \wedge (A = \uplus_{k\in K} E_k)\big). \qquad (2.57)$$

The set of all equivalence sets in A under R is called the quotient set of A by R and is written A/R. We may write a/R for the equivalence set E_a defined in (2.53), and then

$$A/R \;=\; \{a/R\colon a \in A\} \qquad (2.58)$$

(it being remembered that a set is identified by its distinct members, without regard to repetition of equal objects.) It will be noted that the quotient A/R is a family of sets indexed by the set K defined in (2.57).

Consider now a function $f\colon A \to B$. Then for each $b \in B$, the sets $f^{-1}(b)$ are distinct equivalence sets in A under the equivalence relation $\equiv_f$ defined by

$$x \equiv_f y \Leftrightarrow f(x) = f(y). \qquad (2.59)$$

The sets $f^{-1}(a)$ are disjoint because f is a function, and their union is A because A is the domain of f (compare (2.28).) This result indicates how we can *factor* f into the composition of a surjection g from A onto $A/\equiv_f$ (defined by $g(a) = a/\equiv_f$) and an injection h from $a/\equiv_f$ into B (defined by $h(a/\equiv_f) = f(a)$; so that, for any a in A,

$$g(a) = a/\equiv_f, \quad h(a/\equiv_f) = f(a), \quad f(a) = h(g(a)) = (h \circ g)(a). \qquad (2.60)$$

Returning to the general Cartesian product defined in (2.8) and (2.9), we note that, for any $\alpha \in J$, we can define the function P_α from $\cup_{\lambda\in J} E_\lambda$ into E_α for which

$$\mathrm{P}_\alpha(\{x_\lambda\}_{\lambda\in J}) = x_\alpha. \qquad (2.61)$$

This function is the projection mapping from the product space to the *factor space* E_α indexed by α: it is clearly surjective; and the quotient set $\cup_{\lambda\in J} E_\lambda/\equiv_{\mathrm{P}_\alpha}$ corresponding to P_α is the set of all *sheets* indexed by E_α: sets of points $\{x_\lambda\}_{\lambda\in J}$ with fixed x_α in E_α.

Just as in deriving (1.45) from the general notation in (1.43) and (1.44), when the index set J of the family F in (1.42) is finite ($J = \{1, 2, \ldots, n\}$) or countably infinite ($J = \{1, 2, 3, \ldots\}$), we may write

$$\left.\begin{array}{ll} & [x_\alpha]_{\alpha=1}^{n} \text{ for } [x_\alpha]_{\alpha\in J}, \text{ and } \mathsf{X}_{\alpha=1}^{n} E_\alpha = \{[x_\alpha]_{\alpha=1}^{n}\colon (\forall\alpha \in J)\ x_\alpha \in E_\alpha\}, \\ \text{or} & [x_\alpha]_{\alpha=1}^{\infty} \text{ for } [x_\alpha]_{\alpha\in J}, \text{ and } \mathsf{X}_{\alpha=1}^{\infty} E_\alpha = \{[x_\alpha]_{\alpha=1}^{\infty}\colon (\forall\alpha \in J)\ x_\alpha \in E_\alpha\}. \end{array}\right\} \tag{2.62}$$

respectively, using the notation of (2.8) and (2.9).

Finally, we consider the important special case, when all the sets E_α in the family F are the same set E: then we call their Cartesian product a Cartesian power and write

$$\mathsf{X}_{\alpha\in J}\, E = E^{J}. \tag{2.63}$$

It follows that the elements of the set E^J correspond to all possible functions

$$x\colon J \to E, \quad \text{with} \quad x(\alpha) = x_\alpha, \tag{2.64}$$

in accordance with the alternative notations mentioned in (2.20) and (2.21).

It now follows from the definitions (2.9) and (2.63) that

$$E^{(J\times K)} = \mathsf{X}_{[\alpha,\beta]\in J\times K}\, E = \mathsf{X}_{\alpha\in J}\, (\mathsf{X}_{\beta\in K}\, E) = (E^K)^J = (E^J)^K; \tag{2.65}$$

and similarly, that

$$(E \times F)^J = \mathsf{X}_{\alpha\in J}\, (E \times F) = \{\{[x_\alpha, y_\alpha]\colon \alpha \in J\}\colon (\forall\alpha \in J)\ (x_\alpha \in E \wedge y_\alpha \in F)\}. \tag{2.66}$$

From the former result, we see that, in commoner parlance, if f is a function of two variables α and β; then, for each value of α, $f(\alpha, \beta)$ is a function of β, and for each value of β, $f(\alpha, \beta)$ is a function of α. From the latter result, we have that any two functions $x\colon J \to E$ and $y\colon J \to F$ correspond uniquely to a function $f\colon J \to (E \times F)$; the bijective correspondence being given by

$$(\forall\alpha \in J)\ f(\alpha) = [x_\alpha, y_\alpha]. \tag{2.67}$$

EXERCISES 2

Again, we strongly recommend that the reader verify in detail all results quoted without proof in the text of §2.

⟨2.1⟩ If $A = \{0, 1, 2\}$ and $B = \{7, 8, 9\}$, list all possible functions from A into B, together with their inverse relations, and classify them as surjective, injective, bijective, or none of these.

⟨2.2⟩ Suppose that $f: X \to Y$ and $g: Y \to Z$ both have left (or both have right) inverses. Prove that $g \circ f$ then also has a left (or, respectively, right) inverse. Hence, show that $g \circ f$ is, respectively, a surjection, injection, or bijection, if both f and g are surjections, injections, or bijections.

⟨2.3⟩ If $A \subseteq P$ and $B \subseteq Q$, then prove that

$$(P \times Q)\ (A \times B)^{c} = ((PA^{c}) \times Q) \cup (A \times (QB^{c})). \tag{2.68}$$

⟨2.4⟩ Prove that, for f, g, and h functions such that the compositions below are meaningful,

$$\left.\begin{aligned} &f \circ g = f \circ h \Rightarrow g = h \quad \text{if } f \text{ is injective,} \\ \text{and} \quad &g \circ f = h \circ f \Rightarrow g = h \quad \text{if } f \text{ is surjective.} \end{aligned}\right\} \tag{2.69}$$

⟨2.5⟩ Let $A = \{0, 1, 2\}$ and $R = \{[0, 0], [1, 2], [2, 2]\}$. Determine (and prove) whether R is reflexive (2.41), symmetric (2.42), transitive (2.43), or antisymmetric (2.45).

⟨2.6⟩ Let R be a symmetric, transitive relation on A. What is wrong with the following "proof" that R must be reflexive also? By symmetry, $(\forall a, b \in A)\ a\ R\ b \Rightarrow b\ R\ a$; and by transitivity, $a\ R\ b \wedge b\ R\ a \Rightarrow a\ R\ a$, which is reflexivity (!?)

⟨2.7⟩ Construct all possible equivalence relations on $A = \{\lambda, \mu, \nu\}$.

⟨2.8⟩ Let $\mathbb{N}$ be the set of all positive integers and let $f: \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n^2$. Show that f has no right-inverse and exhibit two different left-inverses of f. What is the corresponding situation if f is similarly defined with $\mathbb{N}$ replaced by the set $\mathbb{R}$ of all real numbers?

⟨2.9⟩ The axiom of choice may be stated as follows: if $\mathsf{F} = \{F_\alpha : \alpha \in J\}$ with $F_\alpha \neq \emptyset$ for all α, and $F_\alpha F_\beta = \emptyset$ whenever $\alpha \neq \beta$, then there exists a set & such that $(\forall \alpha \in J)$ & F_α is a singleton. In other words, *given a family* F *of disjoint non-empty sets, we can always construct a set having exactly one member from each set in* F.

Prove that the axiom of choice is equivalent to the assertion that every surjection has a right-inverse. [Hint: let $\Omega = \biguplus_{\alpha\in J} E_\alpha$ and $f\colon \Omega \to \mathsf{F}$ be defined by $f(\omega) = E_\alpha$ whenever $\omega \in E_\alpha$.]

⟨2.10⟩ Prove that max A exists iff $A \ulcorner A \neq \emptyset$, and min A exists iff $A \llcorner A \neq \emptyset$. (2.70)

⟨2.11⟩ Prove that any partition of A as $\biguplus_{\alpha\in J} E_\alpha$ can be made the quotient set of A by an equivalence relation R, and define R explicitly.

⟨2.12⟩ The characteristic function of a set A is defined as $\chi_A\colon W \to \{0, 1\}$ such that $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. Show that the family of all characteristic functions, $\{\chi_A : A \in \mathsf{P}(W)\}$, corresponds to $\{0, 1\}^W$.

⟨2.13⟩ If $f\colon A \to B$ and $A \neq \emptyset$, construct an $e\colon B \to A$, such that $f \circ e \circ f = f$. [Hint: use the factorization in (2.59) and (2.60).]

⟨2.14⟩ Prove that
$$\left(\times_{\alpha\in J} E_\alpha\right) \cap \left(\times_{\alpha\in J} F_\alpha\right) = \times_{\alpha\in J} (E_\alpha \cap F_\alpha). \tag{2.71}$$

⟨2.15⟩ Prove that
$$W^J \cap \left(\times_{\alpha\in J} E_\alpha\right)^c = \biguplus_{K\subset J} \left(\times_{\alpha\in J} Q_\alpha^{(K)}\right); \tag{2.72}$$
where
$$(\forall \alpha \in K)\ Q_\alpha^{(K)} = E_\alpha, \quad (\forall \alpha \in JK^c)\ Q_\alpha^{(K)} = W(E_\alpha)^c. \tag{2.73}$$
[Note that the sets K are proper subsets of J: $K \subset J$.]

⟨2.16⟩ Prove that
$$\left(\times_{\alpha\in J} E_\alpha\right) \cup \left(\times_{\alpha\in J} F_\alpha\right) = \biguplus_{K\subset J} \left\{\left(\times_{\alpha\in J} (Q_\alpha^{(K)} \cap F_\alpha)\right) \uplus \left(\times_{\alpha\in J} (E_\alpha \cap R_\alpha^{(K)})\right)\right\} \uplus \times_{\alpha\in J} (E_\alpha \cap F_\alpha); \tag{2.74}$$
where the $Q_\alpha^{(K)}$ are defined as in (2.73) and the $R_\alpha^{(K)}$ are similarly defined, with respect to the F_α instead of the E_α. [Hint: use (1.68) with (2.71) and (2.72).]