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by

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Under the supervision of Professor Lawrence H. Landweber

ABSTRACT

We present fast algorithms for six NP-hard problems. These algorithms are shown to be optimal or near-optimal with probability one (i.e., almost surely).

First we design an algorithm for the Euclidean traveling salesman problem in any k-dimensional Lebesgue set E of zero-volume boundary. For n points independently, uniformly distributed in E, we show that, in probability, the time taken by the algorithm is of order less than $n\ \sigma(n)$, as $n \to \infty$, for any choice of an increasing function σ (however slow its rate of increase). The resulting solution will, with probability one, be asymptotic, as $n \to \infty$, to the optimal solution.

In addition, by applying a uniform method, we design algorithms for five NP-hard problems: the vertex set cover of an undirected graph, the set cover of a collection of sets, the clique of an undirected graph, the set pack of a collection of sets, and the k-dimensional matching of an undirected graph. Each algorithm has its worst case running time bounded by a polynomial or a function slightly greater than a polynomial on the size of the problem in-

stance. Furthermore, we show, as corollaries of main theorems, that each algorithm gives an optimal or near-optimal solution with probability one, as the size of the corresponding problem instance increases.

ACKNOWLEDGMENTS

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I wish to express my deepest gratitude to my wife Mary, my son Randy, and my parents.

to Randy

Contents

Chapter I

Introduction and Summary

This introductory chapter begins with some basic definitions in the area of design and analysis of algorithms, and some of the motivations for this thesis. Section 2 contains a brief review of previous work in the area, and a summary of our results. Most of the details are left for later chapters. Section 3 contains a description of notation and some basic concepts in elementary probability theory.

1. Preliminary Definitions and Motivations

Algorithms can be evaluated by a variety of criteria. Frequently, we are interested in the rate of growth of the time required to solve larger and larger instances of a problem. More specifically, let P be a computational problem, i.e., a collection of computational tasks each of which is called an *instance* of P. With each instance I in P we associate an integer $|I|$, called the *size* of I. Generally speaking, we take the size $|I| = n$ to be correlated to the amount of information required to specify I. If, for all sufficiently large n, the time needed by an algorithm A to solve instances of size n has a least upper-bound proportional to f(n), we say that the *worst case running time* of A is f(n) or that *A solves P in f-time*. In particular, when f(.) is a polynomial on n, we say A solves P in *polynomial time*, and if $f(n) = c^{p(n)}$, where c is a constant and p(.) is a polynomial, we say A solves P in exponential-time, and A is not fast.

There is a general agreement that if a problem P cannot be solved by a polynomial-time algorithm, then P should be considered intractable. Of course, in some applications, just a subset of all the problem instances is of interest and can be shown to be tractable.

There is evidence that a certain class of problems, the non-deterministic polynomial time complete problems ("NP-complete" for short), is likely to contain only intractable problems (see, e.g., Aho, Hopcroft and Ullman[1975]). Many "classical" problems in combinatorics, such as the traveling salesman problem, the Hamiltonian circuit problem, and integer linear programming are NP-complete. All problems in the class can be shown "equivalent", in the sense that if one is tractable, then all are tractable (Cook[1971], Karp[1972]).

We will consider a second class of problems, called the "NP-hard" problems, which are at least as hard to solve as the NP-complete problems in the sense that the existence of a polynomial time algorithm to solve an NP-hard problem implies that all NP-complete problems can also be solved in polynomial time (Cook[1971], Karp[1972]).

Since many of the NP-complete and NP-hard problems have been studied by mathematicians and computer scientists for decades, and all known algorithms to solve any of them require at least exponential time, it is natural to conjecture that no algorithm requiring less than exponential time exists , and consequently, to regard all the problems in these classes as being intractable.

But in many real-world applications, exact solutions for NP-hard problems are not required. As a result, some researchers have developed <u>approximation algorithms</u> for these applications, which attempt to guarantee near-optimal solutions to all instances of a problem (cf. Garey and Johnson[1976]). The definitional set-up is as follows. Consider a minimization(resp., maximization) problem which, for each problem instance I, asks for a solution with minimum (resp. maximum) cost m(I). Consider an algorithm A that, on problem instance I, produces a solution of cost A(I). Then, given a real number $r > 1$, we say that <u>A solves the problem within ratio r</u> if , for all I,

$$A(I) \leq r\, m(I) \qquad (1.1)$$

(resp., $$A(I) \geq (1/r)\, m(I) \qquad (1.2)$$)

This "guaranteed approximation" approach has yielded a number of successes, particularly in connection with various packing problems (cf. Garey and Johnson[1976a]). However, some important NP-hard problems seem to be not well suited to this approach. For example, Garey and Johnson[1976] prove that it is NP-hard to solve the coloring of a graph problem within a ratio $r<2$ (the cost in this case is the number of colors). Moreover , no polynomial-time algorithm is known which solves the coloring problem within <u>any</u> fixed ratio r. Another example is the problem of finding the largest clique (i.e., complete subgraph) in a graph. Garey and Johnson [1976] suggest how to prove that the following statements are equivalent (the cost here is the clique size):

(a) for some $r>1$, there is an polynomial-time algorithm to solve the largest clique problem within r;

(b) for every $r>1$, there is a polynomial-time algorithm for solving the largest clique problem within r.

Recently, such negative results and the conjecture that all NP-complete and NP-hard problems are intractable have motivated the design of the so-called "probabilistic algorithms". In this thesis we are interested in the design of a particular type of probabilistic algorithms for NP-hard problems, those which are fast and are guaranteed to give optimal or near-optimal solutions with "probability one", as the size of the problem instance increases. This is the strongest type of probabilistic algorithm we can look for (cf. Feller [1968], or Chung[1974]).

To formulate what "probability one" means, a probabilistic distribution over all problem instances is assumed. Let $\{I_j, j \geq 1\}$ be a sequence of problem instances such that the size $|I_j| = j$, and $\{I_j, j \geq 1\}$ is sampled incrementally according to the probabilistic distribution assumed, in the sense that I_j is obtained from I_{j-1} by adding one component or element of the problem to I_{j-1}, according to the probabilistic distribution. For example, if the underlying problem structure is a graph, the incremental change might be the addition of a new node and some edges incident to it, with all the edges of the previous graph unchanged. Using the same notation of (1.1) and (1.2), given a real number $r \geq 1$, an algorithm A solves a minimization (resp., maximization) problem within ratio r with probability one iff for every $\epsilon > 0$, we have (cf. Feller[1968] or Chung[1974])

$$\lim_{n \to \infty} \Pr\{ 1 \leq \frac{A(I_j)}{m(I_j)} < r + \epsilon, \text{ for all } j \geq n\} = 1 \tag{1.3}$$

$$\text{(resp., } \lim_{n \to \infty} \Pr\{ 1 \geq \frac{A(I_j)}{m(I_j)} > 1/r - \epsilon, \text{ for all } j \geq n\} = 1\text{)}$$

When $r = 1$, we will say that A is an <u>optimal algorithm with probability one</u>, and when $r > 1$, we will say A is a <u>near-optimal algorithm with probability one</u>.

2. Previous Work and Summary of the Thesis

One of the earliest results on probabilistic algorithms is a fast algorithm by Solovay and Strassen [1977] and Rabin [1976] for testing whether a number n is prime. This problem becomes infeasible to solve for n larger than 10^{60}. Rabin claims that the probability of error (i.e., guessing that a composite number is prime) is halved at each step of the algorithm, regardless of the size of n.

Posa[1976], and Angluin and Valiant[1977] give polynomial-time probabilistic algorithms to find Hamiltonian circuits in graphs. This problem is known to be NP-complete (Karp[1972]). Their algorithms find a solution with probability tending to one, as the size of the problem instance increases, if the graphs are sufficiently dense.

Grimmett and McDiarmid [1975] describe a polynomial time probabilistic algorithm to color a graph within any ratio $r > 2$

with probability one. As we mentioned in Section 1 (Garey and Johnson[1976]), the coloring of a graph is an NP-hard problem. They also have a polynomial-time algorithm to find the largest subset of the set of vertices of a graph such that no two vertices in the subset are connected. This algorithm is near-optimal (within $r = 2$) with probability one, as we comment in Section 4.1 of Chapter III.

In this thesis, we study algorithms for three minimization and three maximization NP-hard problems.

In Chapter II we give a fast algorithm to solve the k-dimensional Euclidean traveling salesman problem (k-TSP for short) which is optimal with probability one. For the particular case of k=2 (i.e., the TSP in the plane), Garey et al.[1976], and Papadimitriou [1977] proved that the TSP is NP-hard. The best known polynomial-time approximation algorithms for this problem, by Christofides[1976], solves it within $r = 3/2$.

On the other hand, there has been some research on heuristic methods for the solution of the 2-TSP. For example, computer programs to find near optimal solutions for 2-TSP instances of up to 300 points in an acceptable amount of time were described by Krolak et al.[1970] and by Lin and Kernighan[1973]. Their programs seem to give good results but no rigorous analysis of the algorithms are available.

Karp[1977] gives an algorithm whose expected running time is bounded by $n \log^2 n$. He claims it solves the 2-TSP within any $r > 1$ with probability one, but, as we comment in Section 7 of Chapter II, the proof of this claim is incomplete.

In Chapter III we give polynomial-time algorithms for two minimization NP -hard problems which are optimal with probability one. The problems are the vertex set cover of an undirected graph and the set cover of a collection of sets. The best known polynomial-time approximation algorithm for the vertex set cover problem, by Gavril (cf. Garey and Johnson [1978], p.134), solves it within r = 2. Also in Chapter III we give polynomial-time algorithms for three maximization NP-hard problems which solves each of them within r = 2 with probability one. The problems are: the clique of an undirected graph , the set pack of a collection of sets, and the k-dimensional matching of a graph. So far, no polynomial-time approximation algorithm is known to solve any of these three problems within <u>any fixed ratio r</u>. All the algorithms presented in Chapter III are derived from a central algorithm , Algorithm C.

In Chapter IV we have new algorithms for the three maximization problems considered in Chapter III. These new algorithms are optimal with probability one, but they require more running time than the ones in Chapter III. The algorithms in Chapter IV are also derived from a central algorithm, Algorithm D.

3. Notation and Background Material

This section contains a summary of notation and some elementary probability theory which will be used throughout the remaining chapters. We intend to only provide a basis for the terminology which we will propose and use later. Most of the additional concepts and notations are defined when they arise naturally in later chapters.

When dealing with asymptotic behavior of functions, specific notations are available to describe the relationships between functions $f(n)$ and $g(n)$, all of which are based on the behavior of the ratio $f(n)/g(n)$, for all sufficiently large values of n. We say that

$$f(n) = \underline{o}\,(g(n)) \quad \text{iff} \quad f(n)/g(n) \to 0;$$

$$f(n) = \underline{O}\,(g(n)) \quad \text{iff} \quad f(n)/g(n) \leq c, \text{ for some constant } c;$$

$$f(n) \sim g(n) \quad \text{iff} \quad f(n)/g(n) \to 1.$$

For the basic background material in elementary probability theory, we will follow Feller[1968].

Let Ω be the space of all possible outcomes of a random experiment. Ω is called the *sample space* of the experiment. A function defined on a sample space is called a *random variable*. Let X be a random variable and let x_1, x_2, x_3, ... be the values which it assumes. In general, the same value x_j may correspond to several sample points. This aggregate forms the event that $X = x_j$; its probability is denoted by $\Pr\{ X = x_j \}$. The system of relations

$$\Pr[X = x_j] = f(x_j),\ j=1,2,3,\ \ldots. \tag{3.1}$$

defines the *probabilistic distribution* of the random variable X. Clearly,

$$f(x_j) \geq 0,\ \sum f(x_j) = 1 \tag{3.2}$$

If a value x is never assumed, we write $\Pr\{ X = x \} = 0$.

If two or more random variables $X_1, X_2, \ldots, X_n$ are defined on the same sample space, their joint distribution is given by the system of equations which assigns probabilities to all combinations $X_1 = x_{j1}$, $X_2 = x_{j2}$, etc.. The variables $X_1, \ldots, X_n$ are called *mutually independent* if for any combination of values x_{j1},

$\ldots, x_{jn},$

$$\Pr\{X_1=x_{j1},\ X_2=x_{j2},\ \ldots\ ,\ X_n=x_{jn}\} = \Pr\{X_1=x_{j1}\}\ \Pr\{X_2=x_{j2}\}\ \ldots\ \Pr\{X_n=x_{jn}\} \tag{3.3}$$

Let X be a random variable assuming the values $x_1, x_2, \ldots$, with corresponding probabilities $f(x_1)$, $f(x_2)$, ... The <u>expected value</u> of X is defined by

$$\mathcal{E}\, X = \sum x_k\, f(x_k) \tag{3.4}$$

provided that the series converges absolutely.

The <u>second moment</u> of X is defined by

$$\mathcal{E}\, X^2 = \sum x_k^{\,2}\, f(x_k) \tag{3.5}$$

provided that the series converges absolutely.

The <u>variance</u> of X is defined by

$$\operatorname{var} X = \mathcal{E}\, X^2 - (\mathcal{E}\, X)^2 \tag{3.6}$$

A sequence of random variables $X_1, X_2, \ldots$ is said to converge <u>in probability</u> to X iff for every $\epsilon > 0$, we have

$$\lim_{n \to \infty} \Pr\{\, |X_n - X| > \epsilon \,\} = 0 \tag{3.7}$$

Chapter II

Euclidean Traveling Salesman Problem

1. Introduction and Summary

Given an integer $k \geq 2$, the k-dimensional Euclidean Traveling Salesman Problem (k-TSP) can be defined as follows: given a set of n points distributed in the k-dimensional Euclidean space R^k, determine a tour, i.e., a closed path visiting each of the n points exactly once, so that the tour is the shortest possible one (we take the distance between two points to be the ordinary Euclidean distance).

In Section 2 of this chapter we present Algorithm A , a non-recursive, divide-and-conquer algorithm for the k-TSP, $k \geq 2$. In defining Algorithm A, we assume that a non-zero function $\delta(n)$ is chosen. Furthermore, in Sections 3, and 4 we will use the following.

Condition C:

[1] the points of a sequence $\underset{\sim}{P}$ are distributed uniformly in the k-dimensional unit hypercube $\underline{\underline{C}}$;

[2] the function $\delta(n)$ satisfies

$$\delta(n) \to \infty \quad \text{and} \quad \delta(n)\; e^{\delta(n)}/n \to 0 \quad \text{as } n \to \infty.$$

Let $\underset{\sim}{P}^n$ denote the first n points of $\underset{\sim}{P}$; in Section 3 we prove the following:

<u>Theorem 1</u> : Under Condition C, if Algorithm A is applied to a k-TSP instance $\underset{\sim}{P}^n$, then Algorithm A runs in time

$$R_n \sim A/2\ n\ \delta(n)\ e^{\delta(n)}\quad, \text{ as } n \to \infty, \text{ in probability,}$$

where A is a constant

We are thinking in particular of very slowly increasing functions $\delta(n)$. We notice that, for example, if we let $\delta(n) =$ log log log n in Theorem 1,we would have

$$R_n \sim A/2\ n(\log \log \log n)(\log \log n), \text{ as } n \to \infty, \text{ in probability.}$$

Indeed, by choosing $\delta(n) = \alpha \log \sigma(n)$, for any $0< \alpha <1$, we obtain

<u>Corollary TSP</u> : Under the hypotheses of Theorem 1, we can find a function $\delta(n)$, such that, for any arbitrarily slowly increasing function $\sigma(n)$ the running time of Algorithm A will be

$$R_n = \underline{o}(\ n\ \sigma(n)), \text{ in probability}$$

Let $T_0(n)$ denote the length of an optimal solution for a given k-TSP instance $\underset{\sim}{P}^n$. And let T(n) denote the length of the closed path given by Algorithm A for $\underset{\sim}{P}^n$. In Section 4 of this chapter we characterize the asymptotic performance of Algorithm A by the following

<u>Theorem 2</u> : Under the hypotheses of Theorem 1, we have:

$$T(n)/T_0(n) \to 1, \text{ with probability one, as } n \to \infty.$$

Finally, in Section 5, we consider Condition D, that

[1] the points of a sequence $\tilde{P}$ are distributed uniformly and independently in a Lebesgue subset E of αC, the k-dimensional hypercube of side α;

[2] the boundary of E is of zero k-dimensional Lebesgue measure;

[3] the function $\delta(n)$ satisfies

$\delta(n) \to \infty$ and $\delta(n)e^{\delta(n)}/n \to 0$ as $n \to \infty$.

In this case, we apply Algorithm A to αC (instead of C, as in Section 2) and obtain

Theorem 3: Under Condition D,

(1) Theorem 1 holds, with $\delta(n)$ replaced by $\delta(n)\ \alpha^k/v(E)$, where $v(E)$ is the k-dimensional Lebesgue measure of E;

(2) Theorem 2 holds.

2. Algorithm A

Algorithm A computes a closed path which visits some of the points more than once. We will see later in this section that it is easy to transform such a closed path into a tour with a shorter length.

In specifying Algorithm A, we need a function $\delta(.)$ and an integer m defined as the smallest even integer greater than or equal to :

$$\left(\frac{n}{\delta(n)}\right)^{1/k}$$

where n is the number of points of a k-TSP instance J in C.

Now we are able to specify:

Algorithm A

(For an illustration for the case of $k=2$, see Figures 1 and 2 below)

[1] Divide each side of $\underline{\underline{C}}$ into m equal parts, thus creating a cubic lattice of m^k cells (of side h) in $\underline{\underline{C}}$.

[2] Let B be the set of cell-centers (mid-points of cells created in [1]). Form the union $B \cup J$.

[3] For each of the m^k cells, find the shortest tour through the points of $B \cup J$ in the cell by applying a dynamic programming algorithm (Bellman[1962] and Held and Karp[1962]);

[4] Construct a basic tour through the points of B added in step [2] above, using Algorithm B below.

[5] The closed path consisting of all the subtours constructed in step [3] chain-connected by the basic tour built in step [4] is the result of the algorithm.

To construct the basic tour, we have a cubic lattice of cubic cells of side h, m in each coordinate direction, m^k in all, where m is a positive even integer. Suppose that, for $a_i \in L = \{0,1,2,...,m-1\}$, $1 \le i \le k$, the cell containing the cell-center with coordinates

$((2a_1+1)h/2,\ (2a_2+1)h/2,\ \ldots\ ,(2a_k+1)h/2)$

is identified by the vector

$a = (a_1,a_2,\ \ldots\ ,a_k)$.

Let e_i denote the unit vector in the i-th coordinate direction and write

$$r_i = r_i(a) = (-1)^{1+a_1+a_2+\ldots+a_{i-1}}, \text{ for } 2\leq i\leq k. \qquad (3.1)$$

Algorithm B: Given cell a, find its successor b according to the basic tour. (For an illustration, see Figures 3 and 4 below.)

[1] If there exists one value d such that

$$d\geq 3,\ a_d+r_d \in L \text{ and } a_i + r_i \notin L \text{ for } d+1 \leq i\leq k; \qquad (3.2)$$

then the successor of a is

$$b = a + r_d\, e_d \qquad (3.3)$$

(i.e., for all $i \neq d$, $b_i = a_i$, and $b_d = a_d + r_d$).

[2] Otherwise, if (3.2) cannot be satisfied by any d , the successor is determined as follows:

$$b = a - e_1, \text{ if } a_1 = 1,\ a_2 = 0,$$
$$\text{or } a_1>1,\ a_2 \text{ even}; \qquad (3.4)$$

$$b = a + e_1, \text{ if } a_1 = 0,\ a_2 = m-1,$$
$$\text{or } 0 < a_1 < m - 1,\ a_2 \text{ odd}; \qquad (3.5)$$

$$b = a - e_2, \text{ if } a_1 = 1,\ a_2 \text{ even},\ a_2 \neq 0,$$
$$\text{or } a_1 = m - 1,\ a_2 \text{ odd}; \qquad (3.6)$$

$$b = a + e_2, \text{ if } a_1 = 0,\ a_2 < m - 1 \qquad (3.7)$$

Having defined Algorithm B, we observe that the step [2] above is executed only when

$$a_4 = a_5 = \ldots = a_k = 0; \text{ and } a_1 + a_2 \text{ is odd and } a_3 = m-1,$$
$$\text{or } a_1 + a_2 \text{ is even and } a_3 = 0.$$

This is so because step [2] is executed when

$$a_i + r_i \notin L \quad \text{for } 3 \leq i \leq k;$$

thus $a_3 + (-1)^{1+a_1+a_2} \notin L$;

whence $a_1 + a_2$ is odd and $a_3 = m-1$ (odd),

or $a_1 + a_2$ is even and $a_3 = 0$ (even).

If $a_1 + a_2$ is odd, then

$$r_3 = (-1)^{1+a_1+a_2} = +1,$$

and $a_3 = m - 1$, whence $r_4 = (-1)^{a_3} r_3 = -1$;

$a_4 + r_4 \notin L$, whence $a_4 = 0$, so $r_5 = (-1)^{a_4} r_4 = r_4 = -1$;

$a_5 + r_5 \notin L$, whence $a_5 = 0$, so $r_6 = r_5 = -1$;

....

$$a_k = 0.$$

Also, if $a_1 + a_2$ is even, then

$$r_3 = (-1)^{1+a_1+a_2} = -1;$$

and $a_3 = 0$, whence $r_4 = (-1)^{a_3} r_3 = r_3 = -1$;

$a_4 = 0$, whence $r_5 = r_4 = -1$;

....

$a_k = 0$.

For $k = 2$, step [2] prevails and from the observation made above it is easy to verify that (3.4) - (3.7) prescribe the entire set of successors b of possible vectors a and is in accordance with the basic tour in Figure 3. For $k = 3$, Figure 4 shows the basic tour as an illustration.

$m = 6$

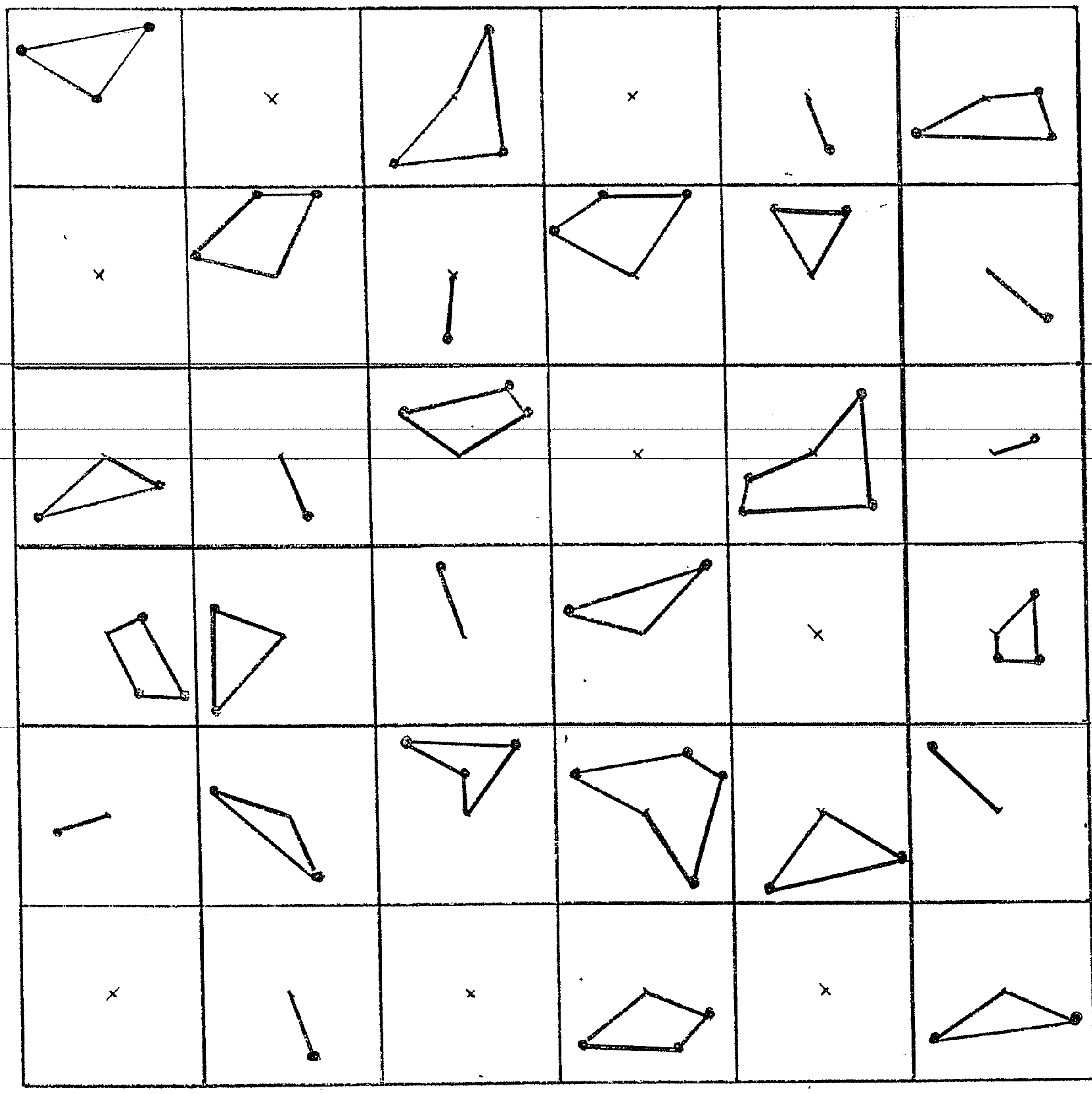


Figure 1: An illustration of steps (1) - (3) of Algorithm A

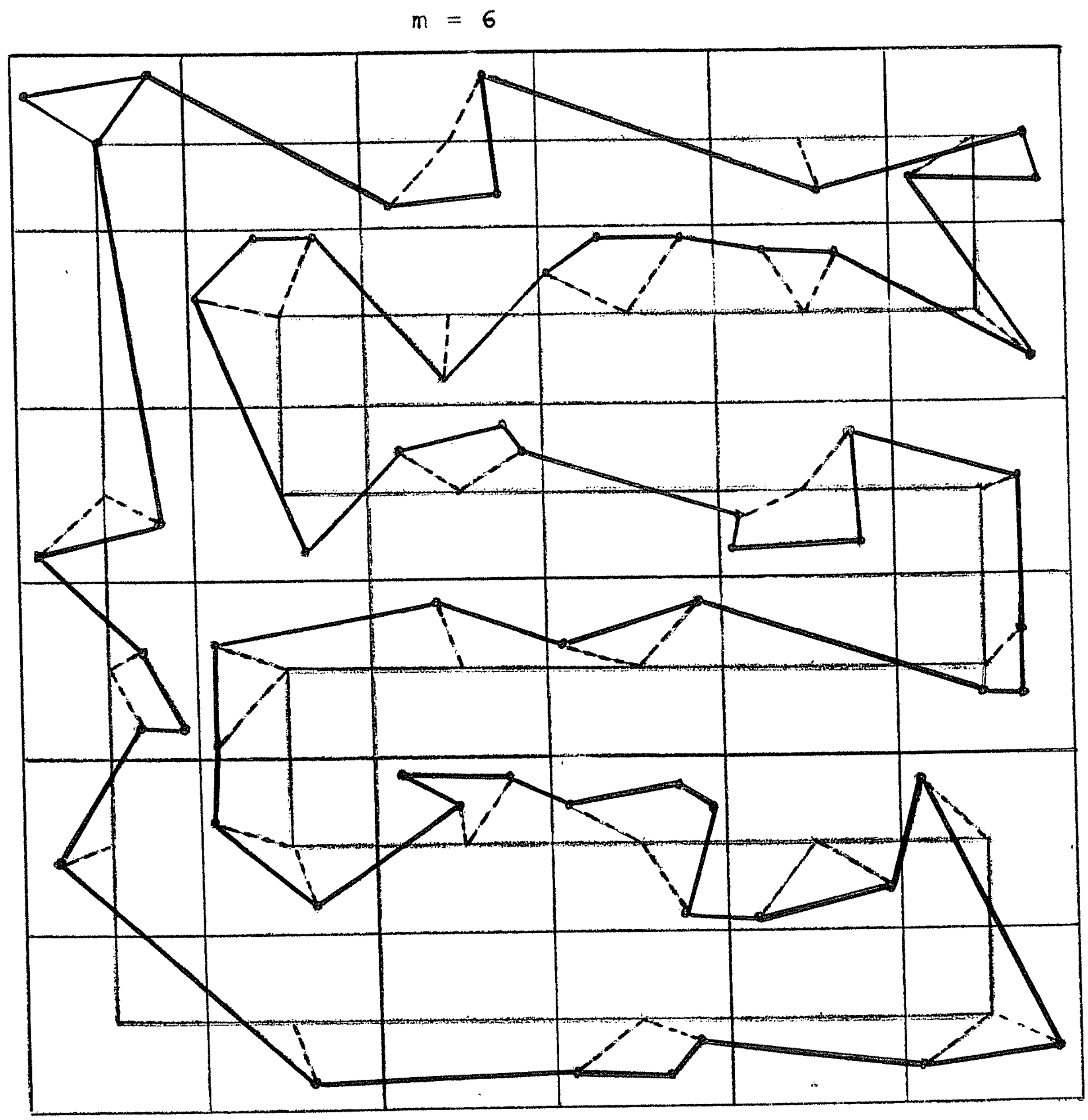


Figure 2: An illustration of steps (4) - (5) of Algorithm A and the final tour

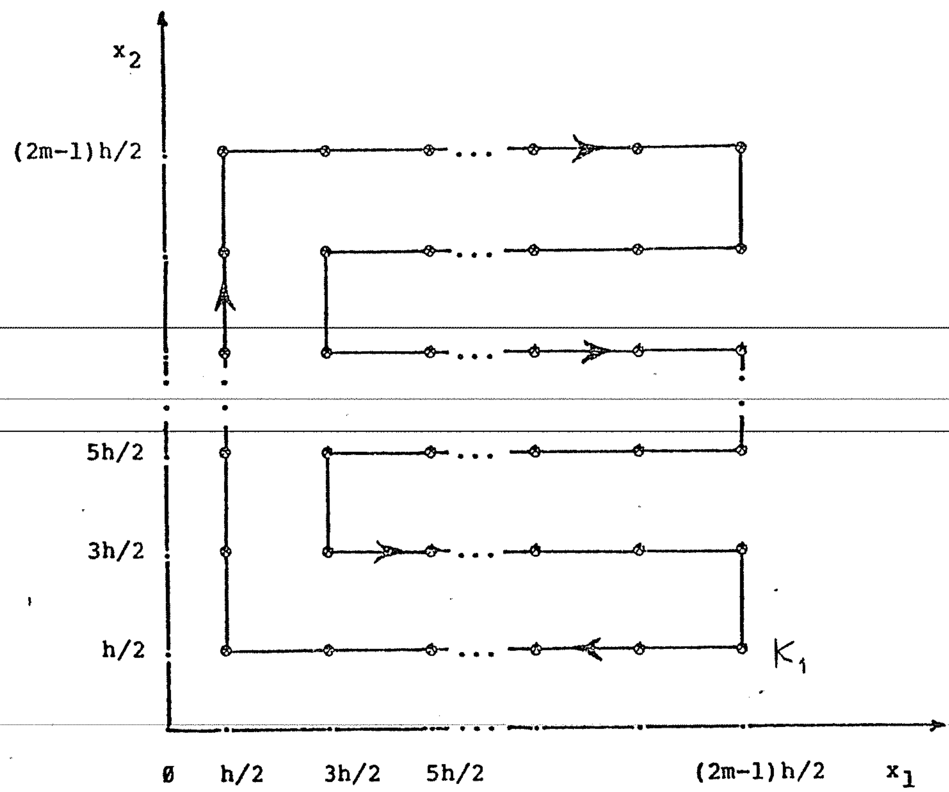
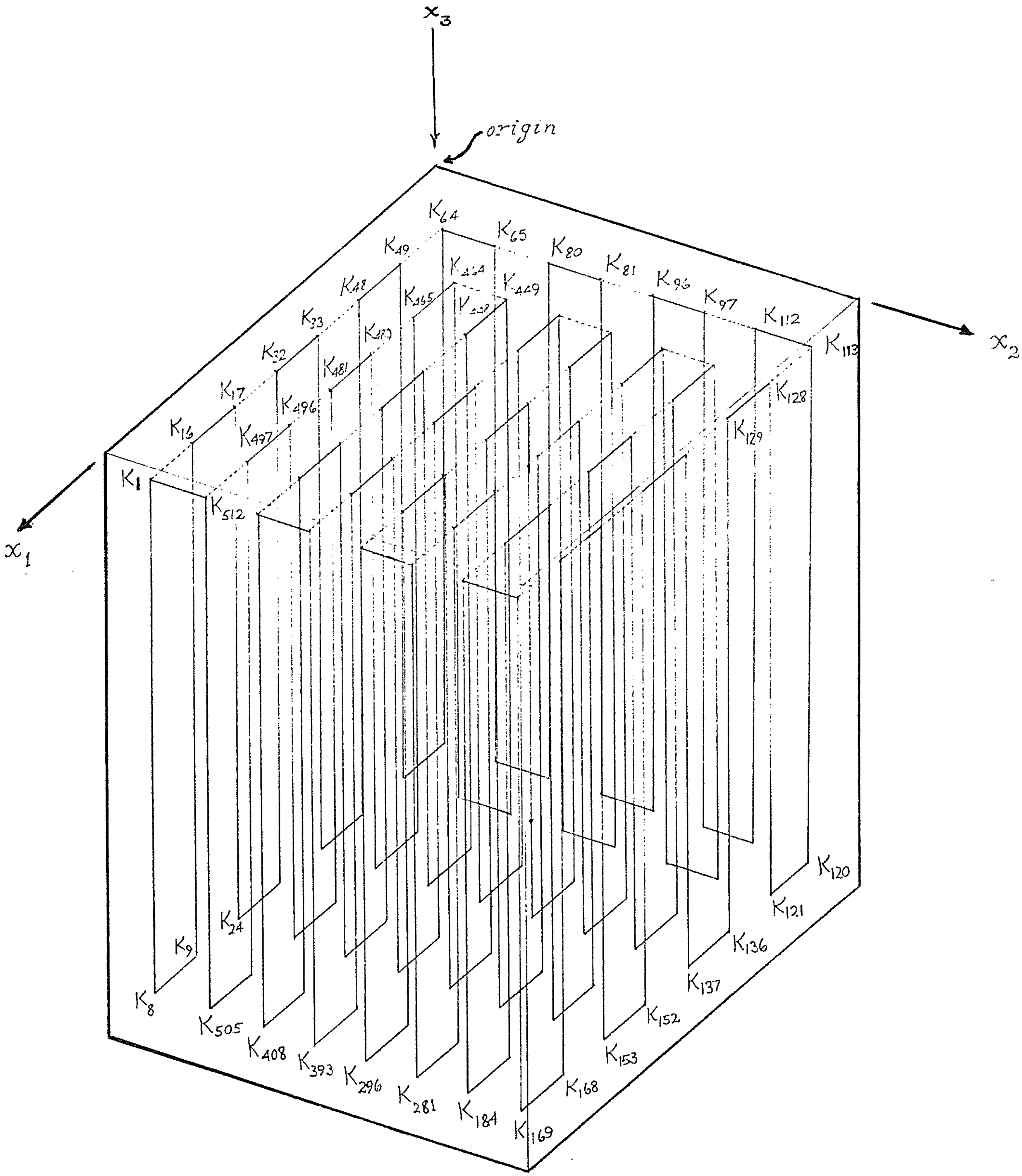


Figure 3:

The "basic tour" of cell-centers for

the case of $k = 2$.



<u>Figure 4</u>:

The "basic tour" of cell-centers for the case of k=3 and m=8. The dotted segments indicate the basic tour for k=2, in the top layer of cells, illustrating Algorithm B.

Now we want to show how the closed path constructed by Algorithm A can be tranformed into a tour with a shorter length.

First, if any cell has no points of J, then the basic tour can be shortened by connecting the previous cell-center to the next one. This may be repeated until the basic tour contains only cell-centers from cells containing points of J (without changing the sequential order of cell-centers in the original basic tour). This does not affect steps [4] and [5] of Algorithm A. Moreover, this can clearly be done in time proportional to m^k, i.e., $\underline{O}(n/\delta(n))$.

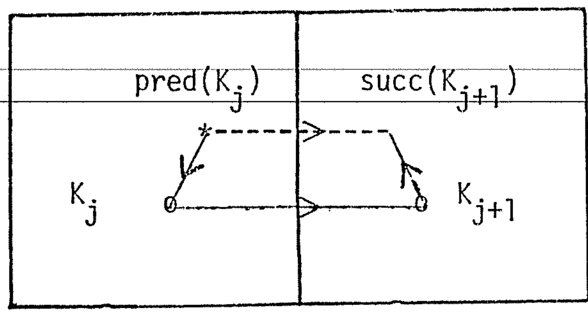


Figure 5

Secondly, let K_j and K_{j+1} be two consecutive cell-centers and let pred(K) and succ(K) denote the predecessor and the successor of a cell-center K, respectively, according to an order assigned to the closed path. Then, if $K_j \notin J$ and $K_{j+1} \notin J$, replace the edges

$(pred(K_j), K_j)$, (K_j, K_{j+1}), and $(K_{j+1}, succ(K_{j+1}))$, by the edge (pred(K_j), succ(K_{j+1})), as illustrated in Figure 5. If $K_j \in J$ and $K_{j+1} \notin J$, replace the edges

(K_j, K_{j+1}) and (K_{j+1}, succ(K_{j+1})),

by the edge (K_j, succ(K_{j+1})). Proceed similarly if $K_j \notin J$ and $K_{j+1} \in J$. After applying the procedure above to all pairs (K_j, K_{j+1}) of cell-centers, we get a tour which is shorter than the original closed path, since each replacement of edges always shortens the length of the closed path. Moreover , this shortening procedure can be clearly executed in time proportional to m^k, i.e., $\underline{O}(n/\delta(n))$.

3. Asymptotic Execution Time

Before giving the proof of Theorem 1, we want to state two lemmas which will be useful in this section. Their proofs will be given in Section 6.

Let S_n denote the time needed to compute the $M = m^k$ shortest tours through the points in each of the cells C_j constructed in Algorithm A, and let $(n)_j$ denote $n(n-1)\ldots(n-j+1)$.

Lemma 3.1: Under Condition C, if Algorithm A is applied to a k-TSP instance P_n, then there is a constant A such that

$$\mathcal{E}\, S_n \sim A\, n\, \delta(n)\, e^{\delta(n)}\, (1 - 1/\delta(n)) \text{ , as } n \to \infty .$$

Lemma 3.2: Under the same conditions as in Lemma 3.1, we have that

$$\operatorname{var} S_n \leqslant A^2 e^{2\delta(n)} \left\{ 16n\delta(n)^3 e^{\delta(n)} + \frac{n^2}{4\delta(n)^2} \right\} [1+\underline{O}(1/\delta(n))]$$

as $n \to \infty$, where A is the constant in Lemma 3.1.

Theorem 1 : Under Condition C, if Algorithm A is applied to a k-TSP instance $\underset{\sim}{P}^n$ then Algorithm A runs in time

$R_n \sim A/2\ n\ \delta(n)\ e^{\delta(n)}$ in probability,

where A is a constant

Proof

We have three terms to consider for the execution time of Algorithm A:

(i) the time to determine which points are in each of the $M = m^k$ cells;

(ii) the time to compute the shortest tours through the points in each of the M cells (step [3] of Algorithm A)

(iii) the time to construct the basic tour (Algorithm B).

We assume that $\underline{O}(n)$ (on- or off-line) memory space is available and a hashing technique may be used to determine the points in each cell and term (i) is then $\underline{O}(n)$ (otherwise, a sorting requiring $\underline{O}(n \log n)$ would be needed).

We estimate term (ii) as follows.

Since, for any $\varepsilon > 0$ and for all sufficiently large n, by Lemma 3.1,

$$|\mathcal{E}S_n - An\delta(n)e^{\delta(n)}| < \frac{\varepsilon}{2} An\delta(n)e^{\delta(n)},$$

we see, by the Chebyshev inequality with Lemma 3.2, for any $\varepsilon > 0$ and all sufficiently large n, that

$$\Pr[An\delta(n)e^{\delta(n)}(1-\varepsilon) \leqslant S_n \leqslant An\delta(n)e^{\delta(n)}(1+\varepsilon)]$$

$$= \Pr[|S_n - An\delta(n)e^{\delta(n)}| \leqslant \varepsilon\, An\delta(n)e^{\delta(n)}]$$

$$\geqslant \Pr[|S_n - \mathcal{E}S_n| \leqslant \frac{\varepsilon}{2} An\delta(n)e^{\delta(n)}]$$

$$\geqslant 1 - \operatorname{var} S_n / \frac{\varepsilon^2}{4} A^2 n^2 \delta(n)^2 e^{2\delta(n)}$$

$$\geqslant 1 - \{64n^{-1}\delta(n)e^{\delta(n)} + \delta(n)^{-4}\}\, [1+\underline{O}(1/\delta(n))]/\varepsilon^2 \to 1 \quad \text{as} \quad n \to \infty .$$

Therefore, since $n^{-1}\delta(n)e^{\delta(n)} \to 0$ as $n \to \infty$ (for example, we might choose $\delta(n) = \underline{O}$ (log n/log log n))

$$S_n \sim An\delta(n)e^{\delta(n)}, \quad \text{in probability,} \quad \text{as} \quad n \to \infty .$$

Finally, the basic tour can be constructed by using Algorithm B M times so that the term (iii) is clearly

$$\underline{O}(M) = \underline{O}(n/\delta(n)).$$

The proof is now complete, since the term (ii) dominates the others.

QED

4. Asymptotic Performance

Before proving Theorem 2, we need to prove three auxiliary lemmas. First, let us establish a notation for some concepts used in this section (following the notation in Beardwood, Halton, and Hammersley [1959]).

We have already stated that $\underset{\sim}{P}$ denotes a sequence of points, $\underset{\sim}{P}^n$ denotes the first n points of $\underset{\sim}{P}$ and $\underline{\underline{C}}$ denotes the unit hypercube. Let $\underline{\underline{E}}$ denote any bounded Lebesgue-measurable subset of R^k (we shall suppose that the boundary of $\underline{\underline{E}}$ has zero measure); $\underset{\sim}{P}^n\,\underline{\underline{E}}$ denote the subset of $\underset{\sim}{P}^n$ which lies in $\underline{\underline{E}}$; $N(\underset{\sim}{P}\underline{\underline{E}})$ denote the (possibly infinite) number of points of $\underset{\sim}{P}$ in $\underline{\underline{E}}$; $\ell\,(\underset{\sim}{P}\underline{\underline{E}})$ denote the length of the shortest tour through the points of $\underset{\sim}{P}\underline{\underline{E}}$; $\underline{\underline{C}}_1, \underline{\underline{C}}_2, \ldots$ denote semiclosed hypercubes (i.e., hypercubes open on their lower-left faces and closed on their upper-right faces) in different positions in R^k; and $v(\underline{\underline{E}})$ denote the volume (k-dimensional Lebesgue measure) of $\underline{\underline{E}}$. If ξ is a positive real number, we write $\xi\underline{\underline{E}}$ for the set of all points with coordinates $(\xi X_1, \xi X_2, \ldots, \xi X_k)$ such that the $(X_1, X_2, \ldots, X_k)$ are points of $\underline{\underline{E}}$. Thus $\xi\underline{\underline{E}}$ is a ξ-fold linear magnification of $\underline{\underline{E}}$, which leaves the origin of R^k invariant, and $v(\xi\underline{\underline{E}}) = \xi^k\, v(\underline{\underline{E}})$. We will use $\xi\underset{\sim}{P}\underline{\underline{E}}$ to denote the magnification of $\underset{\sim}{P}\underline{\underline{E}}$, whereas $\underset{\sim}{P}\,\xi\,\underline{\underline{E}}$ will denote the intersection of the unmagnified $\underset{\sim}{P}$ with the magnified $\underline{\underline{E}}$.

The phrase '$\underset{\sim}{P} \in u(\underline{\underline{E}})$', where $\underline{\underline{E}}$ is a Lebesgue set of strictly positive measure, means that $\underset{\sim}{P} = P_1, P_2, \ldots$ is a sample of random points independently distributed over $\underline{\underline{E}}$ with uniform probability density.

The phrase '$\underset{\sim}{P} \in W_\xi$' means that $\underset{\sim}{P} = P_1, P_2, \ldots$ is a sample from a Poisson process of density ξ over R^k; that is to say, for arbitrary disjoint Lebesgue sets $\underline{\underline{E}}_1, \underline{\underline{E}}_2, \ldots, \underline{\underline{E}}_m$,

$$\Pr\left\{ N(\underset{\sim}{P}\,\underline{\underline{E}}_j) = N_j \;;\; j = 1,2,\ldots,m \right\} = \prod_{j=1}^{m} \frac{\{\xi\, v(\underline{\underline{E}}_j)\}^{N_j}}{N_j!} \exp\{-\xi\, v(\underline{\underline{E}}_j)\}$$

Finally, we adopt the abbreviation $q = 1 - 1/k$, where $k \geqslant 2$.

With these notational conventions in mind, we are now able to state and prove the following lemmas.

Lemma 4.1: Let $M = m^k$ (where m is a positive even integer) be an integer value (but not a function of n as in Algorithm A) and let $\underline{\underline{C}}_j$, $j = 1,2,\ldots,M$, be the cubic cells, congruent to $(1/m)\underline{\underline{C}}$, obtained by dissecting $\underline{\underline{C}}$, as in Algorithm A. If $\underset{\sim}{P} \in W_\xi$, then

$$\mathcal{E}\, \ell(\underset{\sim}{P}\, \underline{\underline{C}}_j) \sim \beta \xi^q/M \quad , \text{ as } \quad \xi/M \to \infty \quad , \tag{4.1}$$

where β is an absolute constant (independent of ξ,M and $\underset{\sim}{P}$; but depending on k, the dimension of the space).

Proof: If ζ is a positive real number, Lemma 5 of Beardwood, Halton, and Hammersley [1959] says that

$$\mathcal{E}\, \ell\, (\underset{\sim}{P}' \zeta \underline{\underline{E}}) \sim \beta\, \zeta^k\, v(\underline{\underline{E}}) \quad \text{as} \quad \zeta \to \infty \,, \text{ for } \underset{\sim}{P}' \in W_1 . \tag{4.2}$$

We notice that, by scaling, to each $\underset{\sim}{P}' \in W_1$ in $\zeta\underline{\underline{E}}$ corresponds a $\underset{\sim}{P} \in W_\xi$ in $\zeta\xi^{-1/k}\underline{\underline{E}}$ (and this correspondence is one-to-one). By the same scaling we have

$$\ell(\underset{\sim}{P}\, \zeta\, \xi^{-1/k}\, \underline{\underline{E}}) = \xi^{-1/k}\, \ell(\underset{\sim}{P}'\, \zeta\, \underline{\underline{E}}).$$

Thus, from (4.2) we have

$$\mathcal{E}\, \ell(\underset{\sim}{P}\, \zeta\, \xi^{-1/k}\, \underline{\underline{E}}) \;\sim\; \xi^{-1/k}\, \beta\, \zeta^k\, v(\underline{\underline{E}}) \text{ as } \zeta \to \infty . \tag{4.3}$$

Let us take $\zeta\xi^{-1/k} = 1/m$ and $\underline{\underline{E}} = \underline{\underline{C}}$, so that $\underline{\underline{C}}_j$ is a linear translation of $(1/m)\underline{\underline{C}} = \zeta\xi^{-1/k}\underline{\underline{E}}$. Then

$v(\underline{\underline{E}}) = v(\underline{\underline{C}}) = 1$, $\zeta^k = \xi/m^k = \xi/M$. Thus, as $\zeta \to \infty$, $\xi/M \to \infty$; and $\xi^{-1/k}\,\zeta^k = \xi^q/M$. Since W_ξ is homogeneous in R^k, so that translation of sets has no effect on the statistics, from (4.3) we get (4.1).

QED

Lemma 4.2: Under the same conditions as in Lemma 4.1, we have

$$\operatorname{var}\ \ell(\underset{\sim}{P}\underline{\underline{C}}_j) = \underline{o}(1)\ \xi^{-2/k}\ (\xi/M)^{2-2/k^2}, \quad \text{as } \xi/M \to \infty, \tag{4.4}$$

where $\underline{o}(1)$ depends only on k.

Proof: If ζ is a positive real number and if $\underline{\underline{E}} \subseteq \underline{\underline{C}}$, Lemma 6 of Beardwood, Halton, and Hammersley [1959] implies that

$$\operatorname{var}\ \ell(\underset{\sim}{P}'\zeta\underline{\underline{E}}) = \underline{O}\ (\zeta^{2k-2/(k-1)}\ \log^2 \zeta), \text{ as } \zeta \to \infty, \quad \text{for } \underset{\sim}{P}' \in W_1. \tag{4.5}$$

We notice that

$$\zeta^{2/k}\zeta^{-2/(k-1)}\ \log^2\zeta = \zeta^{-2/(k(k-1))}\ \log^2\zeta = \underline{o}(1), \quad \text{as } \zeta \to \infty, \text{ for all } k \geq 2,$$

Thus, from (4.5) we have that

$$\operatorname{var}\ \ell(\underset{\sim}{P}'\zeta\ \underline{\underline{E}}) = \underline{o}(1)\ \zeta^{2k-2/k}, \text{ as } \zeta \to \infty. \tag{4.6}$$

If $\underset{\sim}{P} \in W_\xi$ and we consider the set $\zeta\, \xi^{-1/k} \underline{\underline{E}}$; by scaling as in the proof of Lemma 4.1 above, we have from (4.6) that

$$\text{var}\ \ell(\underset{\sim}{P}\ \zeta\xi^{-1/k}\underline{\underline{E}}) = \xi^{-2/k}\ \text{var}\ \ell\ (\underset{\sim}{P}'\zeta\ \underline{\underline{E}}) =$$

$$= \underline{o}(1)\ \ \xi^{-2/k}\ \zeta^{2k-2/k}\ ,\ \text{as}\ \zeta \to \infty\ . \qquad (4.7)$$

As before, if $\zeta\xi^{-1/k} = 1/m$ and $\underline{\underline{E}} = \underline{\underline{C}}$, then $\zeta = (\xi/M)^{1/k}$.

Thus, from (4.7) we have that

$$\text{var}\ \ \ell(\underset{\sim}{P}(1/m)\underline{\underline{C}}) = \text{var}\ \ \ell(\underset{\sim}{P}\ \underline{\underline{C}}_j) =$$

$$= \underline{o}(1)\ \xi^{-2/k}\ (\xi/M)^{2-2/k^2}\ ,$$

$$\text{as}\ \xi/M \to \infty\ .$$

QED

Let us now introduce $U_{\xi,M}$, a random variable conditional on ξ and M as parameters with $M>1$:

$$U_{\xi,M} = \sum_{j=1}^{M} \ell(\underset{\sim}{P}\ \underline{\underline{C}}_j) \quad , \quad \underset{\sim}{P} \in W_\xi\ ,\ \underline{\underline{C}}_j \text{ a translation of } (1/m)\underline{\underline{C}}\ .$$

(sum of the shortest tours in each cell)

Then, by the independence of $\underset{\sim}{P} \in W_\xi$ in the disjoint $\underline{\underline{C}}_j$'s we have from Lemmas 4.1 and 4.2 that

$$\mathcal{E}\ U_{\xi,M} = \sum_{j=1}^{M} \mathcal{E}\ell(\underset{\sim}{P}\ \underline{\underline{C}}_j) \sim \beta\ \xi^q\ ,\ \text{as}\ \xi/M \to \infty\ . \qquad (4.8)$$

$$\operatorname{var} U_{\xi,M} = \sum_{j=1}^{M} \operatorname{var} \ell(\underset{\sim}{P}\, \underset{=}{C}_j) = \underline{o}(1)\, M\xi^{-2/k}(\xi/M)^{2-2/k^2}, \quad \text{as } \xi/M \to \infty. \tag{4.9}$$

Lemma 4.3: Given any set $\underset{\sim}{P}^n$ of n points in $\underset{=}{C}$, let $M_1 = m_1{}^k$ and $M_2 = m_2{}^k$, where $m_1 < m_2$ and m_1, m_2 are positive even integers. Consider the dissections of $\underset{=}{C}$, into M_1 cells $\underset{=}{C}_{1i}$ congruent to $m_1{}^{-1}\underset{=}{C}$, and into M_2 cells $\underset{=}{C}_{2j}$ congruent to $m_2{}^{-1}\underset{=}{C}$, as in Algorithm A. Then

$$\sum_{j=1}^{M_2} \ell(\underset{\sim}{P}^n \underset{=}{C}_{2j}) \leqslant \sum_{i=1}^{M_1} \ell(\underset{\sim}{P}^n \underset{=}{C}_{1i}) + \underline{O}\left[M_2{}^{1/k(k-1)} n^{1-1/(k-1)}\right] + \underline{O}\left[M_2{}^{1-1/k}\right]. \tag{4.10}$$

Proof: Since $M_1 < M_2$, the cells $\underset{=}{C}_{2j}$ are smaller than the cells $\underset{=}{C}_{1i}$ (sides are $m_2{}^{-1}$ and $m_1{}^{-1}$, respectively); thus any cell $\underset{=}{C}_{2j}$ can contain at most one corner of the dissection into cells $\underset{=}{C}_{1i}$. Therefore, $\underset{=}{C}_{2j}$ contains all or part of at most 2^k minimal cell-tours T_i (say) of $\underset{\sim}{P}^n \underset{=}{C}_{1i}$. We distinguish two cases: k=2 and k>2.

Case (i): k=2. We form a tour of $\underset{\sim}{P}^n \underset{=}{C}_{2j}$ as follows. Any pieces of T_i $(i=1,\ldots,M_1)$ intersecting $\underset{=}{C}_{2j}$ can be formed into a simple closed polygon by tracing parts of the perimeter of $\underset{=}{C}_{2j}$. (See Fig. 6) This perimeter is of length $4m_2{}^{-1} = 4M_2{}^{-1/2}$. Any T_i contained entirely in $\underset{=}{C}_{2j}$ can be connected to the above polygon by a double chord of length less than $m_2{}^{-1}$ (See Fig. 7) Such included tours cannot be more than 4 in number. Since each part of every T_i will lie in exactly one of the $\underset{=}{C}_{2j}$, the sum of the tours constructed above will not exceed

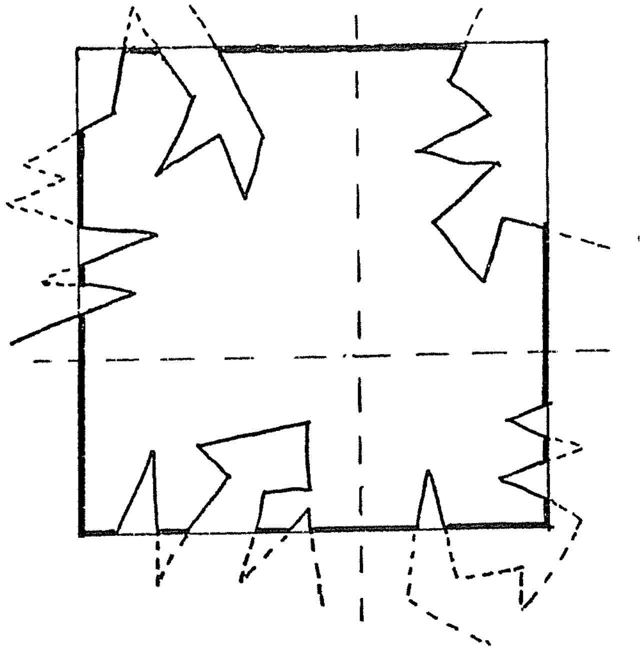


Figure 6

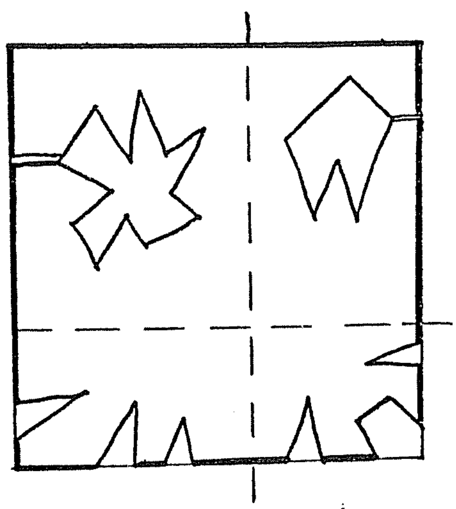


Figure 7

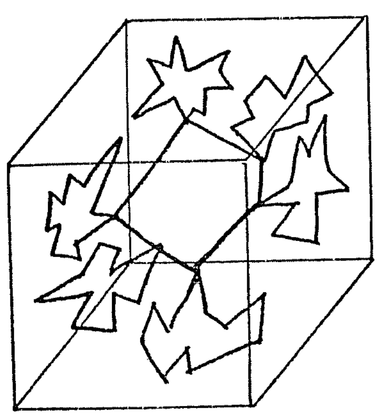


Figure 8

$\sum_{i=1}^{M_1} \ell(\mathbf{P}^n\mathbf{C}_{1i}) + 8\,M_2^{1/2}$, and will not be less than the minimal sum

$\sum_{j=1}^{M_2} \ell(\mathbf{P}^n\mathbf{C}_{2j})$. This proves (4.10) for $k=2$.

Case (ii): $k \geqslant 3$. The cell $\mathbf{C}_{2j}$ now has $2k$ faces (of $k-1$ dimensions), of $(k-1)$-dimensional volume $m_2^{-(k-1)} = M_2^{-(k-1)/k}$. Various tours T_i will cross a particular face (say) F times; and so, we may form a tour of these F intersections by a polygon L, of length not exceeding $\alpha'_{k-1}\,M_2^{-1/k}\,F^{1-1/(k-1)}$ (by Lemma 4 of Beardwood, Halton, and Hammersley [1959]; with $\alpha'_{k-1} \geqslant \alpha_{k-1}$ independent of M_2, F, or $\mathbf{P}^n$); and therefore, all pieces of tours T_i entering into $\mathbf{C}_{2j}$ by the given face may be connected into a simple closed polygon by parts of such a polygon L, rather as in Case (i). All $2k$ such paths belonging to $\mathbf{C}_{2j}$ may then be joined into a single simple closed polygon by $2k$ segments of total length not exceeding $2k^{3/2}M_2^{-1/k}$ (Figure 8), since the diagonal of $\mathbf{C}_{2j}$ is $k^{1/2}M_2^{-1/k}$. As in Case (i), we see that there are at most 2^k tours T_i entirely contained in $\mathbf{C}_{2j}$, and these can be incorporated into our tour of $\mathbf{P}^n\mathbf{C}_{2j}$ by double chords of length less than $M_2^{-1/k}$. Again, each part of every T_i will lie in exactly one $\mathbf{C}_{2j}$, and the sum of all the numbers F of intersections of faces with tours cannot exceed $4n$, since each point of $\mathbf{P}^n$ is connected to its successor, in its T_i, by just one chord, and this can only cross at most two faces of the finer dissection; and every such intersection is counted twice.

Thus the sum of the tours constructed above cannot exceed

$$\sum_{i=1}^{M_1} \ell(\mathbf{P}^n\mathbf{C}_{1i}) + \alpha'_{k-1}\,M_2^{-1/k} \sum_{\text{faces}} F^{1-1/(k-1)}$$

$$+ 2k^{3/2} M_2^{1-1/k} + 2^k M_2^{1-1/k} . \qquad (4.11)$$

By Hölder's inequality, since every face intersected at all will be counted twice, and there are at most $(M_2 - M_2^{1-1/k})2k$ such faces,

$$\sum_{\text{faces}} (1)^{1/(k-1)} F^{1-1/(k-1)} \leqslant \Big[\sum_{\text{faces}} (1)\Big]^{1/(k-1)} \Big[\sum_{\text{faces}} F\Big]^{1-1/(k-1)}$$

$$= \Big[4k(M_2 - M_2^{1-1/k})\Big]^{1/(k-1)} (4n)^{1-1/(k-1)}$$

$$= \underline{O}\Big[M_2^{1/(k-1)} n^{1-1/(k-1)}\Big]$$

Thus the upper bound given by (4.11) is

$$\sum_{i=1}^{M_1} \ell(\underset{\sim}{P}^n \underset{=}{C}_{1i}) + \underline{O}\Big[M_2^{1/(k-1)-1/k} n^{1-1/(k-1)}\Big]$$

$$+ \underline{O}\Big[M_2^{1-1/k}\Big] .$$

Since the sum of the tours constructed above cannot be less than $\sum_{j=1}^{M_2} \ell(\underset{\sim}{P}^n \underset{=}{C}_{2j})$, we obtain (4.10) for $k \geqslant 3$.

Q.E.D.

Finally, we are now able to proceed to:

Proof of Theorem 2:

First, assume the conditions of Lemmas 4.1 and 4.2.

From (4.8) we know that for all sufficiently large ξ/M and for any arbitrary $\varepsilon > 0$ we have

$$|\mathcal{E}\, U_{\xi,M} - \beta\, \xi^q \,| < \frac{1}{2}\, \varepsilon\, \beta\, \xi^q ;$$

and then by Chebyshev's inequality, much as in the proof of Theorem 1,

$$\Pr\{\,|U_{\xi,M} - \beta\,\xi^q\,| \leqslant \varepsilon\,\beta\,\xi^q\,\} \geqslant \Pr\{\,|U_{\xi,M} - \mathcal{E}\;U_{\xi,M}\,| \leqslant \tfrac{1}{2}\,\varepsilon\,\beta\,\xi^q\,\}$$

$$\geqslant 1 - \underline{o}(1)\; M\,\xi^{-2/k}(\xi/M)^{2-2/k^2} \Big/ \left(\tfrac{1}{2}\,\varepsilon\beta\xi^q\right)^2 \qquad \text{(by (4.9))}$$

$$= 1 - \frac{\underline{o}(1)}{\varepsilon^2}\;\frac{1}{\xi^{2/k^2}\,M^{1-2/k^2}}\,, \quad \text{as } \frac{\xi}{M} \to \infty\,. \tag{4.12}$$

Also, if $N(\underset{\sim}{P}\,\underline{\underline{C}}) = n_\xi$, then by Chebyshev's inequality,

$$\Pr\{\;|\,n_\xi - \xi\,| \leqslant \varepsilon\,\xi\,\}\;\geqslant 1 - \frac{1}{\varepsilon^2\xi}\,, \tag{4.13}$$

since $\mathcal{E}\,n_\xi = \operatorname{var} n_\xi = \xi$.

Thus, from (4.12) and (4.13) we have for all sufficiently large ξ/M that

$$\Pr\left[\beta\,\frac{1-\varepsilon}{(1+\varepsilon)^q} \leqslant \frac{U_{\xi,M}}{n_\xi{}^q} \leqslant \beta\,\frac{1+\varepsilon}{(1-\varepsilon)^q}\right] \geqslant 1 - \left[\frac{1}{\xi^{2/k^2}M^{1-2/k^2}} + \frac{1}{\xi}\right]\varepsilon^{-2}\,. \tag{4.14}$$

Now, let $V_{n,M} = \sum_{j=1}^{M} \ell(\underset{\sim}{P}^n\underline{\underline{C}}_j)$ where n is a positive integer value and $\underset{\sim}{P} \in u(\underline{\underline{C}})$.

Next, define $f(n,M)$ by

$$1 - f(n,M) = \Pr\left[\beta\,\frac{1-\varepsilon}{(1+\varepsilon)^q} \leqslant \frac{V_{n,M}}{n^q} \leqslant \beta\,\frac{1+\varepsilon}{(1-\varepsilon)^q}\right]. \tag{4.15}$$

Since the conditional probability distribution of $U_{\xi,M}$ given $n_\xi = n$ is the unconditional probability distribution of $V_{n,M}$, we have

$$\sum_{n=0}^{\infty} e^{-\xi} \frac{\xi^n}{n!} [1-f(n,M)] = \Pr\left[\beta \frac{1-\varepsilon}{(1+\varepsilon)^q} \leqslant \frac{U_{\xi,M}}{n_{\xi}q} \leqslant \beta \frac{1+\varepsilon}{(1-\varepsilon)^q}\right]$$

$$\geqslant 1-\left[\frac{1}{\xi^{2/k^2} M^{1-2/k^2}} + \frac{1}{\xi}\right] \varepsilon^{-2}, \tag{4.16}$$

for all sufficiently large ξ/M.

Since $0 \leqslant 1-f(n,M) \leqslant 1$, (4.16) gives us that

$$\sup_{|t-\xi| \leqslant \varepsilon\xi} [1-f(t,M)] \sum_{|n-\xi| \leqslant \varepsilon\xi} e^{-\xi} \frac{\xi^n}{n!} + \sum_{|n-\xi| > \varepsilon\xi} e^{-\xi} \frac{\xi^n}{n!}$$

$$\geqslant 1-\left[\frac{1}{\xi^{2/k^2} M^{1-2/k^2}} + \frac{1}{\xi}\right] \varepsilon^{-2}. \tag{4.17}$$

By observing that the first summation above is less than 1 and the second summation is less than $1/(\varepsilon^2\xi)$, by (4.13), we have that

$$\sup_{|t-\xi| \leqslant \varepsilon\xi} [1-f(t,M)] \geqslant 1-\left[\frac{1}{\xi^{2/k^2} M^{1-2/k^2}} + \frac{2}{\xi}\right] \varepsilon^{-2},$$

for all sufficiently large ξ/M. (4.18)

Since by hypothesis $M > 1$, for all sufficiently large ξ/M we have

$$\sup_{|t-\xi| \leqslant \varepsilon\xi} [1-f(t,M)] \geqslant 1-\left[\frac{1}{\xi^{2/k^2}} + \frac{2}{\xi}\right] \varepsilon^{-2}$$

$$\geqslant 1- C\xi^{-2/k^2}, \tag{4.19}$$

for all sufficiently large ξ/M, where C is a constant (depending upon ε and k but <u>not</u> on M.)

The supremum in (4.19) is taken over the range:

$$(1-\varepsilon)\,\xi \leqslant t \leqslant (1+\varepsilon)\,\xi$$

If $\xi = \dfrac{(1+\varepsilon)^m}{(1-\varepsilon)^{m+1}}$, and if J_m is the set of integers t satisfying: $\left(\dfrac{1+\varepsilon}{1-\varepsilon}\right)^m \leqslant t \leqslant \left(\dfrac{1+\varepsilon}{1-\varepsilon}\right)^{m+1}$, $m = 1,2,\ldots..,$ then J_m becomes the range of the supremum in (4.19).

We observe that, for any M and for sufficiently large m, ξ/M can be made as large as we like, and so can ensure that (4.19) above holds true. In particular, if we let n be any member of J_m, for fixed ε, and let $M = M(n) = n/\delta(n)$. We have

$$\frac{\xi}{M(n)} = \frac{(1+\varepsilon)^m}{(1-\varepsilon)^{m+1}}\ \frac{\delta(n)}{n} \geqslant \frac{(1+\varepsilon)^m}{(1-\varepsilon)^{m+1}}\ \frac{(1-\varepsilon)^{m+1}}{(1+\varepsilon)^{m+1}}\,\delta(n) = \frac{\delta(n)}{(1+\varepsilon)}$$

Since $\delta(\cdot)$ is an increasing function; for fixed ε and for sufficiently large m, $\xi/M(n)$ can be made as large as we like; so that from (4.19) we have

$$\sup_{t\in J_m}\ [1-f(t,M(n))] \geqslant 1 - C'\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{-2m/k^2}, \tag{4.20}$$

for all sufficiently large m,

where $C' = C(1-\varepsilon)^{2/k^2}$ is a constant (depending only on ε and k). That is, there is an integer m_0 (depending on ε and k) such that (4.20) holds for all $m \geq m_0$. Further, since J_m contains only a finite number of integers, it contains an integer n_m (depending on ε, k and n) such that

$$1-f(n_m,M(n)) = \sup_{t\in J_m} [1-f(t,M(n)]; \quad \text{whence}$$

$$\sum_{m=0}^{\infty} \left\{ 1-\Pr\left[\beta\frac{1-\varepsilon}{(1+\varepsilon)^q} \le \frac{V_{n_m},M(n)}{n_m{}^q} \le \beta\frac{1+\varepsilon}{(1-\varepsilon)^q}\right]\right\}$$

$$= \sum_{m=0}^{m_0-1} f(n_m,M(n)) + \sum_{m=m_0}^{\infty} f(n_m,M(n)) \le m_0 + \sum_{m=m_0}^{\infty} C'\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{-2m/k^2} < \infty . \tag{4.21}$$

By the Borel-Cantelli lemma, (4.21) implies that, with probability one, for any choices of n (and consequent values of M and n_m) in each J_m,

$$\beta\frac{1-\varepsilon}{(1+\varepsilon)^q} \leqslant \lim_{m\to\infty} \inf \frac{V_{n_m},M(n)}{n_m{}^q} \leqslant \lim_{m\to\infty} \sup \frac{V_{n_m},M(n)}{n_m{}^q} \leqslant \beta\frac{1+\varepsilon}{(1-\varepsilon)^q} . \tag{4.22}$$

Next, for choices n', n, and n'' in J_{m-1}, J_m, and J_{m+1}, respectively, write $\mu_n = n_{m-1}$, $\nu_n = n_{m+1}$. From the definition of J_m and n_m we have

$$\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{m-1} \leqslant \mu_n \leqslant \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{m} \leqslant n \leqslant \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{m+1} \leqslant \nu_n \leqslant \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{m+2} . \tag{4.23}$$

From (4.23),
$$0 \leqslant n-\mu_n \leqslant n - \frac{n}{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{m+1}}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{m-1}$$
$$= n\left[1-\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2\right]$$
$$= n\frac{4\varepsilon}{(1+\varepsilon)^2} \leqslant 4\,\varepsilon\, n . \tag{4.24}$$

Similarly, from (4.23), $0 \leqslant \nu_n - n \leqslant \dfrac{n}{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^m} \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{m+2} - n$

$$= n\left[\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2 - 1\right]$$

$$= n \, \frac{4\varepsilon}{(1-\varepsilon)^2} \leqslant 5\,\varepsilon\, n \qquad (4.25)$$

for sufficiently small ε ($< \frac{1}{5+\sqrt{20}}$).

Thus $\left(\underset{\sim}{P}^{\mu_n}\right)^c \underset{\sim}{P}^n$ consists of a set of not more than $4\,\varepsilon\, n$ points in $\underline{\underline{C}}$ and $\underset{\sim}{P}^{\nu_n} (\underset{\sim}{P}^n)^c$ consists of a set of not more than $5\,\varepsilon\, n$ points in $\underline{\underline{C}}$.

Now, if $\bar{\underline{\underline{E}}}$ denotes the closure of $\underline{\underline{E}}$, by Lemma 4 of Beardwood, Halton and Hammersley [1959] there is an α such that $\limsup_{n\to\infty} n^{-q}\ell(\underset{\sim}{P}^n \underline{\underline{E}}) \leq \alpha \; v^{1/k}(\bar{\underline{\underline{E}}})$ i.e. there is an α' such that

$$(\forall n)\ n^{-q}\ell(\underset{\sim}{P}^n \underline{\underline{E}}) \leqslant \alpha' \; v^{1/k}(\bar{\underline{\underline{E}}}) \,, \qquad (4.26)$$

where α and α' are absolute constants (depending on k). If $a_j = N(\underset{\sim}{P}^{\nu_n}(\underset{\sim}{P}^n)^c \underline{\underline{C}}_j)$, by applying (4.26) to $\underline{\underline{C}}_j$ we have

$$(\forall a_j)\ \ a_j^{-q}\, \ell(P^{\nu_n}(\underset{\sim}{P}^n)^c \underline{\underline{C}}_j) \leqslant \alpha' M(n)^{-1/k} \,, \qquad (4.27)$$

since $v(\underline{\underline{C}}_j) = M(n)^{-1}$.

From (4.25) we have

$$V_{\nu_n, M(n)} \leqslant V_{n, M(n)} + \sum_{j=1}^{M(n)} [\ell(\underset{\sim}{P}^{\nu_n}(\underset{\sim}{P}^n)^c \underline{\underline{C}}_j) + 2\sqrt{k}\, M(n)^{-1/k}] \qquad (4.28)$$

($\sqrt{k}\, M(n)^{-1/k}$ is the diameter of $\underline{\underline{C}}_j$).

Since $\sum_{j=1}^{M(n)} a_j^q \leqslant M(n)^{1/k}\left(\sum_{j=1}^{M(n)} a_j\right)^q$ (Hölder's inequality), from (4.27)

we have

$$\sum_{j=1}^{M(n)} [\ell(\underset{\sim}{P}^{\nu_n}(\underset{\sim}{P}^n)^c \underline{\underline{C}}_j)] \leqslant \sum_{j=1}^{M(n)} a_j^q \, \alpha' M(n)^{-1/k}$$

$$\leqslant \alpha' M(n)^{-1/k} \, M(n)^{1/k} \left[\sum_{j=1}^{M(n)} a_j\right]^q$$

$$\leqslant \alpha' \ (5\varepsilon n)^q \ . \tag{4.29}$$

From (4.28) and (4.29), we have

$$n^{-q} \, V_{\nu_n, M(n)} \leqslant n^{-q} \, V_{n,M(n)} + n^{-q}[\alpha'(5\varepsilon n)^q + 2\sqrt{k}\, M(n)^q]$$

$$= n^{-q} V_{n,M(n)} + \alpha'(5\varepsilon)^q + 2\sqrt{k}\left(\frac{M(n)}{n}\right)^q \tag{4.30}$$

and the last term in (4.30) is $\underline{o}(1)$, as $n \to \infty$. On the other hand, since $M(n) < M(n'')$ and $\nu_n^{-q} \leqslant n^{-q}$, by Lemma 4.3 we have that

$$\nu_n^{-q} \, V_{\nu_n, M(n'')} \leqslant n^{-q} V_{\nu_n, M(n)} + n^{-q}\left[\underline{O}\left(M(n'')^{\frac{1}{k(k-1)}} \nu_n^{1-\frac{1}{k-1}}\right) + \underline{O}(M(n'')^q)\right] . \tag{4.31}$$

Since, by (4.25), $\nu_n \leqslant (1+5\varepsilon)n$ and similarly $n'' \leqslant (1+5\varepsilon)n$, we have that

$$n^{-q}\underline{O}\left(M(n'')^{\frac{1}{k(k-1)}}\ \nu_n^{\,1-\frac{1}{k-1}}\right) = \underline{O}\left[\left(\frac{(1+5\varepsilon)n}{\delta(n'')}\right)^{\frac{1}{k(k-1)}} \frac{((1+5\varepsilon)n)^{1-\frac{1}{k-1}}}{n^{1-\frac{1}{k}}}\right]$$

$$= \underline{O}\left(\delta(n'')^{\frac{-1}{k(k-1)}}\right) = \underline{o}(1), \text{ as } n \to \infty, \tag{4.32}$$

and also

$$n^{-q}\underline{O}(M(n'')^q) = \underline{O}\left[\left(\frac{(1+5\varepsilon)n}{n\ \delta(n'')}\right)^q\right] = \underline{o}(1), \text{ as } n \to \infty . \tag{4.33}$$

We have from (4.30), (4.31), (4.32), and (4.33)

$$\nu_n^{-q}\ V_{\nu_n, M(n'')} \leqslant n^{-q}\ V_{n,M(n)} + \alpha'(5\varepsilon)^q + \underline{o}(1), \text{ as } n \to \infty .$$

We see that, in this inequality, the independent variables are ε, n (in J_m, which determines m), and n'' (chosen in J_{m+1}, which determines $\nu_n = n_{m+1}$). Applying (4.22), we thus get that

$$\frac{\beta(1-\varepsilon)}{(1+\varepsilon)^q} \leqslant \liminf_{n\to\infty}\ n^{-q}\ V_{n,M(n)} + \alpha'(5\varepsilon)^q,$$

$$\text{with probability one.} \tag{4.34}$$

Similarly, if $b_j = N(\mathcal{L}^n(\mathcal{L}^{{}^{\mu}n})^c\underline{\underline{C}}_j)$, by applying (4.26) to $\underline{\underline{C}}_j$, we get

$$(\forall b_j)\quad b_j^{-q}\ \ell(\mathcal{L}^n(\mathcal{L}^{{}^{\mu}n})^c\underline{\underline{C}}_j) \leqslant \alpha' M(n)^{-1/k} . \tag{4.35}$$

From (4.24) we have

$$V_{\mu_n,M(n)} \geqslant V_{n,M(n)} - \sum_{j=1}^{M(n)} [\ell(\underset{\sim}{P}^n(\underset{\sim}{P}^{\mu_n})^c \underline{\underline{C}}_j) + 2\sqrt{k}\, M(n)^{-1/k}] . \quad (4.36)$$

From (4.35) we have

$$\sum_{j=1}^{M(n)} [\ell(\underset{\sim}{P}^n(\underset{\sim}{P}^{\mu_n})^c \underline{\underline{C}}_j)] \leqslant \sum_{j=1}^{M(n)} b_j^q\, \alpha' M(n)^{-1/k}$$

$$\leqslant \alpha'\, M(n)^{-1/k}\, M(n)^{1/k} \left[\sum_{j=1}^{M(n)} b_j\right]^q$$

$$\leqslant \alpha'(4\varepsilon n)^q \quad \text{(by (4.24))} . \qquad (4.37)$$

From (4.36) and (4.37) we have

$$n^{-q}\, V_{\mu_n,M(n)} \geqslant n^{-q}\, V_{n,M(n)} - n^{-q}[\alpha'(4\varepsilon n)^q + 2\sqrt{k}\, M(n)^q]$$

$$= n^{-q}\, V_{n,M(n)} - \alpha'(4\varepsilon)^q + \underline{o}(1) , \qquad (4.38)$$

as $n \to \infty$.

On the other hand, since $M(n') < M(n)$ and $n^{-q} \leqslant \mu_n^{-q}$, by Lemma 4.3 we have

$$\mu_n^{-q}\, V_{\mu_n,M(n')} + \mu_n^{-q}\left[\underline{0}\left(M(n)^{\frac{1}{k(k-1)}}\, \mu_n^{1-\frac{1}{k-1}}\right) + \underline{0}(M(n)^q)\right]$$

$$\geqslant n^{-q}\, V_{\mu_n,M(n)} . \qquad (4.39)$$

Since, by (4.24), $\mu_n \geqslant (1-4\varepsilon)n$ we have

$$\mu_n^{-q}\, \underline{0}\left(M(n)^{\frac{1}{k(k-1)}}\ \mu_n^{1-\frac{1}{k-1}}\right) = \underline{0}\left[\left(\frac{(1-4\varepsilon)^{-1}\mu_n}{\delta(n)}\right)^{\frac{1}{k(k-1)}} \frac{\mu_n^{1-\frac{1}{k-1}}}{\mu_n^{1-\frac{1}{k}}}\right]$$

$$= \underline{0}\ \left(\delta(n)^{\frac{-1}{k(k-1)}}\right) = \underline{o}(1),\ \text{ as } n \to \infty, \tag{4.40}$$

and also

$$\mu_n^{-q}\ \underline{0}(M(n)^q) = \underline{0}\ \left[\left(\frac{(1-4\varepsilon)^{-1}\mu_n}{\delta(n)\mu_n}\right)^q\right] = \underline{o}(1),\ \text{ as } n \to \infty\ . \tag{4.41}$$

We have from (4.38), (4.39), (4.40), and (4.41) that

$$\mu_n^{-q}\ V_{\mu_n, M(n')} \geqslant n^{-q}\ V_{n,M(n)} - \alpha'(4\varepsilon)^q + \underline{o}(1),\ \text{ as } n \to \infty\ .$$

As in obtaining (4.34), we note that the independent variables in this inequality are ε, n (which determines m), and n' (which determines $\mu_n = n_{m-1}$). Applying (4.22) again, we get

$$\frac{\beta(1+\varepsilon)}{(1-\varepsilon)^q} \geqslant \limsup_{n\to\infty}\ n^{-q}\ V_{n,M(n)} - \alpha'(4\varepsilon)^q\ , \tag{4.42}$$

with probability one.

Since ε is arbitrary and $n^{-q}\ V_{n,M(n)}$ does not depend on ε, (4.34) and (4.42) imply that

$$\lim_{n\to\infty}\ n^{-q}\ V_{n,M(n)} = \beta,\ \text{ with probability one.} \tag{4.43}$$

Now let $X_{n,M(n)} = \sum_{j=1}^{M(n)} \ell(\underset{\sim}{P}^n \underline{\underline{C}}_j \cup K_j)$ where K_j is the singleton containing the cell-center of $\underline{\underline{C}}_j$. Then we have

$$\begin{aligned} X_{n,M(n)} &\leq V_{n,M(n)} + M(n) \quad [2\sqrt{k}\, M(n)^{-1/k}] \\ &= V_{n,M(n)} + \underline{O}[M(n)^q] \\ &\sim \beta \quad n^q + \underline{O}[n^q / \delta(n)^q] \qquad \text{(by (4.43))} \\ &= \beta \quad n^q + \underline{o}(n^q) . \end{aligned}$$

Thus: $X_{n,M(n)} \leq \beta \quad n^q + \underline{o}(n^q)$, as $n \to \infty$, with probability one. (4.44)

Since the basic tour has length $M(n)\, h$ (there are $M(n)$ cell centers being connected by edges of length h), where $h = 1/m = 1/M(n)^{1/k}$, we have that the length of the closed path given by Algorithm A is, by (4.44)

$$\begin{aligned} T(n) &= X_{n,M(n)} + \underline{O}[M(n)^q] \\ &= X_{n,M(n)} + \underline{o}(n^q) \\ &\leq \beta\, n^q + \underline{o}(n^q) , \quad \text{as } n \to \infty , \quad \text{with probability one.} \qquad (4.45) \end{aligned}$$

On the other hand, by Lemma 7 of Beardwood, Halton, and Hammersley [1959], the length $T_0(n)$ of the optimal tour is such that

$$T_0(n) \sim \beta\, n^q , \quad \text{as } n \to \infty , \quad \text{with probability one.} \qquad (4.46)$$

From (4.45) and (4.46) we have

$$1 \leqslant \frac{T(n)}{T_0(n)} \leqslant \frac{\beta n^q + \underline{o}(n^q)}{\beta n^q}$$

Thus $$\frac{T(n)}{T_0(n)} \sim 1 \quad \text{as} \quad n \to \infty \text{ , with probability one.}$$

QED

5. A Generalization of the Results

As we mentioned in Section 1, Algorithm A can be applied to $\lambda\underline{\underline{C}}$, the k-dimensional hypercube of side λ, instead of $\underline{\underline{C}}$. In this section, we want to show how Theorems 1 and 2 can be modified so that, under Condition D, Theorem 3 holds true.

Under Condition D, we partition $\lambda\underline{\underline{C}}$ into $M = n/\delta(n)$ cubic cells, $\underline{\underline{C}}_j$ say. Let us define index sets H_0, H_1, H_2 as follows, for $1 \leq j \leq M$,

$$j \in H_0 \text{ , iff } \underline{\underline{C}}_j \subseteq \underline{\underline{E}}^c \text{ ;}$$
$$j \in H_1 \text{ , iff } C_j \subseteq \underline{\underline{E}} \text{ ;}$$
$$j \in H_2 \text{ , iff } j \notin H_0 \text{ and } j \notin H_1 \text{ .}$$

Let $N(H_0) = N_0$, $N(H_1) = N_1$, $N(H_2) = N_2$, and $M' = v(\underline{\underline{E}})M/\lambda^k$.

Since each cell is of volume λ^k/M and $\underline{\underline{E}}$ has volume $v(\underline{\underline{E}})$; under Condition D, the probability of s points falling into $\underline{\underline{C}}_j$ is

$$\binom{n}{s} \left\{\frac{\lambda^k/M}{v(\underline{\underline{E}})}\right\}^s \left\{1 - \frac{\lambda^k/M}{v(\underline{\underline{E}})}\right\}^{n-s} = \binom{n}{s} \; (1/M')^s \; (1-1/M')^{n-s} \qquad (5.1)$$

if $j \in H_1$; while if $j \in H_0$, the probability is zero; and if $j \in H_2$, the probability will have λ^k/M replaced by $v(\underline{\underline{C}}_j\underline{\underline{E}}) \leqslant \lambda^k/M$.

Since the boundary of $\underline{\underline{E}}$ has zero k-dimensional Lebesgue measure, asymptotically, only $M\, v(\underline{\underline{E}})/\lambda^k$ cells contain some of the n points. More precisely, we have

$$N_1/M' \to 1 \text{ and } N_2/N_1 \to 0 \text{ , as } n \to \infty . \qquad (5.2)$$

In the proof of Lemma 3.1 (in Section 6) we have, under Condition D,

$$\mathcal{E}\, S_n = \left(\sum_{j \in H_0} + \sum_{j \in H_1} + \sum_{j \in H_2} \right) \sum_{s=0}^{n} \Pr\,[\text{s points in } \underline{\underline{C}}_j]\; t_s$$

$$= N_1 \sum_{s=0}^{n} \binom{n}{s} (1/M')^s (1-1/M')^{n-s}\, t_s$$

$$+ \sum_{j \in H_2} \sum_{s=0}^{n} \binom{n}{s} (1/M_j)^s (1-1/M_j)^{n-s}\, t_s$$

$$= N_1\, \psi(M',n) + \sum_{j \in H_2} \psi(M_j,n)\,,$$

where, for each $j \in H_2$, $M_j = v(\underline{\underline{E}})/v(\underline{\underline{C}}_j \underline{\underline{E}}) \geqslant M'$. (5.3)

Then the proof of Lemma (3.1) shows that

$$\psi(M,n) \sim A(n/M)^2\, e^{(n/M)}\, (1-M/n)\,, \quad \text{as} \quad n \to \infty. \tag{5.4}$$

Hence, if $M_j \geqslant M'$, $\psi(M_j,n) = \underline{O}[\psi(M',n)]$. (5.5)

Applying (5.2) and (5.5) to (5.3) we have

$$\mathcal{E}\, S_n = N_1\, \psi(M',n) + \sum_{j \in H_2} \psi(M_j,n)$$

$$\sim M'\psi(M',n) + M' \left(\frac{N_1}{M'}\right) \left(\frac{N_2}{N_1}\right) \underline{O}[\psi(M',n)]$$

$$\sim M'\psi(M',n)\,, \quad \text{as} \quad n \to \infty\,; \tag{5.6}$$

so that, by (5.4), we have, if $\delta^*(n) = n/M'$,

$$\mathcal{E}\, S_n \sim An\delta^*(n)\, e^{\delta^*(n)}[1-1/\delta^*(n)]$$

$$= An[\delta(n)\lambda^k/v(\underline{\underline{E}})]e^{[\delta(n)\lambda^k/v(\underline{\underline{E}})]}[1-v(\underline{\underline{E}})/\delta(n)\lambda^k]$$

$$\text{as } n \to \infty\,; \qquad (5.7)$$

i.e., Lemma 3.1 holds true under Condition D, after replacing each $\delta(n)$ by $\delta^*(n) = n/M'$.

Similarly, if we denote by $\psi_1(n/\delta(n),n) = \psi_1(M,n)$ the right-hand side of (6.9)(in Section 6), and we denote by $\psi_2(M,n)$ the expression for $\mathcal{E}(t_{n_i} t_{n_j} | i \neq j)$ in (6.11), and we denote by $\psi_3(M,n)$ the expression for $(\mathcal{E}\, S_n)^2/M^2$ in (6.12); then we have, under Condition D,

$$\begin{aligned}
\operatorname{var} S_n \leqslant\ & N_1\psi_1(M',n) + \underline{o}(N_1)\psi_1(M',n) \\
& + N_1^2\psi_2(M',n) + \underline{o}(N_1^2)\, \underline{O}[\psi_2(M',n)] \\
& - \left\{N_1^2\psi_3(M',n) + \underline{o}(N_1^2)\, \underline{O}[\psi_3(M',n)]\right\} \text{ by (6.9), (6.11) and (6.12)).} \\
\sim\ & M'\psi_1(M',n) \\
& + M'^2\psi_2(M',n) \\
& - M'^2\psi_3(M',n) \qquad \text{(by (5.2))}
\end{aligned}$$

$$= M'\psi_1(M',n) + M'^2[\psi_2(M',n) - \psi_3(M',n)]$$

$$\leqslant A^2 \left\{16n\delta^*(n)^3 e^{3\delta^*(n)} + \frac{n^2}{4\delta^*(n)^2} e^{2\delta^*(n)}\right\} \{1+\underline{O}[1/\delta^*(n)]\} \quad (5.8)$$

(by the paragraph following (6.13)); i.e. Lemma 3.2 holds true under Condition D, when $\delta(n)$ is replaced by $\delta^*(n)$.

By (5.7) and (5.8), the proof of Theorem 1 holds under Condition D if we replace each $\delta(n)$ by $\delta^*(n)$; so that the first part of Theorem 3 is proved.

Now we want to show that the second part of Theorem 3 is true.

~~As in the proof of Lemma 4.1, we have from Lemma 5 of Beardwood, Halton, and Hammersley [1959] that, for~~ $\cancel{\underset{\sim}{P} \in W_\xi}$ ~~,~~

$$\mathcal{E}\, \ell(\underset{\sim}{P}\zeta\xi^{-1/k}\underline{\underline{E}}') \sim \xi^{-1/k}\, \beta\zeta^k v(\underline{\underline{E}}'), \quad \text{as } \zeta \to \infty , \qquad (5.9)/(4.3)^*$$

for any bounded Lebesgue-measureable subset $\underline{\underline{E}}'$ of R^k, with boundary of zero measure.

Now, under Condition D, take $\zeta = \xi^{1/k}$ and $\underline{\underline{E}}' = \underline{\underline{C}}_j\, \underline{\underline{E}} \subseteq \underline{\underline{C}}_j$ (note that $\underline{\underline{C}}_j$ is congruent to $\lambda M^{-1/k}\underline{\underline{C}}$). Then, as $\zeta \to \infty$, $\xi \to \infty$, and from (5.9), we have

$$\mathcal{E}\, \ell(\underset{\sim}{P}\,\underline{\underline{C}}_j\underline{\underline{E}}) \sim \beta\xi^q v(\underline{\underline{C}}_j\underline{\underline{E}}), \quad \text{as } \xi \to \infty . \qquad (5.10)/(4.1)^*$$

As in the proof of Lemma 4.2, we have from Lemma 6 of Beardwood, Halton, and Hammersley [1959] that, for $\underset{\sim}{P} \in W_\xi$,

$$\operatorname{var} \ell(\underset{\sim}{P}\,\zeta\xi^{-1/k}\underline{\underline{E}}') = \underline{o}(1)\xi^{-2/k}\, \zeta^{2k-2/k}, \quad \text{as } \zeta \to \infty, \qquad (5.11)/(4.7)^*$$

uniformly in $\underline{\underline{E}}' \subseteq \{$any set congruent to $\underline{\underline{C}}\}$.

Now, under Condition D, take $\zeta\xi^{-1/k} = \lambda/M^{-1/k}$ and $\underline{\underline{E}}' = \lambda^{-1}M^{1/k}\,\underline{\underline{C}}_j\,\underline{\underline{E}} \subseteq \lambda^{-1}M^{1/k}\,\underline{\underline{C}}_j$ (note that $\lambda^{-1}M^{1/k}\,\underline{\underline{C}}_j$ is congruent to

to $\underline{\underline{C}}$). Then $\zeta^k = \lambda\xi/M$ and thus, as $\zeta \to \infty$, $\xi/M \to \infty$, and from (5.11), we have

$$\text{var}\ \ell(\underset{\sim}{P}\,\underline{\underline{C}}_j\,\underline{\underline{E}}) = \underline{o}(1)\xi^{-2/k}(\xi/M)^{2-2/k^2}\ , \quad \text{as} \quad \xi/M \to \infty, \qquad (5.12)/(4.4)^*$$

uniformly in j.

Under Condition D, $U_{\xi,M}$ is defined as follows:

$$U_{\xi,M} = \sum_{j=1}^{M} \ell(\underset{\sim}{P}\,\underline{\underline{C}}_j\,\underline{\underline{E}}) \quad .$$

Then we have, as in Section 4, from (5.10) and (5.12), that

$$\mathcal{E}\, U_{\xi,M} = \sum_{j=1}^{M} \mathcal{E}\ \ell(\underset{\sim}{P}\,\underline{\underline{C}}_j\,\underline{\underline{E}}) \sim \beta\xi^q \sum_{j=1}^{M} v(\underline{\underline{C}}_j\underline{\underline{E}})$$

$$= \beta\xi^q\ v(\underline{\underline{E}})\ , \quad \text{as}\ \xi \to \infty\ ; \qquad (5.13)/(4.8)^*$$

and, by the uniformity of (5.12) over $j = 1,2,\ldots,M$,

$$\text{var}\ \ell(\underset{\sim}{P}\,\underline{\underline{C}}_j\,\underline{\underline{E}}) = \sum_{j=1}^{M} \text{var}\ (\underset{\sim}{P}\,\underline{\underline{C}}_j\,\underline{\underline{E}})$$

$$= M\ \underline{o}(1)\ \xi^{2q-2/k^2}/M^{2-2/k^2}$$

$$= \underline{o}(1)\ \xi^{2q-2/k^2}/M^{1-2/k^2}, \quad \text{as} \quad \xi/M \to \infty\ . \qquad (5.14)/(4.9)^*$$

As in the proof of Theorem 2, we have from (5.13) that, for sufficiently large ξ and for any $\varepsilon > 0$,

$$|\mathcal{E}\, U_{\xi,M} - \beta\xi^q\, v(\underline{\underline{E}})| \leqslant \tfrac{1}{2}\, \varepsilon\beta\xi^q\, v(\underline{\underline{E}})\,;$$

and then, by Chebyshev's inequality,

$$\Pr\{|U_{\xi,M} - \beta\xi^q\, v(\underline{\underline{E}})| \leqslant \varepsilon\, \beta\xi^q\, v(\underline{\underline{E}})\}$$

$$\geqslant \Pr\{|U_{\xi,M} - \mathcal{E}\, U_{\xi,M}| \leqslant \tfrac{1}{2}\, \varepsilon\beta\xi^q\, v(\underline{\underline{E}})\}$$

$$\geqslant 1 - \underline{o}(1)[\xi^{2q-2/k^2}/M^{1-2/k^2}]/(\tfrac{1}{2}\,\varepsilon\beta\xi^q\, v(\underline{\underline{E}}))^2 \qquad \text{(by 5.14))}$$

$$= 1 - \frac{\underline{\underline{o}}(1)}{\varepsilon^2}\left[\frac{1}{\xi^{2/k^2}M^{1-2/k^2}}\right], \quad \text{as} \quad \xi/M \to \infty \quad \text{and} \quad \xi \to \infty. \qquad (5.15)/(4.12)^*$$

Also, as in the proof of Theorem 2, if $N(\underset{\sim}{P}\,\underline{\underline{E}}) = n_\xi$ then

$$\Pr\{|n_\xi - \xi v(\underline{\underline{E}})| \leqslant \varepsilon\xi v(\underline{\underline{E}})\} \geqslant 1 - \frac{1}{\varepsilon^2 \xi v(\underline{\underline{E}})}, \qquad (5.16)/(4.13)^*$$

since $\mathcal{E}\, n_\xi = \text{var}\; n_\xi = \xi v(\underline{\underline{E}})$.

From (5.15) and (5.16), for sufficiently large ξ/M and ξ, we have

$$\Pr\left[\beta\,\frac{\xi^q v(\underline{\underline{E}})}{\xi^q v(E)^q}\,\frac{(1-\varepsilon)}{(1+\varepsilon)^q} \leqslant \frac{U_{\xi,M}}{n_\xi{}^q} \leqslant \frac{\beta\xi^q v(\underline{\underline{E}})}{\xi^q v(\underline{\underline{E}})^q}\,\frac{(1+\varepsilon)}{(1-\varepsilon)^q}\right]$$

$$= \Pr\left[\beta v(\underline{\underline{E}})^{1/k}\,\frac{(1-\varepsilon)}{(1+\varepsilon)^q} \leqslant \frac{U_{\xi,M}}{n_\xi{}^q} \leqslant \beta v(\underline{\underline{E}})^{1/k}\,\frac{(1+\varepsilon)}{(1-\varepsilon)^q}\right]$$

$$\geqslant 1 - \left[\frac{1}{\xi^{2/k^2}M^{1-2/k^2}} + \frac{1}{\xi v(\underline{\underline{E}})}\right]\varepsilon^{-2} \qquad (5.17)/(4.14)^*$$

Under Condition D, $V_{n,M}$ is defined as follows:

$$V_{n,M} = \sum_{j=1}^{M} \ell(\underset{\sim}{P}^n C_j \underline{\underline{E}})$$

where n is a positive integer value and $\underset{\sim}{P} \in u(\underline{\underline{C}})$.

The remaining part of the proof of Theorem 2 holds true here if we replace each occurrence of β by $\beta v(\underline{\underline{E}})^{1/k}$ (as (5.17) above suggests), and if we replace each occurrence of the condition "sufficiently large ξ/M" by "sufficiently large ξ/M and ξ".

The only additional point to observe is that, since we take $\xi = \frac{(1+\varepsilon)^m}{(1-\varepsilon)^{m+1}}$ just before the definition of the sets J_m in the proof of Theorem 2, the condition "sufficiently large ξ" is satisfied for sufficiently large m.

6. Proofs of the Basic Lemmas

First we want to prove a remark which will be used in the proofs of Lemmas 3.1 and 3.2.

Remark 1 (A)

If $x, q \geqslant 0$ are fixed and $M \to \infty$ in such a way that $M/n \to 0$, then $0 \leqslant e^{xn/M} - (1+x/M)^{n-q} \leqslant e^{xn/M}\,[\frac{xq}{M} + \frac{x^2 n}{2M^2}]$.

Proof: $e^{xn/M} = 1 + \frac{xn}{M} + \frac{1}{2}\frac{x^2n^2}{M^2} + \ldots + \frac{1}{m!}\frac{x^m n^m}{M^m} + \ldots$, $(1+\frac{x}{M})^{n-q} = 1 + \frac{x(n-q)}{M}$ $+ \frac{1}{2}\frac{x^2(n-q)_2}{M^2} + \ldots + \frac{1}{m!}\frac{x^m(n-q)_m}{M^m} + \ldots$. So $e^{xn/M} - (1+x/M)^{n-q}$

$$= \sum_{m=1}^{\infty} \frac{1}{m!}\frac{x^m}{M^m}[n^m - (n-q)_m].$$

Now $n^m \geqslant (n-q)_m \geqslant n^m - n^{m-1}\sum_{i=0}^{m-1}(i+q)$

$= n^m - n^{m-1} m[\frac{1}{2}(m-1)+q)]$.

The first inequality holds because each factor of $(n-q)_m$ is less than or equal to n, and there are m factors on each side; the second is seen by induction: $n-q = n^1 - n^0 q$ $(m=1)$. If true for $m = h-1$, then $(n-q)_h = (n-q)_{h-1}\,(n-q-h+1) \geqslant \{n^{h-1} - n^{h-2}(h-1)[\frac{1}{2}(h-2)+q]\}\,(n-q-h+1)$

$\geqslant n^h - n^{h-1}(h-1)[\frac{1}{2}(h-2)+q] - (q+h-1)n^{h-1} = n^h - n^{h-1}\,h[\frac{1}{2}(h-1)+q]$ and induction is complete. Thus $0 \leqslant n^m - (n-q)_m \leqslant n^{m-1}\,m[\frac{1}{2}(m-1)+q]$, whence

$$0 \leqslant e^{xn/M} - (1+x/M)^{n-q} \leqslant \sum_{m=1}^{\infty} \frac{1}{m!}\frac{x^m}{M^m} n^{m-1} m[\frac{1}{2}(m-1)+q]$$

$$\leqslant \sum_{m=1}^{\infty} \frac{1}{(m-1)!}\frac{x^{m-1}n^{m-1}}{M^{m-1}}\frac{xq}{M}$$

$$+ \sum_{m=2}^{\infty} \frac{1}{(m-2)!}\frac{x^{m-2}n^{m-2}}{M^{m-2}}\frac{x^2 n}{2M^2}$$

$$= e^{xn/M}\,[\frac{xq}{M} + \frac{x^2 n}{2M^2}].$$

QED

Remark 1 (B)

If $x < 0$, $q \geqslant 0$ are fixed and $M \to \infty$ in such a way that $M/n \to 0$, then

$$\frac{x^2 n}{4M^2}(u-v) + \frac{xq}{2M}(u+v) \leqslant e^{xn/M} - (1+x/M)^{n-q} \leqslant \frac{x^2 n}{4M^2}(u+v) + \frac{xq}{2M}(u-v),$$

where $u = e^{xn/M}$, $v = 1/u = e^{-xn/M}$.

Proof: As in Remark 1(A), we see that

$$\Delta(x) = e^{xn/M} - (1+x/M)^{n-q} = \sum_{m=1}^{\infty} \frac{1}{m!}\frac{x^m}{M^m}[n^m - (n-q)_m],$$

and that

$$0 \leqslant n^m - (n-q)_m \leqslant n^{m-1} m [\tfrac{1}{2}(m-1)+q].$$

The series for $\Delta(x)$ now alternates. By collecting positive terms only, we obtain that

$$\Delta(x) \leqslant \sum_{\substack{m=2 \\ (m\ \text{even})}}^{\infty} \frac{1}{(m-1)!}\left(\frac{xn}{M}\right)^m \frac{1}{n}\left[\frac{1}{2}(m-1)+q\right]$$

$$= \frac{x^2 n}{2M^2} \sum_{\substack{m=2 \\ (m\ \text{even})}}^{\infty} \frac{1}{(m-2)!}\left(\frac{xn}{M}\right)^{m-2} + \frac{xq}{M} \sum_{\substack{m=2 \\ (m\ \text{even})}}^{\infty} \frac{1}{(m-1)!}\left(\frac{xn}{M}\right)^{m-1}.$$

Now $\displaystyle\sum_{\substack{m=0 \\ (m\ \text{even})}}^{\infty} \frac{1}{m!}\left(\frac{xn}{M}\right)^m = \frac{1}{2}(u+v)$ and $\displaystyle\sum_{\substack{m=1 \\ (m\ \text{odd})}}^{\infty} \frac{1}{m!}\left(\frac{xn}{M}\right)^m = \frac{1}{2}(u-v)$.

Thus $\Delta(x) \leqslant \frac{x^2 n}{4M^2}(u+v) + \frac{xq}{2M}(u-v)$.

Similarly, by collecting negative terms, we get

$$\Delta(x) \geqslant \sum_{\substack{m=1 \\ (m \text{ odd})}}^{\infty} \frac{1}{(m-1)!}\left(\frac{xn}{M}\right)^m \frac{1}{n}\left[\tfrac{1}{2}(m-1)+q\right]$$

$$= \frac{x^2 n}{2M^2} \sum_{\substack{m=3 \\ (m \text{ odd})}}^{\infty} \frac{1}{(m-2)!}\left(\frac{xn}{M}\right)^{m-2} + \frac{xq}{M} \sum_{\substack{m=1 \\ (m \text{ odd})}}^{\infty} \frac{1}{(m-1)!}\left(\frac{xn}{M}\right)^{m-1}$$

$$= \frac{x^2 n}{4M^2}(u-v) + \frac{xq}{2M}(u+v).$$

QED

Lemma 3.1: Under Condition C, if Algorithm A is applied to a k-TSP instance $\underset{\sim}{p}^n$ then there is a constant A, such that

$$\mathcal{E}\, S_n \sim A n\delta(n) e^{\delta(n)}[1-1/\delta(n)], \quad \text{as} \quad n \to \infty .$$

Proof: Let t_s denote the time needed to compute a shortest tour through s points. From Bellman [1962] and Held and Karp [1962], we know there is a constant A (roughly, half the time needed for one addition), such that

$$\begin{aligned} t_s &= 2A(s-1)\,[2^{s-3}(s-2)+1] \\ &= A[2^{s-2}(s)_2-2^{s-1}s+2s+2^{s-1}-2] = t^*(s), \text{ for } s \geqslant 1, \text{ and } t_0 = 0. \quad (6.1) \end{aligned}$$

If k,p, and $q \geqslant 0$ are fixed, and $n \to \infty$, we see that

$$\begin{aligned} f(n;k,p,q) &= \sum_{s=0}^{n} \binom{n}{s} (1/M)^s (1-k/M)^{n-s} p^{s-q} (s)_q \\ &= (n)_q (1/M)^q \sum_{s=q}^{n} \binom{n-q}{s-q} (p/M)^{s-q} (1-k/M)^{n-s} \\ &= (n)_q (1/M)^q\, [1+(p-k)/M]^{n-q} . \end{aligned} \qquad (6.2)$$

By Remark 1 (A) we have that, if $p \geqslant k$, $q \geqslant 0$, then

$$\begin{aligned} &\delta(n)^q e^{(p-k)\delta(n)} \left(1- \frac{(p-k)q}{M} - \frac{(p-k)^2 n}{2M^2}\right)(1+\underline{0}(1/n)) \\ &\leqslant f(n;k,p,q) \\ &\leqslant \delta(n)^q\, e^{(p-k)\delta(n)} . \end{aligned} \qquad (6.3)$$

So $f(n;k,p,q) \sim \delta(n)^q e^{(p-k)\delta(n)}[1+\underline{O}(\delta(n)^2/n)]$. (6.4)

Now, if n_j denotes the number of points in cell $\underline{\underline{C}}_j$, we have

$$\& S_n = \& \sum_{j=1}^{M} t_{n_j} = M \& t_{n_j} ; \tag{6.5}$$

and, since n_j has (binomial) probability $\binom{n}{s} (1/M)^s(1-1/M)^{n-s}$ of taking the value s,

$$\begin{aligned} \& S_n &= M \sum_{s=0}^{n} \binom{n}{s} (1/M)^s(1-1/M)^{n-s} A[2^{s-2}(s)_2 \\ &\quad -2^{s-1}s + 2s + 2^{s-1}-2] + M(1-1/M)^n(3/2)A \\ &= AM[f(n;1,2,2) - f(n;1,2,1) + 2f(n;1,1,1) \\ &\quad + (1/2) f(n;1,2,0) - 2f(n;1,1,0) + (3/2)(1-1/M)^n] \end{aligned} \tag{6.6}$$

[Note that we use the general formula $t^*(s)$ in (6.1) for t_s even when $s = 0$. This incorrectly yields $t^*(0) = -(3/2)A$; forcing us to make the corresponding correction, $+M(1-1/M)^n(3/2)A$, above.] Thus

$$\begin{aligned} \& S_n &\sim A \frac{n}{\delta(n)} [\delta(n)^2 e^{\delta(n)} - \delta(n)e^{\delta(n)} + 2\delta(n) \\ &\quad + (1/2)e^{\delta(n)} - 2 + (3/2)e^{-\delta(n)}][1+\underline{O}(\delta(n)^2/n)] \quad \text{(by Remark 1 (A) \& (B))} \\ &\sim An\delta(n)e^{\delta(n)}[1-1/\delta(n) + 1/2\delta(n)^2 + \underline{O}(\delta(n)^2/n)], \end{aligned}$$

as $n \to \infty$, and the lemma follows , since

$[\delta(n)^2/n]/[1/\delta(n)] \to 0$, as $n \to \infty$.

QED

Now we want to prove a second remark which will be used in the proof of Lemma 3.2.

Remark 2

$$0 \leqslant (1+1/M)^{2n-k} - (1+2/M)^{n-k} \leqslant e^{2n/M}\,\frac{n}{M^2}\left(1+\frac{kM}{n}\right) \quad \text{if} \quad k \geqslant 0 \quad \text{is fixed and}$$

$M \to \infty$ in such a way that $M/n \to 0$.

Proof: The first inequality above is true since clearly

$$(1+1/M)^{2n-k} - (1+2/M)^{n-k}$$

$$= 1 + \frac{2n-k}{M} + \frac{(2n-k)(2n-k-1)}{2M^2} + \ldots + \frac{(2n-k)\ldots(2n-k-j+1)}{j!M^j} + \ldots$$

$$- 1 - \frac{2n-2k}{M} - \frac{(2n-2k)(2n-2k-2)}{2M^2} - \ldots - \frac{(2n-2k)\ldots(2n-2k-2j+2)}{j!M^j} - \ldots$$

$$\geqslant 0 .$$

Now, the j-th term in the difference above is:

$$T_j = \frac{(2n-k)\ldots(2n-k-j+1) - (2n-2k)\ldots(2n-2k-2j+2)}{j!M^j}$$

$$= \frac{1}{j!M^j}\left\{\sum_{i=k}^{k+j-1}(2n)^{j-1}i - 3\sum_{\text{pairs}}(2n)^{j-2}\,ii' + 7\sum_{\text{triplets}}(2n)^{j-3}\,ii'i'' - \ldots\right\}.$$

By induction on j, we want to show that

$$T_j \leqslant \frac{1}{j!M^j}(2n)^{j-1}\sum_{i=k}^{k+j-1} i . \tag{6.7}$$

For $j=1$, $T_1 = \frac{k}{M} = \frac{1}{M}\;(2n)^0 \sum_{i=k}^{k} i = \frac{k}{M}$. Assume the inequality above is true for $j = h-1$.

Then $T_h h! M^h = [(2n-k)\ldots(2n-k-h+2)](2n-k-h+1)$

$$- [(2n-2k)\ldots(2n-2k-2h+4)]\,(2n-2k-2h+2)$$

$$= T_{h-1}(h-1)!M^{h-1}(2n-k-h+1) + (2n-2k)\ldots(2n-2k-2h+4)(k+h-1)\,.$$

$$\leqslant (2n)^{h-2}\left(\sum_{i=k}^{k+h-2} i\right)\,(2n-k-h+1) + (2n-2k)\ldots(2n-2k-2h+4)(k+h-1)$$

$$\leqslant (2n)^{h-1}\sum_{i=k}^{k+h-2} i + (2n)^{h-1}(k+h-1)$$

$$= (2n)^{h-1}\sum_{i=k}^{k+h-1} i \quad ,$$

and induction is complete.

~~From (6.7) we have that~~

$$T_j \leqslant \frac{1}{j!M^j}\,(2n)^{j-1}[\tfrac{1}{2}\,(k+j-1)(k+j) - \tfrac{1}{2}\,(k-1)k]$$

$$= \frac{1}{j!M^j}\,(2n)^{j-1}\,\frac{1}{2}\,[2jk+j(j-1)]\;;$$

so that

$$\sum_{j=1}^{\infty} T_j \leqslant \frac{k}{M}\sum_{j=1}^{\infty}\frac{M^{-(j-1)}}{(j-1)!}\,(2n)^{j-1} + \frac{2n}{2M^2}\sum_{j=2}^{\infty}\frac{M^{-(j-2)}}{(j-2)!}\,(2n)^{j-2}$$

$$= \frac{k}{M}\,e^{2n/M} + \frac{n}{M^2}\,e^{2n/M}\,.$$

QED

Lemma 3.2: Under the same conditions as in Lemma 3.1, we have that

$$\operatorname{var} S_n \leqslant A^2 e^{2\delta(n)} \left\{ 16n\delta(n)^3 e^{\delta(n)} + \frac{n^2}{4\delta(n)^2} \right\} [1+\underline{O}(1/\delta(n))]$$

as $n \to \infty$, where A is the constant in Lemma 3.1.

Proof: $$\mathcal{E} S_n^{\;2} = \mathcal{E} \sum_{i=1}^{M} \sum_{j=1}^{M} t_{n_i} t_{n_j} = \mathcal{E} \left\{ \sum_{i=1}^{M} t_{n_i}^{\;2} + 2 \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} t_{n_i} t_{n_j} \right\}$$

$$= M\mathcal{E}t_{n_i}^{\;2} + M(M-1)\, \mathcal{E}\, (t_{n_i} t_{n_j} | i \neq j) \tag{6.8}$$

Since $[(s)_2]^2 = (s)_4 + 4(s)_3 + 2(s)_2$, $s(s)_2 = (s)_3 + 2(s)_2$, and $s^2 = (s)_2 + s$, using (6.1) and (6.3), we see that

$$\begin{aligned} \mathcal{E}\, t_{n_i}^{\;2} = \sum_{s=0}^{n} \binom{n}{s} (1/M)^s (1-1/M)^{n-s} A^2 \{ & [4^{s-2}(s)_4 + 4^{s-1}(s)_3 \\ & + 2.4^{s-2}(s)_2] - [4^{s-1}(s)_3 + 2.4^{s-1}(s)_2] + [2^s(s)_3 \\ & + 2^{s+1}(s)_2] + 4^{s-1}(s)_2 - 2^s(s)_2 + [4^{s-1}(s)_2 + 4^{s-1}s] \\ & - [2^{s+1}(s)_2 + 2^{s+1}s] - 2.4^{s-1}s + 2^{s+1}s + [4(s)_2 + 4s] \\ & + 2^{s+1}s - 8s + 4^{s-1} - 2^{s+1} + 4\} - (9/4)A^2(1-1/M)^n . \end{aligned}$$

[The last terms above is a correction similar to that in (6.6)]. So

$$\begin{aligned} \mathcal{E}\, t_{n_i}^{\;2} = A^2 [& 16f(n;1,4,4) + 8f(n;1,2,3) + 2f(n;1,4,2) \\ & -4f(n;1,2,2) + 4f(n;1,1,2) - f(n;1,4,1) \\ & +4f(n;1,2,1) - 4f(n;1,1,1) + (1/4)f(n;1,4,0) \\ & -2f(n;1,2,0) + 4f(n;1,1,0) - (9/4)(1-1/M)^n] \end{aligned}$$

$$\leqslant A^2\{16\delta(n)^4 e^{3\delta(n)} + 8\delta(n)^3 e^{\delta(n)}$$
$$+ 2\delta(n)^2 e^{3\delta(n)} + 4\delta(n)^2 + 4\delta(n) e^{\delta(n)}$$
$$+ (1/4) e^{3\delta(n)+4}\} . \tag{6.9}$$

Just as in (6.2), we note that

$$\sum_{s=0}^{v} \binom{v}{s} p^{s-q}(s)_q(v-s)_r = (v)_{q+r} \sum_{s=q}^{v-r} \binom{v-q-r}{s-q} p^{s-q}$$
$$= (v)_{q+r}(p+1)^{v-q-r} = \sum_{s=0}^{v} \binom{v}{s} p^{v-s-r} (s)_q(v-s)_r ; \tag{6.10}$$

whence, by putting $v = s+u$, we get that

$$\mathcal{E}(t_{n_i}t_{n_j}|i\neq j) = \sum_{s=0}^{n}\sum_{u=0}^{n-s} \binom{n}{s+u}\binom{s+u}{s} (1/M)^{s+u}(1-2/M)^{n-s-u} t_s t_u$$
$$= A^2 \sum_{v=0}^{n} \binom{n}{v} (1/M)^v(1-2/M)^{n-v} \left\{\sum_{s=0}^{v} \binom{v}{s} [2^{v-4}(s)_2(v-s)_2\right.$$
$$-2^{v-3}(s)_2(v-s) + 2^{s-1}(s)_2(v-s) + 2^{v-3}(s)_2 - 2^{s-1}(s)_2$$
$$-2^{v-3}s(v-s)_2 + 2^{v-2}s(v-s) - 2^s s(v-s) - 2^{v-2}s$$
$$+2^s s + 2^{v-s-1} s(v-s)_2 - 2^{v-s}s(v-s) + 4s(v-s)$$
$$+2^{v-s}s - 4s + 2^{v-3}(v-s)_2 -2^{v-2}(v-s) + 2^s(v-s)$$
$$+2^{v-2} - 2^s - 2^{v-s-1}(v-s)_2 + 2^{v-s}(v-s) -4(v-s)$$
$$\left.-2^{v-s}+4] + 3(2^{v-2}(v)_2 -2^{v-1}v + 2v + 2^{v-1}-2)\right\}$$
$$+(9/4)A^2(1-2/M)^n$$

[The terms $3(2^{v-2}(v)_2-2^{v-1}v+2v+2^{v-1}+2)$ and $(9/4)A^2(1-2/M)^n$ at the end, above, are corrections for the use of $t^*(0)$ instead of t_0, similar to those in (6.6) and (6.9) above.]

$$= A^2 \sum_{v=0}^{n} \binom{n}{v} (1/M)^v(1-2/M)^{n-v}[2^{v-4}(v)_4\, 2^{v-4}$$

$$-2^{v-3}(v)_3 2^{v-3} + 2(v)_3 3^{v-3} + 2^{v-3}(v)_2 2^{v-2}$$

$$-2(v)_2 3^{v-2} - 2^{v-3}(v)_3 2^{v-3} + 2^{v-2}(v)_2 2^{v-2}$$

$$-2(v)_2 3^{v-2} - 2^{v-2}v2^{v-1} + 2v3^{v-1} + 2(v)_3 3^{v-3}$$

$$-2(v)_2 3^{v-2} + 4(v)_2 2^{v-2} + v3^{v-1} -4v2^{v-1}$$

$$+2^{v-3}(v)_2 2^{v-2} - 2^{v-2}v2^{v-1} + v3^{v-1} +2^{v-2}2^v$$

$$-3^v - 2(v)_2 3^{v-2} + 2v3^{v-1} - 4v2^{v-1} - 3^v + 4.2^v$$

$$+3(2^{v-2}(v)_2-2^{v-1}v+2v+2^{v-1}-2)] + (9/4)A^2(1-2/M)^n$$

$$= A^2[f(n;2,4,4) - 2f(n;2,4,3) + 2f(n;2,4,2)$$

$$-f(n;2,4,1) + (1/4)\, f(n;2,4,0) + 4f(n;2,3,3)$$

$$-8f(n;2,3,2) + 6f(n;2,3,1) - 2f(n;2,3,0)$$

$$+7f(n;2,2,2) - 11f(n;2,2,1) + (11/2)f(n;2,2,0)$$

$$+6f(n;2,1,1) - 6f(n;2,1,0) + (9/4)(1-2/M)^n]$$

Then,

$$M^2 \mathcal{E}(t_{n_i} t_{n_j} | i \neq j) = A^2 M^2 \{((n)_4/M^4)(1+2/M)^{n-4} - 2((n)_3/M^3)(1+2/M)^{n-3}$$

$$+ 2((n)_2/M^2)(1+2/M)^{n-2} - (n/M)(1+2/M)^{n-1} + (1/4)(1+2/M)^n$$

$$+ 4((n)_3/M^3)(1+1/M)^{n-3} - 8((n)_2/M^2)(1+1/M)^{n-2}$$

$$+ 6(n/M)(1+1/M)^{n-1} - 2(1+1/M)^n + 7(n)_2/M^2 - 11(n/M) + 11/2$$

$$+ 6(n/M)(1-1/M)^{n-1} - 6(1-1/M)^n + (9/4)(1-2/M)^n\} , \tag{6.11}$$

On the other hand, from (6.6) we have that

$$\mathcal{E} S_n = AM[((n)_2/M^2)(1+1/M)^{n-2} - (n/M)(1+1/M)^{n-1}$$

$$+ 2n/M + (1/2)(1+1/M)^n - 2 + (3/2)(1-1/M)^n].$$

So
$$(\mathcal{E} S_n)^2 = A^2 M^2 \{[(n)_4 + 4(n)_3 + 2(n)_2](1/M)^4 (1+1/M)^{2n-4}$$

$$-2(n)_2\, n(1/M)^3(1+1/M)^{2n-3}$$

$$+ [(n)_2 + n^2](1/M)^2(1+1/M)^{2n-2}$$

$$- (n/M)(1+1/M)^{2n-1} + (1/4)(1+1/M)^{2n}$$

$$+ 4(\tfrac{n}{M} - 1 + (3/4)(1-1/M)^n)[((n)_2/M^2)(1+1/M)^{n-2} - (n/M)(1+1/M)^{n-1}$$

$$+ (1/2)(1+1/M)^n] + 4(\tfrac{n}{M} - 1)^2$$

$$+ 6(\tfrac{n}{M} - 1)(1-1/M)^n + (9/4)(1-1/M)^{2n}\} . \tag{6.12}$$

Observing that, in (6.12), $(n)_2 n = (n)_3 + 2(n)_2$ and $n^2 = (n)_2 + n$, from (6.11) and (6.12), using Remarks 1 (A) and 2 we have that

$$
\begin{aligned}
& M^2 \mathcal{E}(t_{n_i} t_{n_j} | i \neq j) - (\mathcal{E}\, S_n)^2 \\
= {} & A^2M^2 \Big\{ ((n)_4/M^4)[(1+2/M)^{n-4} - (1+1/M)^{2n-4}] \\
& + ((n)_3/M^3)[-2(1+2/M)^{n-3} - (4/M)(1+1/M)^{2n-4} + 2(1+1/M)^{2n-3}] \\
& + ((n)_2/M^2)[2(1+2/M)^{n-2} - (2/M^2)(1+1/M)^{2n-4} \\
& \qquad + (4/M)(1+1/M)^{2n-3} - 2(1+1/M)^{2n-2}] \\
& + (n/M)[-(1+2/M)^{n-1} - (1/M)(1+1/M)^{2n-2} + (1+1/M)^{2n-1}] \\
& + (1/4)(1+2/M)^{n} - (1/4)(1+1/M)^{2n} \\
& - 4((n)_3/M^4)(1+1/M)^{n-3} - 4((n)_2/M^3)(1+1/M)^{n-2} \\
& + 2(n/M^2)(1+1/M)^{n-1} - 4n/M^2 + 3(n)_2/M^2 - 3n/M + 3/2 + 6(n/M^2)(1-1/M)^n \\
& + (9/4)(1-2/M)^n - 3((n)_2/M^2)(1-1/M)^n(1+1/M)^{n-2} \\
& + 3(n/M)(1-1/M)^n(1+1/M)^{n-1} - (3/2)(1-1/M)^n(1+1/M)^n \\
& - (9/4)(1-1/M)^{2n} \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\leqslant {} & A^2M^2 \Big\{ \delta(n)^3 \, 2 \, \frac{\delta(n)}{M} \, e^{2\delta(n)} [1 + \frac{3}{\delta(n)}] + \delta(n)^2 \, \frac{4}{M} \, e^{2\delta(n)} \\
& + \delta(n) \, \frac{\delta(n)}{M} \, e^{2\delta(n)} [1 + \frac{1}{\delta(n)}] + \frac{1}{4} \, e^{2\delta(n)}
\end{aligned}
$$

[continued....]

$$+ 2\,\frac{\delta(n)}{M}\,e^{\delta(n)} + 3\delta(n)^2 + 3/2 + (6\delta(n)/M)\ \ [e^{-\delta(n)}(1+1/2M)$$

$$+ e^{\delta(n)}(\delta(n)+2)/4M] + (9/4)\ [e^{-2\delta(n)}+\delta(n)e^{2\delta(n)}/M]$$

$$+ 3\delta(n)[1+\delta(n)e^{2\delta(n)}/4M] \Big\}$$

$$= 2A^2 n\delta(n)^3\, e^{2\delta(n)} \Bigg\{ 1 + \frac{3}{\delta(n)} + \frac{5}{2\delta(n)^2} + \frac{1}{2\delta(n)^3} + \frac{n}{8\delta(n)^5}$$

$$+ \frac{1}{\delta(n)^3}\, e^{-\delta(n)} + \frac{ne^{-2\delta(n)}}{2} \Bigg[\frac{3}{\delta(n)^3} + \frac{3}{2\delta(n)^5}$$

$$+ \frac{6}{n\delta(n)^3}\ (e^{-\delta(n)}(1+1/2M)+e^{\delta(n)}(\delta(n)+2)/4M)$$

$$+ \frac{9}{4\delta(n)^5}\ (e^{-2\delta(n)}+\delta(n)e^{2\delta(n)}/M)$$

$$+ \frac{3}{\delta(n)^4}\ (1+\delta(n)e^{2\delta(n)}/4M) \Bigg] \Bigg\} \tag{6.13}$$

From (6.8), (6.9) and (6.13) we have that

$$\text{var}\ S_n = \mathcal{E}\, S_n{}^2 - (\mathcal{E}\, S_n)^2$$

$$= M\mathcal{E}t_{n_i}{}^2 + M(M-1)\ \mathcal{E}\ (t_{n_i} t_{n_j} | i \neq j) - (\mathcal{E}\, S_n)^2$$

$$\leqslant A^2\ \Big\{ 16n\delta(n)^3\, e^{3\delta(n)} + 2n\delta(n)\, e^{3\delta(n)} + (n/4\delta(n))\, e^{3\delta(n)}$$

$$+ 8n\delta(n)^2\, e^{\delta(n)} + 4n\, e^{\delta(n)} + 4n\delta(n) + 4n/\delta(n)$$

$$+ 2n\delta(n)^3\, e^{2\delta(n)} + 6n\delta(n)^2\, e^{2\delta(n)} + 5n\delta(n)\, e^{2\delta(n)}$$

$$+ n\, e^{2\delta(n)} + \frac{n^2}{4\delta(n)^2}\, e^{2\delta(n)} + 2\, n\, e^{\delta(n)} + n^2 \Bigg[3 + \frac{3}{2\delta(n)^2}$$

[continued...]

$$+ \frac{6}{n} (e^{-\delta(n)}(1+1/2M) + e^{\delta(n)}(\delta(n)+2)/4M)$$

$$+ \frac{9}{4\delta(n)^2} (e^{-2\delta(n)} + \delta(n)e^{2\delta(n)}/M) + \frac{3}{\delta(n)}(1+\delta(n)e^{2\delta(n)}/4M)\Big]\Big\}$$

$$= A^2 \Big\{16n\delta(n)^3\, e^{3\delta(n)} + \frac{n^2}{4\delta(n)^2}\, e^{2\delta(n)} + 3n^2\Big[1+\delta(n)^2e^{\delta(n)}/12n+3e^{2\delta(n)}/4n$$

$$+\delta(n)e^{2\delta(n)}/4n\Big]\Big\}\,(1+\underline{0}(1/\delta(n))); \qquad (6.14)$$

but each of the terms in $3n^2[1+\delta(n)^2e^{\delta(n)}/12n + 3e^{2\delta(n)}/4n + \delta(n)e^{2\delta(n)}/4n]$ is $\underline{o}(\delta(n)^{-3}e^{2\delta(n)})$; since $\delta(n)^3\, e^{-2\delta(n)} \to 0$ ($e^{\delta(n)}$ increases faster than any power of $\delta(n)$), $\delta(n)^5\, e^{-\delta(n)}/n \to 0$ (similarly, $\delta(n)^3/n \to 0$) and $\delta(n)^4/n \to 0$ (similarly). The lemma follows.

QED

7. Concluding Remarks

We presented a very fast algorithm for the Euclidean traveling salesman problem which is optimal with probability one.

A similar algorithm was described by Karp [1977], but his paper is incomplete in the following sense.

In Karp [1977] the points of a 2-TSP instance are assumed to be distributed in a region A according to a two dimensional Poisson distribution with density n. As is noted in Karp [1977], it is then known that the expected number of points in A is $n\,v(A)$, where $v(A)$ denotes the area of A.

But the algorithm in Karp [1977], when applied to a region A with $v(A) = 1$, are analyzed as if the observed number of points in a 2-TSP instance were n, rather than considering n as the expected number of points. We conjecture that one possible way to rescue this part of the analysis in Karp [1977] is to prove that the observed number of points in A is asymptotic to the expected number of points in A, with probability one.

Furthermore, we note that Karp [1977] quotes a result (as Theorem 5 in Section 4 of his paper) from Beardwood, Halton, and Hammersley [1959] as if it held under the assumption of the Poisson distribution of the points with density n. But, actually that theorem is proved in Beardwood, Halton, and Hammersley [1959] (as Lemma 7) only for the uniform distribution of n points. The length of the proof of Theorem 2 in our paper indicates that the connection between the two is far from trivial; and, in fact, we do not believe that the results hold for the Poisson distribution more strongly than in probability.

Moreover, our Algorithm A is a significant improvement upon Karp's algorithm in terms of its simplicity, its dimensional generality and its running time (Theorem 1 and Corollary TSP). Karp has an upperbound for the <u>expected</u> running time of his algorithm which cannot be less than $\underline{O}(\ n\ (\log\ n)^2\ )$.

In constructing the proof of Theorem 2, we had occasion to review Beardwood, Halton, and Hammersley [1959]; and, in particular, to check the proofs of the lemmas there.

In the proof of Lemma 7 of Beardwood, Halton, and Hammersley [1959], equation (7.15) is <u>not</u> valid; because the n_m depend on the corresponding intervals J_m, and these depend on the value of ε.

However, the argument via (7.16)-(7.19) <u>is</u> valid, except that, in each of (7.18) and (7.19), a factor of α should be inserted before $(5\varepsilon n)^q$, coming from Lemma 4 of that paper.

(7.20) now follows from (7.14) in the modified form

$$\frac{\beta-\varepsilon}{(1+\varepsilon)^q}\ v(\underline{\underline{E}})^{1/k} \leqslant \liminf_{n\to\infty}\ n^{-q}\ \ell(\underset{\sim}{P}^n) + \alpha(5\varepsilon)^q[v(\underline{\underline{\bar{E}}})+\varepsilon]^{1-q}, \tag{7.20*}$$

holding with probability one.

Similarly (7.21) and the next-following inequality (unnumbered) should have a factor of α inserted before $(5\varepsilon n)^q$; and, again directly from (7.14), we get a modified form of (7.22), holding with probability one:

$$\frac{\beta+\varepsilon}{(1-\varepsilon)^q} \; v(\underline{\underline{E}})^{1/k} \geqslant \limsup_{n \to \infty} \; n^{-q} \; \ell(\underset{\sim}{P}^n) - \alpha(5\varepsilon)^q[v(\underline{\underline{E}})+\varepsilon]^{1-q} . \tag{7.22*}$$

Now we observe that ε is arbitrary and conclude that

$$\lim_{n\to\infty} \; n^{-q} \; \ell(\underset{\sim}{P}^n) = \beta \; v(\underline{\underline{E}})^{1/k} ,$$

establishing Lemma 7 of Beardwood, Halton, and Hammersley [1959].

Incidentally, equation (5.1) of the same paper should read

$$\mathcal{E} \; \ell(P\xi\underline{\underline{E}}) \sim \beta_k \; k^{1/2}\xi^k v(\underline{\underline{E}}) = \beta\xi^k v(\underline{\underline{E}}) \quad \text{as} \quad \xi \to \infty . \tag{5.1*}$$

Chapter III

Additional NP-hard Problems

1. Introduction

The main result of this chapter is Theorem 5 in Section 2 which characterizes a central algorithm that is near-optimal with probability one. In Section 3 we consider two minimization NP-hard problems: the vertex set cover of a graph and the set cover of a collection of sets. In Section 4 we consider three maximization NP-hard problems: the clique of an undirected graph, the set pack of a collection of sets, and the k-dimensional matching of a graph. For each of these problems we present an algorithm, derived from the algorithm of Theorem 5, with its worst case running time bounded by a polynomial on the size of the problem instance. Furthermore, as corollaries of Theorem 5, we show that each algorithm gives an optimal or near-optimal solution with probability one, as the size of the corresponding problem instance increases.

In the following sections, let $I_n = \{1,2,\ldots,n\}$. For a finite set R, let random (R) be a function which returns an element of R chosen at random with equal probability among the elements of R. Let log denote the natural logarithm function, let $[x]^+$ and $[x]_-$ denote the smallest integer not less than x (i.e., the ceiling of x) and the largest integer not greater than x (i.e., the floor of x), respectively.

2. Basic Lemmas and Theorems

Let $\triangle$ be a finite set, and let $\rho \subseteq \triangle \times \triangle$ be an irreflexive and symmetric relation defined on $\triangle$. Let us say that a subset S of $\triangle$ is a ρ-subset iff for any a, b $\in$ S, a ρ b.

In this section we use the following condition.

Condition E: there is a fixed p, $0<p<1$, such that for any a, b $\in$ $\triangle$, we have a ρ b with probability p, independent of other pairs of elements in $\triangle$ being related by ρ.

To prove Theorem 5, we need one lemma and one theorem as follows.

Lemma 2.1:

For $0<p<1$, $n>0$, $0<\epsilon<1$, and $b(n)=[(1-\epsilon)\log n/|\log p|]_-$ we have that

$$b(n)\ \{1 - 1/\ n^{(1-\epsilon)}\}^{[\ n/b(n)]^+ -1} = \underline{o}\ (1/n^{\ 2}).$$

Proof:

We want to show that

$$n^2\ b(n)\ \{1 - 1/n^{(1-\epsilon)}\}^{[\ n/b(n)]^+ -1} \to 0, \text{ as } n \to \infty. \qquad (2.1)$$

Consider the log of the left-hand side of (2.1) :

$$2 \log n + \log b(n) + \{[n/b(n)]^+ -1\} \log (1 - 1/n^{(1-\epsilon)})$$

$$\leq 2 \log n + \log b(n) + \{[n/b(n)]^+ -1\} (-1/n^{(1-\epsilon)})$$

$$2 \log n + \log[(1-\epsilon)\log n/|\log p|] - \frac{n^{\epsilon}\ |\log p|}{(1-\epsilon)\log n} + 1/n^{(1-\epsilon)} \qquad (2.2)$$

since $\log(1-x) \leq -x$, for $0<x<1$.

Expression (2.2) tends to $-\infty$, as $n \to \infty$ (the third term increases faster than the other terms) so that (2.1) is true.

Let M_n denote the cardinality of the largest existing ρ-subset of Δ, for $|\Delta| = n$. Our probabilistic model will be assumed to be incremental (cf. Section 1 of Chapter I) in the sense that the sequence $M_0, M_1, M_2, \ldots$ of random variables is sampled as follows: [1] we increase $|\Delta|$ by one by augmenting Δ to get, say, $\Delta' = \Delta \sqcup \{a\}$ where a is not in Δ; [2] the relation ρ is also augmented to get $\rho \subseteq \Delta' \times \Delta'$, $\rho' = \rho \sqcup \rho_0$, where $\rho_0 \subseteq \Delta \times \{a\}$ and ρ_0 is sampled according to Condition E. Then, the following theorem is proved in Matula[1976].

Theorem 4 Under Condition E, if M_n denotes the cardinality of the largest existing ρ-subset of Δ

$$\lim_{n \to \infty} M_n/\log n = 2/|\log p|, \text{ with probability one.}$$

Theorem 5: Under Condition E, if $|\Delta| = n$, there is an algorithm whose worst case running time is $O(q(n)n^2)$, where $q(.)$ is a polynomial, such that

$$1 \geq \frac{A_n}{M_n} \geq \frac{1}{2} \text{ ,as } n \to \infty, \text{ with probability one,}$$

where A_n denotes the cardinality of a ρ-subset of Δ computed by the algorithm, and M_n denotes the cardinality of the largest existing ρ-subset of Δ.

Proof:

Let us consider the following algorithm.

Algorithm C

(Let S and T be sets. S is the output)

(1) $S := \text{empty};\ T := \Delta;$

(2) **while** T is not empty **do**

(3) $\quad a :=$ random $(T);\ T := T - \{a\};$

(4) $\quad$ **if** (for all $b \in S$, $a \rho b$) **or** ($S =$ empty)

(5) $\qquad$ **then** $S := S \sqcup \{a\}$ **fi**

(6) $\quad$ **od**

Assuming that there is an integer valued polynomial $q(.)$ such that it takes at most $q(n)$ number of steps to check whether $a \rho b$ on line (4) above, the worst case running time of Algorithm C is $O(q(n)\ n^2)$, since in each iteration of the statements on lines (2)-(6) the cardinality of T decreases by one and the cardinality of S increases at most by one. In each iteration also, it is clear that S is a ρ-subset of Δ

Let $S(\Delta,\rho)$ denote the output S when Algorithm C is applied on $\rho \subseteq \Delta \times \Delta$. As in the statement of this theorem, let $A_m = |S(\Delta,\rho)|$ if $|\Delta| = m$. Our probabilistic model will be assumed to be incremental in the sense that the sequence $A_0, A_1, A_2, \ldots$ of random variables is sampled as follows: [1] we increase $|\Delta|$ by one by augmenting Δ to get, say, $\Delta' = \Delta \sqcup \{a\}$ where a is not in Δ, and a is the element chosen by the function random (at line (3) of Algorithm C above) ; [2] the relation ρ is also augmented to get $\rho' \subseteq \Delta' \times \Delta'$, $\rho' = \rho \sqcup \rho_0$, where $\rho_0 \subseteq \Delta \times \{a\}$ and ρ_0 is sampled according to Condition E; [3] if $S(\Delta,\rho) \sqcup \{a\}$ is a ρ-subset of Δ, then by Algorithm C $S(\Delta',\rho') = S(\Delta,\rho) \sqcup \{a\}$. Otherwise, $S(\Delta',\rho') = S(\Delta,\rho)$. Therefore, $A_{m+1} - A_m \leq 1$ for $m=0,1,2,\ldots$.

Let $s_i = \min\{|\triangle| : |S(\triangle,\rho)| = i\} = \min\{m : A_m = i\}$, for $i=1,2,3,\ldots$, and let $s_0 = 0$. Since the sequence $\{A_m : m \geq 0\}$ is non-decreasing, the sequence $\{s_i : i>0\}$ is also non-decreasing.

We now observe that if Algorithm C, at some iteration of the statements on lines (2)-(6) has $|S| = i$ (i.e., so far it has found i elements of $\triangle$ which constitute a ρ-subset) then p^i is the probability that the next element to be examined by the algorithm is related to <u>all</u> elements in S. Hence, $(1-p^i)^{j-1}$ for $j=1,2,3,\ldots$ is the probability that each of the next (j-1) elements to be examined is <u>not</u> related to at least one element in S. Thus we have, for all integers $i,j \geq 1$,

$$\Pr\{s_{i+1} - s_i = j\} = (1-p^i)^{j-1}p^i, \text{ and } s_1 - s_0 = 1 \qquad (2.3)$$

From (2.3), for any positive integer value k we have

$$\begin{aligned}\Pr\{(s_{i+1} - s_i) < k\} &= \sum_{1 \leq j \leq k-1} \Pr\{(s_{i+1} - s_i) = j\} \\ &= \sum_{1 \leq j \leq k-1} (1-p^i)^{j-1}p^i \\ &= p^i\,\frac{(1-p^i)^{k-1} - 1}{(1-p^i) - 1} \\ &= 1 - (1-p^i)^{k-1} \qquad (2.4)\end{aligned}$$

In the following, we want to show that, for any $\epsilon > 0$,

$$\sum_{0 \leq n \leq \infty} \Pr\{A_n \leq (1-\epsilon) \log n/|\log p|\} \text{ is finite.}$$

For any real x, $A_n \leq x$ implies $A_n \leq [x]_-$, since A_n is an integer value. Thus, for any arbitrary $\epsilon > 0$ we have

$$\Pr\{A_n \leq (1-\epsilon) \log n/|\log p|\} \leq \Pr\{A_n \leq b(n)\} \qquad (2.5)$$

where $b(n) = [(1-\epsilon) \log n/ |\log p|]_-$ (2.6)

For any positive integer i, $A_n \leq i$ implies $s_i \geq n$ (as we noted before, the sequence $\{s_i : i \geq 0\}$ is non-decreasing). Thus we have

$$\Pr\{A_n \leq b(n)\} \leq \Pr\{ s_{b(n)} \geq n\} \tag{2.7}$$

Since $s_{b(n)} = \sum_{0 \leq i \leq b(n)-1} (s_{i+1} - s_i)$ for $b(n) \geq 1$, we have

$$\Pr\{s_{b(n)} \geq n\} \leq \Pr\{ \bigsqcup_{0 \leq i \leq b(n)-1} (s_{i+1} - s_i) \geq n/b(n) \}$$

$$\leq \sum_{0 \leq i \leq b(n)-1} \Pr\{(s_{i+1} - s_i) \geq n/b(n)\} \tag{2.8}$$

For any real x, $(s_{i+1} - s_i) \geq x$ implies $(s_{i+1} - s_i) \geq [x]^+$ since $(s_{i+1} - s_i)$ is an integer value. Thus from (2.4) and (2.8)

$$\sum_{0 \leq i \leq b(n)-1} \Pr\{ (s_{i+1} - s_i) \geq n/b(n) \}$$

$$\leq \sum_{0 \leq i \leq b(n)-1} \Pr \{ (s_{i+1} - s_i) \geq [n/b(n)]^+ \}$$

$$= \sum_{0 \leq i \leq b(n)-1} [1 - \Pr\{ (s_{i+1} - s_i) < [n/b(n)]^+]$$

$$= \sum_{0 \leq i \leq b(n)-1} (1 - p^i)^{[n/b(n)]^+ -1}$$

$$\leq b(n) \{1 - p^{b(n)}\}^{[n/b(n)]^+ -1}$$

$$\sim b(n) \{1 - 1/n^{(1-\epsilon)}\}^{[n/b(n)]^+ -1} \tag{2.9}$$

since $p^{b(n)} \sim p^{(1-\epsilon) \log n/|\log p|} = 1/n^{(1-\epsilon)}$

By Lemma 2.1, the last expression in (2.9) is $o(1/n^2)$ so that there exists a positive integer n_0 such that

$$\sum_{0 \le n \le \infty} \Pr\{ A_n \le (1-\epsilon) \log n / |\log p| \}$$

$$\le \sum_{0 \le n \le n_0 - 1} b(n) \; [1-p^{b(n)}]^{[n/b(n)]^{+} - 1}$$

$$+ \sum_{n_0 \le n \le \infty} 1/n^{2} < \infty \qquad (2.10)$$

By the Borel-Cantelli lemma, (2.10) implies that, with probability one, for any choice of $\epsilon > 0$,

$$\liminf_{n \to \infty} \frac{A_n}{\log n} > (1-\epsilon)/|\log p| \qquad (2.11)$$

Since ϵ is arbitrary, (2.11) implies that

$$\liminf_{n \to \infty} \frac{A_n}{\log n} \ge 1/|\log p| \qquad (2.12)$$

with probability one.

On the other hand, by Theorem 4, M_n is such that

$$\lim_{n \to \infty} \frac{M_n}{\log n} = 2/|\log p| \quad \text{, with probability one.} \qquad (2.13)$$

From (2.12) and (2.13) we have

$$\liminf_{n \to \infty} \frac{A_n}{M_n} \ge \frac{1}{2} \quad \text{, with probability one.} \qquad (2.14)$$

Since we know that $A_n/M_n \le 1$, the theorem follows.

Q.E.D.

3. Minimization Problems

3.1 Vertex Set Cover Problem

Let $G = (V, E)$ be an undirected graph (V is the set of vertices, E is the set of edges). A vertex cover of G is a subset S of V such that each edge of G is incident upon some vertex in S. The vertex cover problem (VC) is to find the smallest vertex cover of G. This problem is known to be NP-hard (cf., e.g., Aho, Hopcroft, and Ullman [1975]).

Algorithm VC

(Let $V = I_n$, and let S, S', and T be sets. S is the output)

```
S := V; S' := V;
T := empty;
while S' is not empty do
   v := random (S');
   S' := S' - {v};
   if (for all u ∈ T, u is not connected to v)
      then T := T || {v}; S := S - {v} fi
 od
```

Clearly, the worst case running time of Algorithm VC is $O(n^2)$, and S is a vertex cover of G.

For the probabilistic analysis of Algorithm VC we assume the following (as in Grimmett and McDiarmid [1975], Matula [1976], Posa [1976], and Angluin and Valiant [1977]).

Condition VC: there is a fixed p, $0<p<1$, such that any pair of vertices $\{v,v'\}$ has probability p of being a member of E, independent of other pairs of vertices being members of E.

Corollary VC: Under Condition VC, let VC(n) denote the cardinality of S computed by Algorithm VC, and let m_n denote the cardinality of the minimal vertex cover of G. Then

$$\frac{VC(n)}{m_n} \sim 1, \text{ as } n \to \infty, \text{ with probability one.}$$

Proof:

For the vertex cover problem, the set Δ of Theorem 5 is interpreted to be the set V, the statement a ρ b to mean a "not connected to" b. Then Condition VC is equivalent to Condition E, and the set T in Algorithm VC is a ρ-subset. Therefore, from (2.12), since $|V| = n$, we have

$$\frac{|T|}{\log n} \geq 1/|\log p|, \text{ as } n \to \infty, \text{ with probability one.} \quad (3.1)$$

In Algorithm VC, $S \cup T = V$, and S and T are disjoint. Then $|S| = n - |T|$, and (3.1) implies that, as $n \to \infty$,

$$\frac{|S|}{\log n} = \frac{n}{\log n} - \frac{|T|}{\log n}$$

$$\leq \frac{n}{\log n} - \frac{1}{|\log p|}$$

$$\leq \frac{n}{\log n}, \text{ with probability one.} \quad (3.2)$$

On the other hand, if M_n denotes the largest existing ρ-subset of V, then Theorem 4 says that

$$\frac{M_n}{\log n} \sim 2/|\log p|, \text{ as } n \to \infty, \text{ with probability one.} \quad (3.3)$$

Since $m_n + M_n = n$, (3.3) implies that

$$\frac{m_n}{\log n} \sim \frac{n}{\log n} - 2/|\log p|, \text{ as } n \to \infty, \text{ with probability one.} \quad (3.4)$$

Then (3.2) and (3.4) imply that

$$\frac{|S|}{m_n} \leq 1 + \underline{o}(1), \text{ as } n \to \infty, \text{ with probability one.} \quad (3.5)$$

Since we know $|S|/m_n \geq 1$, the corollary follows.

Q.E.D.

3.2 Set Cover Problem

Let n and k be positive integers such that $k = \max(3,n)$, and let $C = \{S_1, S_2, \ldots, S_n\}$ be a collection of sets of positive integers such that $|S_i| \leq k$ for $1 \leq i \leq n$. A *set cover* of C is a subcollection $S_{i_1}, S_{i_2}, \ldots, S_{i_h}$ such that

$$\bigsqcup_{1 \leq j \leq h} S_{i_j} = \bigsqcup_{1 \leq j \leq n} S_j .$$

The *set cover problem* (*SC*) is to find the smallest set cover of C. This problem is known to be NP-hard even if (i) $|S_i| \leq m$, for a fixed $m \geq 3$, $1 \leq i \leq n$ (see e.g. Garey and Johnson[1978]); and (ii) if $s \in S_i$ for some i, $1 \leq i \leq n$, then there is at least one $j \neq i$, $1 \leq j \leq n$, such that $s \in S_j$ (see e.g. Aho, Hopcroft, and Ullman[1975]). Throughout this section we assume (ii).

Algorithm SC

(Let S, S', and T be collections of sets. S is the output)

```
S := C; S' := C;
T := empty ;
while S' is not empty do
   Si := random (S'); S' := S' - {Si};
   if (for all R in T, R is disjoint from Si)
      then T := T ⊔ {Si}; S := S - {Si}
      fi
   od
```

Clearly, the worst case running time of Algorithm SC is $O(k^2n^2)$, and S is a set cover of C.

For the probabilistic analysis of Algorithm SC we assume the following:

Condition SC: there is a fixed p, $0<p<1$, such that given any pair of sets S_1 and S_2 in C, we have that Pr{ S_1 and S_2 are disjoint } = p, independent of other pairs of sets in C.

Corollary SC: Under Condition SC, let SC(n) denote the cardinality of the set S computed by Algorithm SC, and let m_n denote the cardinality of the minimal set cover of C. Then

$$\frac{SC(n)}{m_n} \sim 1, \text{ as } n \to \infty, \text{ with probability one.}$$

Proof:

This proof is very similar to the proof of Corollary VC.

For the set cover problem, the set Δ of Theorem 5 is interpreted to be the collection C, the statement a ρ b to mean a "disjoint from" b. Then Condition SC is equivalent to Condition

E, and the collection T in Algorithm SC is a ρ-subset. Moreover, an incremental sampling of an SC-instance, as described in the proof of Theorem 5, is feasible. Therefore, from (2.12), since $|C| = n$, we have

$$\frac{|T|}{\log n} \geq 1/|\log p|, \text{ as } n \to \infty, \text{ with probability one.} \qquad (3.6)$$

In Algorithm SC, $S \sqcup T = C$, and S and T are disjoint. Then $|S| = n - |T|$, and (3.6) implies that, as $n \to \infty$,

$$\frac{|S|}{\log n} = \frac{n}{\log n} - \frac{|T|}{\log n}$$

$$\leq \frac{n}{\log n} - 1/|\log p|$$

$$\leq \frac{n}{\log n}, \text{ with probability one.} \qquad (3.7)$$

On the other hand, if M_n denotes the largest existing ρ-subset of C, then Theorem 4 says that

$$\frac{M_n}{\log n} \sim 2/|\log p|, \text{ as } n \to \infty, \text{ with probability one.} \qquad (3.8)$$

Since $m_n + M_n = n$, (3.8) implies that

$$\frac{m_n}{\log n} \sim \frac{n}{\log n} - 2/|\log p|, \text{ as } n \to \infty, \text{ with probability one.} \qquad (3.9)$$

Then (3.7) and (3.9) imply that

$$\frac{|S|}{m_n} \leq 1 + o(1), \text{ as } n \to \infty, \text{ with probability one.} \qquad (3.10)$$

Since we know $|S|/m_n \geq 1$, the corollary follows.

Q.E.D.

4. Maximization Problems

4.1 Clique Problem.

Let G = (V, E) be an undirected graph. A clique of G is a complete subgraph of G (i.e., any pair of vertices in the subgraph is connected to each other by an edge). The clique problem (CL) is to find the largest clique of G. This problem is known to be NP-hard (cf., e.g., Aho, Hopcroft, and Ullman[1975]).

Algorithm CL

(Let n be a positive integer, let $|V| = n$, and let S and T be sets. S is the output)

```
S := empty; T := V;
while T is not empty do
   v:= random (T); T := T - {v};
   if S ⊔ {v} is a clique
      then S := S ⊔ {v} fi
   od
```

Clearly, the worst case running time of Algorithm CL is $O(n^2)$, and S is a clique of G. By duality, Algorithm CL may be changed to find a feasible solution to the maximum independent set problem, i.e., the problem of finding the largest set S of vertices in G such that no two vertices in S are connected. For the maximum independent set problem, an algorithm which does not select the vertices at random was independently studied in Grimmett and McDiarmid[1975], assuming a sampling model which is not incremental. Their algorithm have the ratio between the computed solution and the optimal solution asymptotic to 1/2, with probability one, while our algorithm has ratio at least 1/2, asymptotically, with probability one.

For the probabilistic analysis of Algorithm CL, we assume Condition VC for the graph G = (V, E).

Corollary CL: Under Condition VC, let CL(n) denote the cardinality of the set S computed by Algorithm CL, and let M_n denote the cardinality of the maximal clique in G. Then

$$1 \geq \frac{CL(n)}{M_n} \geq \frac{1}{2} \text{ ,as } n \to \infty \text{, with probability one.}$$

Proof:

For the clique problem, the set Δ of Theorem 5 is interpreted to be the set V, the statement a ρ b to mean a "connected to" b. Then Condition CL is equivalent to Condition E, and the set S in Algorithm CL is a ρ-subset.

Then Theorem 5 directly implies this corollary.

Q.E.D.

4.2 Set Packing Problem

Let k and n be positive integers such that k = max (3,n), and let C = $\{S_1, S_2, \ldots S_n\}$ be a collection of sets of, let us say, positive integers such that $|S_i| \leq k$ for $1 \leq i \leq n$. A set pack of C is a subcollection $S_{i_1}, S_{i_2}, \ldots, S_{i_h}$ of pairwise disjoint sets. The set packing problem (SP) is to find the largest set pack of C. This problem is known to be NP-hard, even if $|S_i| \leq m$, for a fixed $m \geq 3$ and for $1 \leq i \leq n$. (see, e.g., Garey and Johnson[1978]).

Algorithm SP

(Let S and T be sets. S is the output)

```
S := empty; T := C;
while T is not empty do
   S_i := random (T); T := T - {S_i};
   if S_i is disjoint from all sets in S
      then S := S ⊔ {S_i} fi
   od
```

Clearly, the worst case running time of Algorithm SP is $O(k^2n^2)$, and S is a set pack of C.

For the probabilistic analysis of Algorithm SP, we assume Condition SC for the collection C.

Corollary SP: Under Condition SC, let SP(n) denote the cardinality of the set S computed by Algorithm SP, and let M_n denote the cardinality of the maximal set pack of C. Then

$$1 \geq \frac{SP(n)}{M_n} \geq \frac{1}{2} \quad \text{,as } n \to \infty \text{, with probability one.}$$

Proof:

For the set packing problem, the set Δ of Theorem 5 is interpreted to be the collection C, the statement a ρ b to mean a "disjoint from" b. Then Condition SC (as in the proof of Corollary SC) is equivalent to Condition A, and the collection S in Algorithm SP is a ρ-subset. Moreover, an incremental sampling of an SP-instance, as described in the proof of Theorem 5, is feasible.

Then Theorem 5 directly implies this corollary.

Q.E.D.

4.3 k-Dimensional Matching Problem

Let k and n be positive integers such that $k = \max(3,n)$, and let $A_1 = \{a_{11}, a_{12}, \ldots, a_{1n}\}$, $A_2 = \{a_{21}, a_{22}, \ldots, a_{2n}\}$, $\ldots$, $A_k = \{a_{k1}, a_{k2}, \ldots, a_{kn}\}$ be pairwise disjoint sets, and let T be a subset of $A_1 \times A_2 \times \ldots \times A_k$, with $|T| = n$. A matching of T is a subset S of T such that no two elements of S agree in any coordinate. The k-dimensional matching problem (DM) is to find the largest matching of T. This problem is known to be NP-hard even if we have a fixed $k = 3$ (cf., e.g., Garey and Johnson[1978]).

Algorithm DM

(Let S and U be sets. S is the output)

```
    S := empty; U := T;
    while U is not empty do
        u := random (U); U := U - {u};
        if S ⊔ {u} is a matching of T
            then S := S ⊔ {u} fi
        od
```

Clearly, the worst case running time of Algorithm DM is $O(k\ n^2)$, and the set S is a matching of T.

For the probabilistic analysis of Algorithm DM we assume the following:

Condition DM: there is a fixed p, $0<p<1$, such that, given any pair of elements t_1 and t_2 in T, we have that Pr $\{t_1$ and t_2 disagree in all k coordinates$\}$ = p, independent of other pairs of elements in T.

Corollary DM: Under Condition DM, let DM(n) denote the cardinality of S computed by Algorithm DM, and let M_n denote the cardinality of the maximal matching of T. Then

$$1 \geq \frac{DM(n)}{M_n} \geq \frac{1}{2} \quad \text{,as } n \rightarrow \infty \text{, with probability one.}$$

Proof:

For the matching problem, the set Δ of Theorem 5 is interpreted to be the set T, the statement a ρ b to mean a "disagree in all k coordinates with" b. Then Condition DM is equivalent to Condition E, and the set S in Algorithm DM is a ρ-subset. Moreover, an incremental sampling of a DM-instance, as described in the proof of Theorem 5, is feasible.

Then Theorem 5 directly implies this corollary.

Q.E.D.

Chapter IV

Alternative Algorithms for the Maximization Problems

1. Introduction

This chapter presents new algorithms for the three maximization problems considered in Chapter III. These algorithms will give, following the notation used in Chapter III, ratio $A_n/M_n \sim 1$, as $n \to \infty$, with probability one, but they will require more running time.

As in Chapter III, let Δ be a finite set, and let $\rho \subseteq \Delta \times \Delta$ be an irreflexive and symmetric relation defined on Δ. Let us say that a subset S of Δ is a ρ-subset iff for any a, b $\in$ S, a ρ b. We will also use Condition E as stated in Section 2 of Chapter III.

2. Algorithm D and its Asymptotic Performance

In Algorithm D defined below, we assume that $|\Delta| = n$ and $0<p<1$. B and D denote collections of ρ-subsets of Δ, and Q and R denote subsets of Δ. We assume that the elements in B,D,Q, and R are indexed with positive integers. If X denotes such an indexed set, least (X) denotes the element in X with the lowest index. The collection B will be the result of the algorithm.

Algorithm D

```
(1)   B := { {a,b} : a ≠ b, a,b ∈ △, a ρ b };
(2)   k(n) := [2 log_{1/p} n]^+;
(3)   j := 3;
(4)   while j ≤ k(n) and B is not empty do
(5)       D := B; B := empty;
(6)       while D is not empty do
(7)           Q := least (D); D := D -{Q};
(8)           R := △ - Q;
(9)           while R is not empty do
(10)              r := least (R); R:= R-{r};
(11)              if Q ⊔ {r} is a ρ-subset of △
(12)                 then B := B ⊔ {Q ⊔ {r} } fi
(13)          od
(14)      od;
(15)  j := j+1
(16)  od
(17)  if B is empty then B := D
```

Algorithm D finds all the largest ρ-subsets of cardinality not greater than k(n). This is done iteratively starting from the ρ-subsets of cardinality two obtained in statement (1) above. At the j-th iteration of the statements (4) - (16), D contains all the ρ-subsets of cardinality (j-1), and B may contain ρ-subsets of cardinality j. The sets in B, if any, are found by adding the element r obtained on line (10) to the set Q of D if $Q \sqcup \{r\}$ is a ρ-subset. If there is no ρ-subset larger than the sets in D, B will be empty after the execution of the statements

(6) - (14) and B = D after statement (17).

Since Algorithm D finds all ρ-subsets of Δ of cardinality not greater than $k(n) = [2 \log_{1/p} n]^+$, an immediate corollary of Theorem 4 of Chapter III is that Algorithm D finds a ρ-subset of Δ of cardinality $k(n)$, as $n \to \infty$, with probability one. Therefore, we have the following.

Theorem 6 :

Under Condition E, if A_n denotes the cardinality of a ρ-subset of Δ computed by Algorithm D, and if M_n denotes the cardinality of the largest existing ρ-subset of Δ, then

$$\frac{A_n}{M_n} \sim 1, \text{ as } n \to \infty, \text{ with probability one.}$$

3. Expected Running Time

First we need a lemma.

Lemma 3.1 Under Condition E, if $|\Delta| = n$, and if j is a positive integer, and if $N(n,j)$ denotes the number of ρ-subsets of Δ of cardinality j, then

$$\mathcal{E}\, N(n,j) = \binom{n}{j} p^{j(j-1)/2} \qquad (3.1)$$

Proof:

Let the index $I = 1,2, \ldots, \binom{n}{j}$ correspond to the $\binom{n}{j}$ subsets of Δ of cardinality j. Let E_i denote the event that the i-th. subset is a ρ-subset. Then $\Pr\{ E_i \} = p^{j(j-1)/2}$, for $1 \le i \le \binom{n}{j}$ and $N(n,j)$ is the number of E_i that occur in Δ, so that (3.1) is true.

Q.E.D.

Now we want to find an upper bound for the expected running time of Algorithm D.

Theorem 7 :

Under Condition E of Chapter III, if $|\Delta| = n$, the expected running time $E\ R_n$ of Algorithm D is such that

$$\mathcal{E}\ R_n = \underline{o}\left\{ h(n) \quad n^{3/2} \left[\frac{p\ e^2\ n}{h(n)} \right]^{h(n)/2} \right\} \qquad (3.2)$$

where $h(n) = \log_b n - \log_b \log_b n + 1$, and $b = 1/p$ (3.3).

Proof:

Given $a_1, a_2 \in \Delta$, let α denote the amount of time needed to check whether $a_1\ \rho\ a_2$.

The time to execute statement (1) of Algorithm D is

$$\alpha\ n(n-1)/2 \qquad (3.4)$$

since this is the number of subsets of Δ with cardinality two.

Let e_j denote the expected cardinality of D at the j-th iteration of statements (4) - (16). Since D, as we noted in Section 2, contains all the ρ-subsets of Δ of cardinality (j-1), e_j is equal to the expected number of ρ-subsets of Δ of cardinality (j-1). Then, by Lemma 3.1,

$$e_j = \binom{n}{j-1} \quad p^{(j-1)(j-2)/2} \qquad (3.5)$$

Since Q on line (7) is an element of D, $|Q| = j-1$ so that R on line (8) is such that $|R| = n-j+1$. Thus the statements (9) - (13) are iterated (n-j+1) times for each value of j. In each iteration of the statements (9) - (13), to check whether $Q \cup \{r\}$ is a ρ-subset on line (11) will require time $\alpha|Q| = \alpha(j-1)$.

Therefore, the statements (9) - (13) will require time $\propto (n-j+1)(j-1)$ for each value of j.

Since the statements (6) - (16) are executed $|D|$ times, the expected running time of statements (4) - (16) is

$$\propto \sum_{3 \le j \le k(n)} e_j \, (n-j+1)(j-1) \tag{3.6}$$

From (3.4), (3.5), and (3.6) we have that

$$\& R_n = \propto n(n-1)/2 + \propto \sum_{3 \le j \le k(n)} \binom{n}{j-1} p^{(j-1)(j-2)/2} (n-j+1)(j-1) \tag{3.7}$$

where $k(n) = [2 \log_{1/p} n]^+$.

Let a_j denote the j-th term of the summation in (3.7). It is easy to see that

$$\frac{a_{j+1}}{a_j} = p^{(j-1)} \frac{n-j}{j-1} = F(j,n) \tag{3.8}$$

It is clear that $F(j,n)$ decreases when j increases, with $3 \le j \le k(n)$. Moreover, for $j = [h(n)]^+$ where $h(n)$ is as defined in (3.3) we have that

$$F([h(n)]^+, n) \sim p^{\log_b n - \log_b \log_b n} \left(\frac{n-h(n)}{h(n)-1} \right)$$

$$= \frac{1}{n} \log_b n \; \frac{n-h(n)}{h(n)-1}$$

$$= \frac{1 - h(n)/n}{(h(n)-1)/\log_b n} \rightarrow 1, \text{ as } n \rightarrow \infty \tag{3.9}$$

since $h(n)/n \rightarrow 0$, and $(h(n)-1)/\log_b n \rightarrow 1$, as $n \rightarrow \infty$.

Then we may conclude that, asymptotically,

$$F(3,n) \geq F(4,n) \geq \dots \geq F([h(n)]^{+},n) \sim 1 \qquad (3.10)$$

Since by the definition of $F(j,n)$ in (3.8)

$$a_{j+1} = a_j \, F(j,n),$$

we have that, asymptotically,

$$a_{[h(n)]^{+}} = \max\{a_3, a_4, \dots, a_{k(n)}\} \qquad (3.11)$$

(for an illustration of (3.10) and (3.11), please see tables in Section 6 below)

From (3.7) and (3.11) we have that, for sufficiently large n,

$$\& \; R_n \leq \alpha \, n(n-1)/2 + \alpha \, (k(n)-2) \, a_{[h(n)]^{+}}$$

$$\sim \alpha \, n(n-1)/2 + \alpha \, (2 \log_b n - 2) \, \binom{n}{[h(n)]^{+}-1} \, p^{(h(n)-1)(h(n)-2)/2}$$

$$(n-h(n)+1)([h(n)]^{+} -1)$$

$$\leq \alpha \, n(n-1)/2 + \alpha \, (2\log_b n-2)(n-h(n)+1) \, \frac{n^{h(n)-1}}{([h(n)]^{+}-2)!} \, p p^{h(n)[h(n)-3]/2} \qquad (3.12)$$

Since

$$p^{h(n)} = p^{\log_b n} (1/p)^{\log_b \log_b n} \, p$$

$$= p \log_b n / n$$

we have that the last expression in (3.12) is equal to

$$\alpha \, n(n-1)/2 + \alpha \, p(2\log_b n-2)(n-h(n)+1) \, \frac{n^{h(n)-1}}{([h(n)]^{+}-2)!}$$

$$(p \log_b n / n)^{[h(n)-3]/2}$$

$$= \alpha\, n(n-1)/2 + \alpha\, p(2 \log_b n-2)(n-h(n)+1) \frac{n^{[h(n)+1]/2}}{([h(n)]^{+} -2)!} (p \log_b n)^{[h(n)-3]/2}$$

$$\sim \frac{2\alpha\, p\, n^{1/2}\, (n-h(n)+1) \quad (p\, n \log_b n)^{h(n)/2}}{(\log_b n)^{1/2} \quad ([h(n)]^{+} -2)!} \tag{3.13}$$

By Stirling's formula,

$$[h(n)]^{+}! \sim (2\pi\; h(n))^{1/2} \left(\frac{h(n)}{e} \right)^{h(n)}$$

so that the last expression in (3.13) is asymptotic to

$$\frac{2\alpha\, p\, n^{1/2}\, (n-h(n)+1)\, h(n)\, (h(n)-1)}{(\log_b n)^{1/2}\, [2\pi\; h(n)]^{1/2}} \left[\frac{p\, e^2\, n \log_b n}{h^2(n)} \right]^{h(n)/2} \tag{3.14}$$

Since

$$\frac{(h(n) -1)}{[\log_b n\; h(n)]^{1/2}} \to 1, \quad \text{and}$$

$$\frac{\log_b n}{h(n)} \to 1, \text{ as } n \to \infty,$$

(3.14) is asymptotic to

$$\frac{2\alpha\, p\, h(n)}{(2\pi)^{1/2}}\, n^{3/2} \left[\frac{n\, p\, e^2}{h(n)} \right]^{h(n)/2} \tag{3.15}$$

(for an illustration of (3.15), please see tables in Section 6 below)

From (3.12) - (3.15), we conclude (3.2).

Q.E.D.

4. Worst Case Running Time

This section presents an upperbound for the running time of Algorithm D.

Theorem 8 :

If $|\Delta| = n$, and $0<p<1$, the running time R_n of Algorithm D is such that

$$R_n = \underline{o}\left\{ (\log_b n)^{5/2} \left[\frac{e\ n}{2 \log_b n} \right]^{2 \log_b n} \right\} \tag{4.1}$$

where $b = 1/p$.

Proof:

As in the proof of Theorem 6, the time to execute statement (1) of Algorithm D is

$$\alpha\ n(n-1)/2 \tag{4.2}$$

where α denotes the time needed to check whether $a_1\ \rho\ a_2$, for any given $a_1, a_2 \in \Delta$.

As we noted in Section 2, D contains all the ρ-subsets of cardinality (j-1) at the j-th iteration of the statements (4) - (16). Hence, $|D|$ can be at most $\binom{n}{j-1}$.

As in the proof of Theorem 7, the statements (9) - (13) require time α (n-j+1) (j-1) for each value of j.

Since the statements (6) - (14) are executed |D| times, the running time of statements (4) - (16) is at most

$$\alpha \sum_{3 \leq j \leq k(n)} \binom{n}{j-1} (n-j+1)\ (j-1) \tag{4.3}$$

From (4.2) and (4.3) we have that

$$R_n \leq \alpha\, n(n-1)/2 + \alpha \sum_{3<j<k(n)} \binom{n}{j-1} (n-j+1)(j-1) \tag{4.4}$$

where $k(n) = [2 \log_{1/p} n]^+$.

Let b_j denote the j-th term of the summation in (4.4). It is easy to see that

$$\frac{b_{j+1}}{b_j} = \frac{n-j}{j-1} = G(j,n) \tag{4.5}$$

$G(j,n)$ clearly decreases when j increases, but for the interval $3 \leq j \leq k(n)$ being considered, $G(j,n) = 1$ occurs only for finitely many values of n. This is so because $G(j,n) = 1$ implies $j=(n+1)/2$ and $(n+1)/2 \leq k(n)$ only for finitely many values of n.

Therefore, for sufficiently large n,

$$G(3,n) \geq G(4,n) \geq \dots \geq G(k(n),n) > 1 \tag{4.6}$$

so that

$$b_{k(n)} = \max\{ b_3, b_4, \dots, b_{k(n)} \} \tag{4.7}$$

From (4.4) and (4.7) we have that, for sufficiently large n,

$$R_n \le \alpha\, n(n-1)/2 + \alpha\,(k(n)-2)\binom{n}{k(n)-1}(n-k(n)+1)(k(n)-1)$$

$$\le \alpha\, n(n-1)/2 + \frac{\alpha\, n^{k(n)-1}}{(k(n)-1)!}(n-k(n)+1)(k(n)-2)(k(n)-1)$$

$$\sim \alpha\, n\,(n-1)/2 + \frac{\alpha\, n^{k(n)-1}}{k(n)!}(n-k(n)+1)k^3(n) \qquad (4.8)$$

By Stirling's formula,

$$k(n)! \sim [2\pi\, k(n)]^{1/2} \left(\frac{k(n)}{e}\right)^{k(n)}$$

so that (4.8) is asymptotic to

$$\alpha\, \frac{n^{k(n)}k^3(n)}{(2\pi\, k(n))^{1/2}} \left[\frac{e}{k(n)}\right]^{k(n)}$$

$$= \frac{\alpha k^3(n)}{(2\pi\, k(n))^{1/2}} \left[\frac{e\,n}{k(n)}\right]^{k(n)}$$

$$\sim \frac{\alpha k^3(n)}{(2\pi\;\, 2\log_b n)^{1/2}} \left[\frac{e\,n}{2\log_b n}\right]^{2\log_b n} \qquad (4.9)$$

From (4.9) we conclude (4.1)

Q.E.D

5. Applications of the Main Results

In this section, we want to present new algorithms for the three maximization problems studied in Section 3 of Chapter III. These new algorithms are derived from Algorithm D and their asymptotic behavior will be established as corollaries of Theorems 6, 7, and 8.

5.1 Clique Problem

For the clique problem, as defined in Section 4 of Chapter III, we have the following algorithm.

Algorithm CL1

(Let n be a positive integer, and let $|V| = n$, where V is the set of vertices in the undirected graph $G = (V,E)$)

This algorithm is identical to Algorithm D, except for the following:

(i) replace "Δ" by "V", in statements (1) and (8), and "a ρ b" by "a connected to b" in statement (1);

(ii) replace "$Q \sqcup \{r\}$ is a ρ-subset of Δ" by "$Q \sqcup \{r\}$ is a clique of G" on line (11).

As an immediate consequence of Theorems 6, 7, and 8, we have

Corollary CL1 :

Under Condition VC, if $|V| = n$, then

(i) $$\frac{CL1(n)}{M_n} \sim 1, \text{ as } n \to \infty, \text{ with probability one,}$$

where CL1(n) denotes the cardinality of a clique of G = (V, E) computed by Algorithm CL1, and M_n denotes the cardinality of the maximal clique of G;

(ii) the expected running time & R_n of Algorithm CL1 will satisfy (3.2);

(iii) the running time R_n of Algorithm CL1 will satisfy (4.1).

5.2 Set Packing Problem

For the set packing problem, as defined in Section 4 of Chapter III, we have the following algorithm.

Algorithm SP1

(Let n be a positive integer, and let $|C| = n$, where C is the collection of sets, as defined in Section 4 of Chapter III)

This algorithm is identical to Algorithm D, except for the following:

(i) replace "Δ" by "C", in statements (1) and (8), and "a ρ b" by "a disjoint from b", in statement (1);

(ii) replace "$Q \sqcup \{r\}$ is a ρ-subset of Δ" by "r is disjoint from all sets in Q", on line (11).

As an immediate consequence of Theorems 6, 7, and 8, we have

<u>Corollary SP1</u> :

Under Condition SC, if $|C| = n$, then

(i) $$\frac{SP1(n)}{M_n} \sim 1, \text{ as } n \to \infty, \text{ with probability one,}$$

where $SP1(n)$ denotes the cardinality of a set pack computed by Algorithm SP1, and M_n denotes the cardinality of the maximal existing set pack of C;

(ii) the expected running time $\mathcal{E}R_n$ of Algorithm SP1 will satisfy (3.2) with $n^{3/2}$ replaced by $k^2n^{3/2}$, where k is as defined in Section 4 of Chapter III. (This is so because the proof of Theorem 7 will hold for Algorithm SP1 with α replaced by αk^2).

(ii) the running time R_n of Algorithm SP1 will satisfy (4.1) with $(\log_b n)^{5/2}$ replaced by $k^2 (\log_b n)^{5/2}$, where k is as defined in Section 4 of Chapter III. (Because, again, α is replaced by αk^2 in the proof of Theorem 8 for Algorithm SP1).

5.3 k-Dimensional Matching Problem

For the k-dimensional problem, as defined in Section 4 of Chapter III, we have the following algorithm.

Algorithm DM1

(Let n be a positive integer, and let $|T| = n$, where T is a collection of sequences, as defined in Section 4 of Chapter III)

This algorithm is identical to Algorithm 3.1, except for the following:

(i) replace "Δ" by "T", in statements (1) and (8), and "a ρ b" by "a disagree in all k-coordinates with b" , in statement (1);

(ii) replace "$Q \sqcup \{r\}$ is a ρ-subset of Δ" by "$Q \sqcup \{r\}$ is a matching of T" on line (11).

As an immediate consequence of Theorems 6, 7, and 8, we have

Corollary DM1 :

Under Condition DM, if $|T| = n$, then

(i) $$\frac{DM1(n)}{M_n} \sim 1, \text{ as } n \to \infty \text{ with probability one,}$$

where DM1(n) denotes the cardinality of a set pack computed by Algorithm DM1, and M_n denotes the cardinality of the maximal existing matching of T;

(ii) the expected running time & R_n of Algorithm DM1 will satisfy (3.2) with $n^{3/2}$ replaced by $kn^{3/2}$, where k is as defined in Section 4 of Chapter III. (This is so because the proof of Theorem 7 will hold for Algorithm DM1 with α replaced by α k).

(ii) the running time R_n of Algorithm DM1 will satisfy (4.1) with $(\log_b n)^{5/2}$ replaced by $k\,(\log_b n)^{5/2}$, where k is as defined in Section 4 of Chapter II. (Because, again, α is replaced by α k in the proof of Theorem 8 for Algorithm DM1).

6. Some Numerical Tables

In order to illustrate some of the intermediate results obtained in the proof of Theorem 7, we want to show here some numerical tables. (.XXXXXXX+DDD is equal to .XXXXXXX 10^{+DDD} in the tables).

Tables 1 - 5 show, for some values of p and n, the corresponding values of h(n) as defined in (3.3), $P(j) = p^{j-1}$, $Q(j) = (n-j)/(j-1)$, $F(j) = P(j)\ Q(j)$, and $a_{j+1} = F(j)\ a_j$ as defined in (3.8). Next we have the values of (3.7) (without the constant factors) and the final bound (3.2).

The fact that the relative differences between the values of (3.2) and (3.7) (as shown in Tables 1-5) increase, as n increases, suggests that the bound (3.2) is very coarse.

In fact, Tables 6 - 10 show that at least for the interval $10^5 \le n \le 10^{10}$, the upper bound

$$h(n)\quad n^{3/2} \left[\frac{p\ e^2\ n}{h(n)}\right]^{(h(n)-2)/2} \tag{6.1}$$

for (3.7) is tighter than (3.2).

In tables 6 - 10, we have the following correspondence between values and expressions:

column (1) = expression (3.7) (without the constant factors);

column (2) = expression (6.1);

column (3) = expression (3.7);

$$\text{column (4)} = \frac{\text{col.(2)} - \text{col.(1)}}{\text{col.(1)}}$$

$$\text{column (5)} = \frac{\text{col.(3)} - \text{col.(1)}}{\text{col.(1)}}$$

column (6) = (h(n) - 2)/2

column (7) = h(n)/2

N= 1,000 P= .50

H(N) = .76488009+001 CEILING OF H(N) = 8

H(N)/2 = .38244005+001

2 LOG N/LOG(1/P) = .19931569+002 CEILING OF 2 LOG N/LOG(1/P) = 20

		F(J)	P(J)	Q(J)
A(2)=	.49950000+006			
A(3)=	.49850100+009			
A(4)=	.20687791+011	.41500000+002	.12500000+000	.33200000+003
A(5)=	.32163051+012	.15546875+002	.62500000-001	.24875000+003
A(6)=	.19981295+013	.62125000+001	.31250000-001	.19880000+003
A(7)=	.51670381+013	.25859375+001	.15625000-001	.16550000+003
A(8)=	.57206493+013	.11071429+001	.78125000-002	.14171429+003
A(9)=	.27681462+013	.48388672+000	.39062500-002	.12387500+003
A(10)=	.59471892+012	.21484375+000	.19531250-002	.11000000+003
A(11)=	.57439161+011	.96582031-001	.97656250-003	.98900000+002
A(12)=	.25190825+010	.43856534-001	.48828125-003	.89818182+002
A(13)=	.50584604+008	.20080564-001	.24414063-003	.82250000+002
A(14)=	.46834078+006	.92585637-002	.12207031-003	.75846154+002
A(15)=	.20111767+004	.42942592-002	.61035156-004	.70357143+002
A(16)=	.40262815+001	.20019531-002	.30517578-004	.65600000+002
A(17)=	.37744853-002	.93746185-003	.15258789-004	.61437500+002
A(18)=	.16634524-005	.44070973-003	.76293945-005	.57764706+002
A(19)=	.34583342-009	.20790100-003	.38146973-005	.54500000+002
A(20)=	.34022758-013	.98379035-004	.19073486-005	.51578947+002

SUM OF A(J)'S = .16651509+014

FINAL BOUND = H(N) * (N**3/2)*(P E E N/H(N))**(H(N)/2) = .44478762+016

FINAL BOUND - SUM = .44312247+016 , AND (FINAL BOUND - SUM)/SUM = .26611551+003

Table 1

N= 500,000 P= .50

H(N) = .15688847+002 CEILING OF H(N) = 16

H(N)/2 = .78444233+001

2 LOG N/LOG(1/P) = .37863137+002 CEILING OF 2 LOG N/LOG(1/P) = 38

Table 2

		F(J)	P(J)	Q(J)
A(2)=	.12499975+012			
A(3)=	.62499625+017			
A(4)=	.13020651+022	.20833167+005	.12500000+000	.16666533+006
A(5)=	.10172282+026	.78124219+004	.62500000-001	.12499875+006
A(6)=	.31787992+029	.31249625+004	.31250000-001	.99998800+005
A(7)=	.41390045+032	.13020651+004	.15625000-001	.83332167+005
A(8)=	.23096754+035	.55802679+003	.78125000-002	.71427429+005
....				
A(11)=	.29873941+041	.48827051+002	.97656250-003	.49998900+005
A(12)=	.66302433+042	.22194070+002	.48828125-003	.45453455+005
A(13)=	.67444569+043	.10172262+002	.24414063-003	.41665583+005
A(14)=	.31664419+044	.46948806+001	.12207031-003	.38460462+005
A(15)=	.69020886+044	.21797616+001	.61035156-004	.35713214+005
A(16)=	.70209429+044	.10172201+001	.30517578-004	.33332267+005
A(17)=	.33477326+044	.47682095+000	.15258789-004	.31248937+005
A(18)=	.75118393+043	.22438588+000	.76293945-005	.29410706+005
A(19)=	.79595289+042	.10595979+000	.38146973-005	.27776722+005
A(20)=	.39949972+041	.50191377-001	.19073486-005	.26314737+005
A(21)=	.95244154+039	.23840857-001	.95367432-006	.24998950+005
A(22)=	.10812846+038	.11352766-001	.47683716-006	.23808476+005
A(23)=	.58587837+035	.54183548-002	.23841858-006	.22726227+005
A(24)=	.15182346+033	.25913819-002	.11920929-006	.21738087+005
....				
A(31)=	.14727324+006	.15521081-004	.93132257-009	.16665633+005
A(32)=	.11060493+001	.75101852-005	.46566129-009	.16128000+005
A(33)=	.40235184-005	.36377387-005	.23283064-009	.15623969+005
A(34)=	.70964748-011	.17637486-005	.11641532-009	.15150485+005
A(35)=	.60741218-017	.85593509-006	.58207661-010	.14704853+005
A(36)=	.25252498-023	.41573907-006	.29103830-010	.14284686+005
A(37)=	.51034029-030	.20209498-006	.14551915-010	.13887861+005
A(38)=	.50174759-037	.98316279-007	.72759576-011	.13512486+005

SUM OF A(J)'S = .22015874+045

FINAL BOUND = H(N) * (N**3/2)*(P E E N/H(N))**(H(N)/2) = .33314224+050

FINAL BOUND - SUM = .33314004+050 , AND (FINAL BOUND - SUM)/SUM = .15131811+006

N= 10,000,000 P= .50

H(N) = .1971412+002 CEILING OF H(N) = 20

H(N)/2 = .9857060+001

2 LOG N/LOG(1/P) = .4650699+002 CEILING OF 2 LOG N/LOG(1/P) = 47

		F(J)	P(J)	Q(J)
A(2)=	.4999999+014			
A(3)=	.4999999+021			
A(4)=	.2083332+027	.4166665+006	.1250000+000	.3333332+007
A(5)=	.3255204+032	.1562499+006	.6250000-001	.2499999+007
A(6)=	.2034502+037	.6249996+005	.3125000-001	.1999999+007
A(7)=	.5298177+041	.2604165+005	.1562500-001	.1666665+007
A(8)=	.5913140+045	.1116071+005	.7812500-002	.1428570+007
....				
A(15)=	.2262179+064	.4359647+002	.6103516-004	.7142846+006
A(16)=	.4602408+065	.2034502+002	.3051758-004	.6666656+006
A(17)=	.4389191+066	.9536727+001	.1525879-004	.6249989+006
A(18)=	.1969812+067	.4487871+001	.7629395-005	.5882342+006
A(19)=	.4174568+067	.2119272+001	.3814697-005	.5555545+006
A(20)=	.4190706+067	.1003866+001	.1907349-005	.5263147+006
A(21)=	.1998280+067	.4768362+000	.9536743-006	.4999989+006
A(22)=	.4537391+066	.2270648+000	.4768372-006	.4761894+006
A(23)=	.4917254+065	.1083718+000	.2384186-006	.4545444+006
A(24)=	.2548613+064	.5183000-001	.1192093-006	.4347816+006
A(25)=	.6329532+062	.2483521-001	.5960464-007	.4166656+006
A(26)=	.7545371+060	.1192090-001	.2980232-007	.3999990+006
A(27)=	.4324403+058	.5731200-002	.1490116-007	.3846143+006
A(28)=	.1193305+056	.2759467-002	.7450581-008	.3703693+006
....				
A(40)=	.6166984+000	.4664057-006	.1818989-011	.2564092+006
A(41)=	.1402204-006	.2273727-006	.9094947-012	.2499990+006
A(42)=	.1555234-013	.1109135-006	.4547474-012	.2439014+006
A(43)=	.8419470-021	.5413636-007	.2273737-012	.2380942+006
A(44)=	.2225997-028	.2643868-007	.1136868-012	.2325571+006
A(45)=	.2875743-036	.1291890-007	.5684342-013	.2272717+006
A(46)=	.1816293-044	.6315906-008	.2842171-013	.2222212+006
A(47)=	.5611076-053	.3089302-008	.1421085-013	.2173903+006

SUM OF A(J)'S = .1332615+068

FINAL BOUND = H(N) * (N**3/2)*(P*E*E*N/H(N))**(H(N)/2) = .4226339+074

FINAL BOUND - SUM = .4226338+074 , AND (FINAL BOUND - SUM)/SUM = .3171462+007

Table 3

N= 100,000,000 P= .50

H(N) = .2284340+002 CEILING OF H(N) = 23

H(N)/2 = .1142170+002

2 LOG N/LOG(1/P) = .5315085+002 CEILING OF 2 LOG N/LOG(1/P) = 54

Table 4

	A(J)	F(J)	P(J)	Q(J)
A(2)=	.5000000+016			
A(3)=	.5000000+024			
A(4)=	.2083333+031	.4166666+007	.1250000+000	.3333333+008
A(5)=	.3255208+037	.1562500+007	.6250000-001	.2500000+008
A(6)=	.2034505+043	.6250000+006	.3125000-001	.2000000+008
A(7)=	.5298189+048	.2604166+006	.1562500-001	.1666667+008
A(8)=	.5913157+053	.1116071+006	.7812500-002	.1428571+008
....				
A(18)=	.1969842+085	.4487878+002	.7629395-005	.5882352+007
A(19)=	.4174639+086	.2119276+002	.3814697-005	.5555554+007
A(20)=	.4190784+087	.1003868+002	.1907349-005	.5263157+007
A(21)=	.1998321+088	.4768371+001	.9536743-006	.4999999+007
A(22)=	.4537493+088	.2270653+001	.4768372-006	.4761904+007
A(23)=	.4917375+088	.1083721+001	.2384186-006	.4545453+007
A(24)=	.2548681+088	.5183011+000	.1192093-006	.4347825+007
A(25)=	.6329716+087	.2483526+000	.5960464-007	.4166666+007
A(26)=	.7545607+086	.1192093+000	.2980232-007	.3999999+007
A(27)=	.4324549+085	.5731214-001	.1490116-007	.3846153+007
A(28)=	.1193348+084	.2759474-001	.7450581-008	.3703703+007
A(29)=	.1587702+082	.1330460-001	.3725290-008	.3571428+007
A(30)=	.1019767+080	.6422912-002	.1862645-008	.3448275+007
A(31)=	.3165773+077	.3104408-002	.9313226-009	.3333332+007
....				
A(47)=	.5611644-006	.3089315-007	.1421085-013	.2173912+007
A(48)=	.8483640-014	.1511792-007	.7105427-014	.2127659+007
A(49)=	.6279152-022	.7401483-008	.3552714-014	.2083332+007
A(50)=	.2276328-030	.3625216-008	.1776357-014	.2040815+007
A(51)=	.4043569-039	.1776356-008	.8881784-015	.1999999+007
A(52)=	.3520989-048	.8707627-009	.4440892-015	.1960783+007
A(53)=	.1503493-057	.4270086-009	.2220446-015	.1923076+007
A(54)=	.3149456-067	.2094759-009	.1110223-015	.1886791+007

SUM OF A(J)'S = .1517758+089

FINAL BOUND = H(N) * (N**3/2)*(P*E*E*N/H(N))**(H(N)/2) = .4960263+096

FINAL BOUND - SUM = .4960263+096 , AND (FINAL BOUND - SUM)/SUM = .3268151+008

N= 1,000,000,000   P=   .50

H(N) =   .2599541+002   CEILING OF H(N) =   26

H(N)/2 =   .1299770+002

2 LOG N/LOG(1/P) =   .5979471+002   CEILING OF 2 LOG N/LOG(1/P) =   60

		F(J)	P(J)	Q(J)
A(2)=	.5000000+018			
A(3)=	.5000000+027			
A(4)=	.2083333+035	.4166667+008	.1250000+000	.3333333+009
A(5)=	.3255208+042	.1562500+008	.6250000-001	.2500000+009
A(6)=	.2034505+049	.6250000+007	.3125000-001	.2000000+009
A(7)=	.5298191+055	.2604167+007	.1562500-001	.1666667+009
A(8)=	.5913159+061	.1116071+007	.7812500-002	.1428571+009
....				
A(21)=	.1998325+109	.4768371+002	.9536743-006	.5000000+008
A(22)=	.4537503+110	.2270653+002	.4768372-006	.4761905+008
A(23)=	.4917387+111	.1083721+002	.2384186-006	.4545454+008
A(24)=	.2548688+112	.5183012+001	.1192093-006	.4347826+008
A(25)=	.6329734+112	.2483527+001	.5960464-007	.4166667+008
A(26)=	.7545631+112	.1192093+001	.2980232-007	.4000000+008
A(27)=	.4324564+112	.5731216+000	.1490116-007	.3846154+008
A(28)=	.1193352+112	.2759474+000	.7450581-008	.3703704+008
A(29)=	.1587708+111	.1330461+000	.3725290-008	.3571428+008
A(30)=	.1019771+110	.6422914-001	.1862645-008	.3448276+008
A(31)=	.3165787+108	.3104408-001	.9313226-009	.3333333+008
A(32)=	.4755434+106	.1502133-001	.4656613-009	.3225806+008
A(33)=	.3460033+104	.7275957-002	.2328306-009	.3125000+008
A(34)=	.1220609+102	.3527737-002	.1164153-009	.3030303+008
....				
A(53)=	.1503512-004	.4270088-008	.2220446-015	.1923077+008
A(54)=	.3149498-013	.2094760-008	.1110223-015	.1886792+008
A(55)=	.3237634-022	.1027984-008	.5551115-016	.1851852+008
A(56)=	.1633862-031	.5046468-009	.2775558-016	.1818182+008
A(57)=	.4048997-041	.2478176-009	.1387779-016	.1785714+008
A(58)=	.4929045-051	.1217350-009	.6938894-017	.1754386+008
A(59)=	.2948459-061	.5981805-010	.3469447-017	.1724138+008
A(60)=	.8669085-072	.2940209-010	.1734723-017	.1694915+008

SUM OF A(J)'S =   .2265041+113

FINAL BOUND = H(N) * (N**3/2)*(P*E*E*N/H(N))**(H(N)/2) =   .7599492+121

FINAL BOUND - SUM =   .7599492+121   , AND (FINAL BOUND - SUM)/SUM =   .3355123+009

Table 5

P = .50

N	(1)	(2)	(3)	(4)	(5)	(6)	(7)
.5000000+004	.1805983+020	.1302394+020	.2488338+023	-.2788448+000	.1376830+004	.3834284+001	.4834284+001
.1000000+005	.1913053+023	.1529260+023	.5352462+026	-.2006180+000	.2796864+004	.4277846+001	.5277846+001
.1500000+005	.1487488+025	.1259138+025	.6298578+028	-.1535139+000	.4233373+004	.4539250+001	.5539250+001
.2000000+005	.3703829+026	.3261050+026	.2104276+030	-.1195463+000	.5680352+004	.4725505+001	.5725505+001
.2500000+005	.4820039+027	.4372233+027	.3439576+031	-.9290496-001	.7134993+004	.4870396+001	.5870396+001
.3000000+005	.4111798+028	.3820045+028	.3534772+032	-.7095518-001	.8595657+004	.4989041+001	.5989041+001
.3500000+005	.2603465+029	.2467373+029	.2619678+033	-.5227349-001	.1006128+005	.5089531+001	.6089531+001
.4000000+005	.1319890+030	.1272372+030	.1522110+034	-.3600187-001	.1153110+005	.5176706+001	.6176706+001
.4500000+005	.5631474+030	.5509933+030	.7324047+034	-.2158252-001	.1300456+005	.5253695+001	.6253695+001
.5000000+005	.2093189+031	.2075120+031	.3031405+035	-.8632047-002	.1448123+005	.5322638+001	.6322638+001
.5500000+005	.6949372+031	.6971086+031	.1109244+036	.3124553-002	.1596078+005	.5385063+001	.6385063+001
.6000000+005	.2099625+032	.2128792+032	.3662570+036	.1389133-001	.1744292+005	.5442101+001	.6442101+001
.6500000+005	.5856279+032	.5995799+032	.1108501+037	.2382390-001	.1892743+005	.5494610+001	.6494610+001
.7000000+005	.1525008+033	.1575400+033	.3113319+037	.3304373-001	.2041410+005	.5543260+001	.6543260+001
.7500000+005	.3741117+033	.3896925+033	.8194461+037	.4164741-001	.2190278+005	.5588581+001	.6588581+001
.8000000+005	.8709372+033	.9142342+033	.2037499+038	.4971308-001	.2339333+005	.5631000+001	.6631000+001
.8500000+005	.1935742+034	.2046669+034	.4817406+038	.5730483-001	.2488561+005	.5670868+001	.6670868+001
.9000000+005	.4128270+034	.4394444+034	.1089059+039	.6447590-001	.2637953+005	.5708476+001	.6708476+001
.9500000+005	.8483813+034	.9088463+034	.2364946+039	.7127099-001	.2787498+005	.5744066+001	.6744066+001
.1000000+006	.1686107+035	.1817165+035	.4952580+039	.7772802-001	.2937188+005	.5777846+001	.6777846+001

Table 6

P = .50

N	(1)	(2)	(3)	(4)	(5)	(6)	(7)
.5000000+006	.2201587+045	.2829383+045	.3331422+050	.2851559+000	.1513181+006	.6844423+001	.7844423+001
.1000000+007	.1426012+050	.1963770+050	.4366768+055	.3771068+000	.3062215+006	.7307293+001	.8307293+001
.1500000+007	.1248594+053	.1787434+053	.5773221+058	.4315573+000	.4623768+006	.7578908+001	.8578908+001
.2000000+007	.1743476+055	.2563739+055	.1079780+061	.4704751+000	.6193246+006	.7771980+001	.8771980+001
.2500000+007	.8671238+056	.1301397+057	.6736275+062	.5008205+000	.7768518+006	.7921934+001	.8921934+001
.3000000+007	.2216718+058	.3382082+058	.2072268+064	.5257155+000	.9348351+006	.8044579+001	.9044579+001
.3500000+007	.3555028+059	.5499038+059	.3886338+065	.5468338+000	.1093194+007	.8148357+001	.9148357+001
.4000000+007	.4035479+060	.6316245+060	.5051905+066	.5651784+000	.1251871+007	.8238316+001	.9238316+001
.4500000+007	.3508257+061	.5547952+061	.4949541+067	.5813984+000	.1410825+007	.8317711+001	.9317711+001
.5000000+007	.2466300+062	.3936063+062	.3872152+068	.5959385+000	.1570024+007	.8388768+001	.9388768+001
.5500000+007	.1457997+063	.2346087+063	.2521521+069	.6091163+000	.1729441+007	.8453077+001	.9453077+001
.6000000+007	.7462374+063	.1209776+064	.1409684+070	.6211672+000	.1889055+007	.8511809+001	.9511809+001
.6500000+007	.3381322+064	.5519231+064	.6927826+070	.6322701+000	.2048850+007	.8565858+001	.9565858+001
.7000000+007	.1380256+065	.2267160+065	.3048726+071	.6425643+000	.2208811+007	.8615916+001	.9615916+001
.7500000+007	.5146806+065	.8503350+065	.1219240+072	.6521606+000	.2368925+007	.8662533+001	.9662533+001
.8000000+007	.1773039+066	.2945281+066	.4484340+072	.6611481+000	.2529182+007	.8706153+001	.9706153+001
.8500000+007	.5695453+066	.9509130+066	.1531833+073	.6696001+000	.2689571+007	.8747138+001	.9747138+001
.9000000+007	.1719173+067	.2884046+067	.4899793+073	.6775773+000	.2850086+007	.8785790+001	.9785790+001
.9500000+007	.4908094+067	.8270779+067	.1477690+074	.6851306+000	.3010719+007	.8822359+001	.9822359+001
.1000000+008	.1332615+068	.2255188+068	.4226339+074	.6923031+000	.3171462+007	.8857060+001	.9857060+001

Table 7

P = .50

N	(1)	(2)	(3)	(4)	(5)	(6)	(7)
.5000000+007	.2466300+062	.3936063+062	.3872152+068	.5959385+000	.1570024+007	.8388768+001	.9388768+001
.1000000+008	.1332615+068	.2255188+068	.4226339+074	.6923031+000	.3171462+007	.8857060+001	.9857060+001
.1500000+008	.4052302+071	.7088401+071	.1938606+078	.7492281+000	.4783962+007	.9131620+001	.1013162+002
.2000000+008	.1371133+074	.2454135+074	.8780038+080	.7898587+000	.6403488+007	.9326688+001	.1032669+002
.2500000+008	.1356148+076	.2470238+076	.1088738+083	.8215099+000	.8028165+007	.9478140+001	.1047814+002
.3000000+008	.6083297+077	.1123863+078	.5874580+084	.8474577+000	.9656901+007	.9601978+001	.1060198+002
.3500000+008	.1570070+079	.2935178+079	.1772450+086	.8694565+000	.1128899+008	.9706744+001	.1070674+002
.4000000+008	.2692192+080	.5084359+080	.3479367+087	.8885570+000	.1292391+008	.9797543+001	.1079754+002
.4500000+008	.3368279+081	.6418049+081	.4904658+088	.9054387+000	.1456132+008	.9877668+001	.1087767+002
.5000000+008	.3279965+082	.6299391+082	.5313840+089	.9205666+000	.1620090+008	.9949369+001	.1094937+002
.5500000+008	.2604051+083	.5036947+083	.4646265+090	.9342731+000	.1784245+008	.1001425+002	.1101425+002
.6000000+008	.1744771+084	.3396727+084	.3399818+091	.9468040+000	.1948576+008	.1007351+002	.1107351+002
.6500000+008	.1012955+085	.1983718+085	.2140445+092	.9583463+000	.2113069+008	.1012803+002	.1112803+002
.7000000+008	.5201879+085	.1024274+086	.1184837+093	.9690456+000	.2277710+008	.1017852+002	.1117852+002
.7500000+008	.2402061+086	.4753721+086	.5867009+093	.9790173+000	.2442489+008	.1022554+002	.1122554+002
.8000000+008	.1010746+087	.2009723+087	.2635417+094	.9883548+000	.2607396+008	.1026953+002	.1126953+002
.8500000+008	.3918177+087	.7825126+087	.1086284+095	.9971344+000	.2772423+008	.1031086+002	.1131086+002
.9000000+008	.1412083+088	.2831818+088	.4148080+095	.1005419+001	.2937562+008	.1034984+002	.1134984+002
.9500000+008	.4767521+088	.9598274+088	.1479270+096	.1013263+001	.3102807+008	.1038671+002	.1138671+002
.1000000+009	.1517758+089	.3066949+089	.4960263+096	.1020710+001	.3268151+008	.1042170+002	.1142170+002

Table 8

P = .50

N	(1)	(2)	(3)	(4)	(5)	(6)	(7)
.5000000+008	.3279965+082	.6299391+082	.5313840+089	.9205666+000	.1620090+008	.9949369+001	.1094937+002
.1000000+009	.1517758+089	.3066949+089	.4960263+096	.1020710+001	.3268151+008	.1042170+002	.1142170+002
.1500000+009	.1625939+093	.3381594+093	.8009628+100	.1079779+001	.4926155+008	.1069848+002	.1169848+002
.2000000+009	.1346102+096	.2856299+096	.8871480+103	.1121903+001	.6590494+008	.1089506+002	.1189506+002
.2500000+009	.2667064+098	.5746717+098	.2202858+106	.1154697+001	.8259485+008	.1104765+002	.1204765+002
.3000000+009	.2111496+100	.4606377+100	.2097170+108	.1181570+001	.9932149+008	.1117240+002	.1217240+002
.3500000+009	.8812673+101	.1942616+102	.1022961+110	.1204344+001	.1160785+009	.1127792+002	.1227792+002
.4000000+009	.2291970+103	.5097595+103	.3045138+111	.1224111+001	.1328612+009	.1136936+002	.1236936+002
.4500000+009	.4141546+104	.9283596+104	.6198502+112	.1241577+001	.1496664+009	.1145005+002	.1245005+002
.5000000+009	.5603961+105	.1264940+106	.9330113+113	.1257226+001	.1664914+009	.1152224+002	.1252224+002
.5500000+009	.5991907+106	.1361002+107	.1098521+115	.1271401+001	.1833342+009	.1158757+002	.1258757+002
.6000000+009	.5268954+107	.1203617+108	.1054808+116	.1284357+001	.2001931+009	.1164722+002	.1264722+002
.6500000+009	.3928464+108	.9020892+108	.8527384+116	.1296290+001	.2170666+009	.1170211+002	.1270211+002
.7000000+009	.2543376+109	.5868457+109	.5950324+117	.1307349+001	.2339537+009	.1175294+002	.1275294+002
.7500000+009	.1457239+110	.3377377+110	.3655532+118	.1317655+001	.2508533+009	.1180027+002	.1280027+002
.8000000+009	.7503366+110	.1746261+111	.2009136+119	.1327304+001	.2677646+009	.1184455+002	.1284455+002
.8500000+009	.3516141+111	.8215023+111	.1000999+120	.1336375+001	.2846867+009	.1188615+002	.1288615+002
.9000000+009	.1515363+112	.3553427+112	.4570625+120	.1344934+001	.3016191+009	.1192538+002	.1292538+002
.9500000+009	.6059498+112	.1425822+113	.1930321+121	.1353037+001	.3185611+009	.1196249+002	.1296249+002
.1000000+010	.2265041+113	.5347148+113	.7599492+121	.1360729+001	.3355123+009	.1199770+002	.1299770+002

Table 9

P = .50

N	(1)	(2)	(3)	(4)	(5)	(6)	(7)
.5000000+009	.5603961+105	.1264940+106	.9330113+113	.1257226+001	.1664914+009	.1152224+002	.1252224+002
.1000000+010	.2265041+113	.5347148+113	.7599492+121	.1360729+001	.3355123+009	.1199770+002	.1299770+002
.1500000+010	.8641567+117	.2092741+118	.4367789+126	.1421715+001	.5054395+009	.1227621+002	.1327621+002
.2000000+010	.1763510+121	.4347367+121	.1192037+130	.1465178+001	.6759456+009	.1247397+002	.1347397+002
.2500000+010	.7038609+123	.1758949+124	.5960839+132	.1499001+001	.8468774+009	.1262746+002	.1362746+002
.3000000+010	.9878881+125	.2496103+126	.1005814+135	.1526706+001	.1018146+010	.1275292+002	.1375292+002
.3500000+010	.6692256+127	.1706645+128	.7961723+136	.1550179+001	.1189692+010	.1285904+002	.1385904+002
.4000000+010	.2648266+129	.6807496+129	.3605547+138	.1570549+001	.1361475+010	.1295099+002	.1395099+002
.4500000+010	.6930470+130	.1793982+131	.1062763+140	.1588544+001	.1533464+010	.1303212+002	.1403212+002
.5000000+010	.1306243+132	.3402322+132	.2227974+141	.1604662+001	.1705635+010	.1310471+002	.1410471+002
.5500000+010	.1885081+133	.4937519+133	.3540125+142	.1619261+001	.1877970+010	.1317039+002	.1417039+002
.6000000+010	.2179796+134	.5738539+134	.4469567+143	.1632603+001	.2050452+010	.1323037+002	.1423037+002
.6500000+010	.2090938+135	.5530301+135	.4648301+144	.1644890+001	.2223070+010	.1328555+002	.1428555+002
.7000000+010	.1709480+136	.4540850+136	.4095593+145	.1656275+001	.2395812+010	.1333664+002	.1433664+002
.7500000+010	.1217148+137	.3245991+137	.3126449+146	.1666884+001	.2568669+010	.1338422+002	.1438422+002
.8000000+010	.7674828+137	.2055748+138	.2105527+147	.1676815+001	.2741633+010	.1342873+002	.1442873+002
.8500000+010	.4356167+138	.1170133+139	.1269691+148	.1686152+001	.2914698+010	.1347054+002	.1447054+002
.9000000+010	.2247854+139	.6057876+139	.6941052+148	.1694960+001	.3087858+010	.1350997+002	.1450997+002
.9500000+010	.1065816+140	.2881217+140	.3475737+149	.1703298+001	.3261106+010	.1354727+002	.1454727+002
.1000000+011	.4683053+140	.1269675+141	.1608365+150	.1711213+001	.3434438+010	.1358267+002	.1458267+002

Table 10

V

Conclusions and Open Problems

Several algorithms for NP-hard problems have been shown to give optimal or near-optimal solutions with probability one.

By designing and analysing algorithms for many different NP-hard problems, we intend to provide some insight on a uniform and general probabilistic approach to solve all the NP-hard problems derived from NP-complete problems, in spite of their different structural characteristics (derived in the sense that, for example, if the NP-complete problem is to answer the question "given a positive integer k and a graph with n vertices ($k \leq n$) is there a clique of size k?", the derived NP-hard problem would be "given a graph find the largest clique of the graph"). To some extent, we were successful in devising a uniform method to derive fast probabilistic algorithms to solve different NP-hard problems.

For the problems studied in this thesis, we have been unable to find in the literature any result stronger than our algorithms and the corresponding theorems on their probabilistic performances.

Some of the algorithms presented are simple, but their analyses are often difficult. Heuristics may occur to the reader that would improve the performance of the algorithms. But introduction of heuristics seems to introduce probabilistic dependencies that are very difficult to analyse. However, if they can be shown not to reduce the accuracy of the algorithms, they may still be used in practice without weakening the results.

One theoretical conjecture is that an algorithm which is optimal with probability one for one NP-hard problem is "polynomially translatable" (in the sense of Karp[1972]) to another algorithm to solve a second NP-hard problem, preserving the probabilistic properties.

In addition to the general observations above, there are other questions of varying degrees of importance which could be explored in an extension of this work or which remain as open problems. The following is a partial list.

(1) Find experimental results by implementing the algorithms presented;

(2) By experimentation, measure possible improvements of the algorithms by adding some heuristics;

(3) As a consequence of (1) above, get an accurate value of the universal constant β of Theorem 4;

(4) Extend the algorithms and the results of Chapter II to any normed space;

(5) Design algorithms for the problems considered in Chapter IV, faster than the ones presented, but still optimal or within a ratio r, $1<r<2$, with probability one;

(6) Find bounds for the variance of the running time of Algorithm D;

(7) In connection with (6), find a function f(n) such that the

running time of Algorithm D is asymptotic to f(n) in probability or with probability one;

(8) By applying the uniform method presented in Chapters III and IV, find fast algorithms for additional NP-hard problems which are optimal or near-optimal with probability one;

(9) Consider another probability distribution for the problems studied in this thesis, and develop uniform methods to design algorithms which are optimal or near-optimal with probability one under this new distribution.

VI

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