LOCALLY UNIQUE SOLUTIONS OF QUADRATIC PROGRAMS, LINEAR AND NONLINEAR COMPLEMENTARITY PROBLEMS

by

O. L. Mangasarian

Computer Sciences Technical Report #345

February 1979

Locally Unique Solutions of Quadratic Programs, Linear and Nonlinear Complementarity Problems [1)]

O. L. Mangasarian [2)]

ABSTRACT

It is shown that McCormick's second order sufficient optimality conditions are also necessary for a solution to a quadratic program to be locally unique and hence these conditions completely characterize a locally unique solution of any quadratic program. This result is then used to give characterizations of a locally unique solution to the linear complementarity problem. Sufficient conditions are also given for local uniqueness of solutions of the nonlinear complementarity problem.

1) Research supported by National Science Foundation Grant MCS74-20584 A02.

2) Computer Sciences and Industrial Engineering Departments, University of Wisconsin, 1210 West Dayton Street, Madison, Wisconsin 53706.

1. Introduction

In [11] where global uniqueness of a linear programming solution was characterized, it was shown that, among other conditions, McCormick's [8,3] second order sufficient optimality conditions were also necessary for uniqueness and hence these conditions completely characterized a unique solution to a linear program. Curiously, it turns out that the same situation also holds locally for quadratic programs. That is, McCormick's second order sufficeint optimality conditions, which ensure that a solution to a quadratic programming problem is locally unique, are shown to be also necessary for local uniqueness. Thus again in the case of quadratic programming problems, McCormick's conditions completely characterize local uniqueness (Theorem 2.1). This result is used then to characterize locally unique solutions of the linear complementarity problem (Theorems 3.1 and 3.5). This characterization of locally unique solutions of the linear complementarity problem includes as a special case the local uniqueness result of Murty [16] for nondegenerate matrices, that is those with nonsingular principal submatrices. In addition, Robinson's strong regularity condition [17], which among other things ensures the local uniqueness of a linear complementarity problem solution, implies (Corollary 3.6) the hypothesis of one of our local uniqueness characterizations (Theorem 3.5). We also extend Kaneko's [4] uniqueness result for linear complementarity problems with positive semidefinite matrices to a local uniqueness result for linear complementarity problems with arbitrary matrices (Theorem 3.8). Finally in Section 4 of the paper we extend the results of Section 3 to the nonlinear complementarity problem and give sufficient conditions for the local uniqueness of solutions of the nonlinear complementarity problem. Sufficient conditions for the global uniqueness of solutions of the nonlinear complementarity problem have been given in [1,5,13,14,15]. When the functions defining the nonlinear complementarity problem are twice differentiable at the solution point we show that these various sufficient conditions for global uniqueness imply the hypothesis one of our sufficient conditions for local uniqueness (Corollary 4.4). Similarly we show that Kojima's [7] condition which ensures that a certain function is one-to-one and which in turn guarantees the local uniqueness of a nonlinear complementarity problem solution, implies the hypothesis of another one of our sufficient conditions for local uniqueness (Corollary 4.7).

We note that in [9] a slightly sharper second order necessary optimality condition than that of McCormick [8,3] was shown to be also sufficient for a point to be a local but not necessarily strict minimum of a quadratic program. This characterization cannot, of course, be used to establish the local uniqueness of a solution to either a quadratic program or a linear complementarity problem.

2. Locally Unique Solutions of Quadratic Programs

We consider the quadratic program of finding an $\bar{x}$ in R^n that will

$$\text{minimize } \frac{1}{2}\, xQx + px \qquad \text{subject to } Ax \leqq b,\ Cx = d \tag{2.1}$$

where Q, A and C are given $n\times n$, $m\times n$ and $k\times n$ real matrices respectively, with Q being symmetric, and p, b and d are given vectors in R^n, R^m and R^k respectively. A point x in R^n satisfying the constraints $Ax \leqq b$ and $Cx = d$ is called <u>feasible</u>. A feasible point $\bar{x}$ in R^n such that $\frac{1}{2}\bar{x}Q\bar{x} + p\bar{x} \leqq \frac{1}{2} xQx + px$ for all feasible x in some open Euclidean ball around $\bar{x}$ is called a <u>local solution</u> of (2.1), and if equality holds only for $x = \bar{x}$ then $\bar{x}$ is said to be a <u>locally unique solution</u> of (2.1). It is well known [6,10] that if $\bar{x}$ is a local solution of (2.1) then there exists $(\bar{u},\bar{v})$ in R^{m+k} such that $(\bar{x},\bar{u},\bar{v})$ satisfies the Karush-Kuhn-Tucker conditions

$$\begin{aligned} &Q\bar{x} + A^T\bar{u} + C^T\bar{v} + p = 0 \\ &A\bar{x} \leqq b,\ \bar{u} \geqq 0,\ \bar{u}(A\bar{x}-b) = 0,\ C\bar{x} = d \end{aligned} \tag{2.2}$$

where the superscript T denotes matrix transposition, whereas vectors are either row or column vectors depending on the context. If we further define the constraint index sets

$$\left.\begin{aligned} I &= \{i \mid A_i\bar{x}=b_i\} \\ J &= \{i \mid \bar{u}_i>0\} \subset I \\ K &= \{i \mid \bar{u}_i=0,\ A_i\bar{x}=b_i\} \subset I \end{aligned}\right\rangle \tag{2.3}$$

where subscripts denote matrix rows or vector elements, then McCormick's [8,3] second order sufficient optimality conditions consist of (2.2) and the implication that

$$\left.\begin{array}{r} A_i x = 0,\ i \in J \\ A_i x \leqq 0,\ i \in K \\ Cx = 0 \\ x \neq 0 \end{array}\right\rangle \Longrightarrow xQx > 0 \qquad (2.4)$$

These second order conditions (2.2)-(2.4) imply that [8,3] $\bar{x}$ is a locally unique solution of (2.1). We will now show that the converse is also true.

2.1 <u>Theorem</u> (Characterization of locally unique solutions of quadratic programs) The point $\bar{x}$ in R^n is a locally unique solution of the quadratic program (2.1) if and only if $\bar{x}$ and some $(\bar{u},\bar{v})$ in R^{m+k} satisfy the conditions (2.2)-(2.4).

<u>Proof</u> (Necessity) Because $\bar{x}$ is a local solution of (2.1), the Karush-Kuhn-Tucker conditions (2.2) are satisfied [10]. We will now show that if (2.4) does not hold then $\bar{x}$ is not a locally unique solution of (2.1). If (2.4) does not hold then there exists an x in R^n such that

$$A_i x = 0,\ i \in J \qquad (2.5)$$

$$A_i x \leqq 0,\ i \in K \qquad (2.6)$$

$$Cx = 0 \qquad (2.7)$$

$$x \neq 0 \qquad (2.8)$$

$$xQx \leqq 0 \qquad (2.9)$$

It follows then that

$$(\bar{x}Q+p)x = -(\bar{u}A+\bar{v}C)x$$

$$= -\sum_{i\in J} \bar{u}_i A_i x - \sum_{i\in K} \bar{u}_i A_i x - \bar{v}Cx = 0 \tag{2.10}$$

Define

$$\bar{\delta} = \underset{i\notin I,\, A_i x > 0}{\text{minimum}} \{-(\frac{A_i\bar{x}-b_i}{A_i x}),\ 1\} > 0 \tag{2.11}$$

Then for $0 < \delta \leqq \bar{\delta}$, it follows that $\bar{x} + \delta x \neq \bar{x}$ and

$$A_i(\bar{x}+\delta x) - b_i \leqq 0 \qquad \text{for } i \notin I \qquad \text{(By (2.11))}$$

$$A_i(\bar{x}+\delta x) - b_i \leqq 0 \qquad \text{for } i \in I \qquad \text{(By (2.5) and (2.6))}$$

$$C\,(\bar{x}+\delta x) - d = 0 \qquad \text{(By (2.7))}$$

$$\frac{1}{2}(\bar{x}+\delta x)\,Q\,(\bar{x}+\delta x) + p(\bar{x}+\delta x) - (\frac{1}{2}\,\bar{x}Q\bar{x}+p\bar{x})$$

$$= \delta(\bar{x}Q+p)x + \frac{1}{2}\,\delta^2 xQx \leqq 0 \qquad \text{(By (2.10) and (2.9))}$$

Hence for each $\delta\in(0,\bar{\delta}]$, $\bar{x} + \delta x$ is a feasible point distinct from $\bar{x}$ and

$$\frac{1}{2}(\bar{x}+\delta x)\,Q\,(\bar{x}+\delta x) + p(\bar{x}+\delta x) = \frac{1}{2}\,\bar{x}Q\bar{x} + p\bar{x}$$

for $\delta\in(0,\min\{\bar{\delta},\delta_0/\|x\|\})$, where δ_0 is the radius of the open Euclidean ball around $\bar{x}$ for which $\bar{x}$ is a local minimum of (2.1) and $\|x\|$ is the Euclidean norm of x. It follows that $\bar{x}$ cannot be a locally unique solution of (2.1).

(Sufficiency) See [8,3]. $\square$

3. Locally Unique Solutions of the Linear Complementarity Problem

We consider the linear complementarity problem [2,16] of finding an x in R^n such that

$$Mx + q \geqq 0,\ x \geqq 0,\ x(Mx+q) = 0 \tag{3.1}$$

where M is a given $n\times n$ real matrix and q is a given vector in R^n. A solution $\bar{x}$ is a <u>locally unique solution</u> of (3.1) if there exists an open Euclidean ball around $\bar{x}$ which contains no other solution of (3.1). It is obvious that $\bar{x}$ is a solution of the linear complementarity problem if and only if $M\bar{x} + q \geqq 0$, $\bar{x} \geqq 0$ and

$$0 = \bar{x}(M\bar{x}+q) = \text{minimum } \{x(Mx+q)\,|\,Mx+q\geqq 0,\ x\geqq 0\} \tag{3.2}$$

The Karush-Kuhn-Tucker conditions for the quadratic program (3.2) are

$$\begin{aligned} &(M+M^T)\bar{x} + q - M^T\bar{u} - \bar{v} = 0 \\ &M\bar{x} + q \geqq 0,\ \bar{u} \geqq 0,\ \bar{u}(M\bar{x}+q) = 0 \\ &\bar{x} \geqq 0,\ \bar{v} \geqq 0,\ \bar{v}\bar{x} = 0 \end{aligned} \tag{3.3}$$

for some $(\bar{u},\bar{v})$ in R^{2n}. It immediately follows from (3.2) that

$$\bar{u} = \bar{x} \quad \text{and} \quad \bar{v} = M\bar{x} + q$$

satisfy the conditions (3.3) by rendering them into

$$M\bar{x} + q \geqq 0,\ \bar{x} \geqq 0,\ \bar{x}(M\bar{x}+q) = 0 \tag{3.4}$$

which is precisely the linear complementarity problem (3.1) satisfied by $\bar{x}$. We can now apply Theorem 2.1 to obtain a local uniqueness result for (3.2) and equivalently for (3.1). We first define the index sets for $\bar{u} = \bar{x}$ and $\bar{v} = M\bar{x} + q$

$$\begin{aligned}
I &= \{i \,|\, M_i\bar{x}+q_i>0\} = \{i \,|\, \bar{v}_i>0\} \\
J &= \{i \,|\, \bar{x}_i>0\} = \{i \,|\, \bar{u}_i>0\} \qquad\qquad (3.5) \\
K &= \{i \,|\, M_i\bar{x}+q_i=0,\ \bar{x}_i=0\} = \{i \,|\, \bar{v}_i=0,\ \bar{u}_i=0\}
\end{aligned}$$

We further define submatrices of M such as M_J as that submatrix of M with rows M_i, $i \in J$, and M_{JK} as that submatrix of M with elements M_{ij}, $i \in J$, $j \in K$. Similarly we define the vector x_J as that vector with elements x_i, $i \in J$. The implication (2.4) when applied to the quadratic program (3.2) and its optimality conditions (3.3) becomes, upon recalling that $\bar{u} = \bar{x}$ and $\bar{v} = M\bar{x} + q$,

$$\left.\begin{aligned}
M_J x &= 0 \\
M_K x &\geqq 0 \\
x_I &= 0 \\
x_K &\geqq 0 \\
x &\neq 0
\end{aligned}\right\rangle \Longrightarrow xMx > 0$$

which is equivalent to

$$\left.\begin{aligned}
M_{JJ}x_J + M_{JK}x_K &= 0 \\
M_{KJ}x_J + M_{KK}x_K &\geqq 0 \\
x_K &\geqq 0 \\
(x_J,\ x_K) &\neq 0
\end{aligned}\right\rangle \Longrightarrow x_K(M_{KJ}x_J+M_{KK}x_K) > 0 \qquad (3.6)$$

Direct application now of Theorem 2.1 to the quadratic program (3.2) gives the following characterization of locally unique solutions of

(3.2) and equivalently of (3.1).

3.1 Theorem (Characterization of locally unique solutions of the linear complementarity problem) A point $\bar{x}$ in R^n is a locally unique solution of the linear complementarity problem (3.1) if and only if conditions (3.4)-(3.6) are satisfied.

The following corollaries are straightforward consequences of Theorem 3.1.

3.2 Corollary (Characterization of locally unique nondegenerate solutions of the linear complementarity problem) A nondegenerate solution $\bar{x}$ of the linear complementarity problem (3.1) (that is K in (3.5) is empty) is locally unique if and only if M_{JJ} is nonsingular, where J is defined in (3.5).

3.3 Corollary (Local uniqueness of the origin as a solution of the linear complementarity problem) The origin in R^n is a locally unique solution of the linear complementarity problem (3.1) if and only if $q \geqq 0$ and there exists no x_K such that

$$M_{KK}x_K \geqq 0,\ 0 \neq x_K \geqq 0,\ x_K M_{KK} x_K = 0 \tag{3.7}$$

where

$$K = \{i \mid q_i = 0\}$$

If $q > 0$ then the origin is a locally unique solution of the linear complementarity problem.

Note that if implication (3.6) does not hold, then there exists a partition $\{K_0, K_1\}$ of K, that is $K_0 \cup K_1 = K$ and $K_0 \cap K_1 = \phi$, such that $x_{K_0} = 0$ and

$$\begin{aligned} M_{JJ}x_J + M_{JK_1}x_{K_1} &= 0 \\ M_{K_1J}x_J + M_{K_1K_1}x_{K_1} &= 0 \\ (x_J, x_{K_1}) &\neq 0 \end{aligned} \tag{3.8}$$

Conditions (3.8) cannot be true if M has nonsingular principal submatrices. Thus from Theorem 3.1 follows a corollary which is also a consequence of Murty's result [16, Theorem 3.2] that the number of solutions of the linear complementarity problem is finite for all q in R^n if and only if M has nonsingular principal submatrices.

3.4 <u>Corollary</u> (Murty [16]) If M has nonsingular principal submatrices then each solution of the linear complementarity problem (3.1) is locally unique.

We now state and prove a paraphrase of Theorem 3.1 which may be more convenient at times. The hypothesis of this paraphrase is implied by Robinson's strong regularity condition [17].

3.5 <u>Theorem</u> (Characterization of locally unique solutions of the linear complementarity problem) A point $\bar{x}$ in R^n is a locally unique solution of the linear complementarity problem (3.1) if and only if $\bar{x}$ satisfies (3.4) and for J and K defined by (3.5)

$$\text{Columns of } \begin{pmatrix} M_{JJ} \\ M_{KJ} \end{pmatrix} \text{ are linearly independent} \tag{3.9}$$

and for each partition $\{K_0, K_1\}$ of K the following system has a solution (u_J, u_{K_0}, u_{K_1})

$$\begin{aligned} M_{JJ}^T u_J + M_{K_0 J}^T u_{K_0} + M_{K_1 J}^T u_{K_1} &= 0 \\ M_{JK_1}^T u_J + M_{K_0 K_1}^T u_{K_0} + M_{K_1 K_1}^T u_{K_1} &< 0 \\ u_{K_0} &> 0 \end{aligned} \tag{3.10}$$

<u>Proof</u> We need to show that the implication (3.6) is equivalent to the condition (3.9) and to (3.10) having a solution for each partition $\{K_0, K_1\}$ of K. The implication (3.6) is equivalent to this: For each partition $\{K_0, K_1\}$ of K the following system has <u>no</u> solution

$$\begin{aligned} M_{JJ} x_J + M_{JK_0} x_{K_0} + M_{JK_1} x_{K_1} &= 0 \\ M_{K_0 J} x_J + M_{K_0 K_0} x_{K_0} + M_{K_0 K_1} x_{K_1} &\geqq 0 \\ M_{K_1 J} x_J + M_{K_1 K_0} x_{K_0} + M_{K_1 K_1} x_{K_1} &= 0 \\ x_{K_0} &= 0 \\ x_{K_1} &\geqq 0 \\ (x_J \quad , \quad x_{K_0} \quad , \quad x_{K_1}) &\neq 0 \end{aligned} \tag{3.11}$$

This in turn is equivalent to the following system not having a solution for each partition $\{K_0, K_1\}$ of K and for each (c_J, c_{K_1})

$$\begin{aligned}
M_{JJ}\, x_J + M_{JK_1}\, x_{K_1} &= 0 \\
M_{K_0J}\, x_J + M_{K_0K_1} x_{K_1} &\geqq 0 \\
M_{K_1J}\, x_J + M_{K_1K_1} x_{K_1} &= 0 \\
x_{K_1} &\geqq 0 \\
- c_J\, x_J - c_{K_1} x_{K_1} &> 0
\end{aligned} \tag{3.12}$$

By Motzkin's theorem of the alternative [10] this is equivalent to the following system having a solution for each partition $\{K_0, K_1\}$ of K and for each (c_J, c_{K_1})

$$\begin{aligned}
M_{JJ}^T u_J + M_{K_0J}^T u_{K_0} + M_{K_1J}^T u_{K_1} &= c_J \\
M_{JK_1}^T u_J + M_{K_0K_1}^T u_{K_0} + M_{K_1K_1}^T u_{K_1} &\leqq c_{K_1} \\
u_{K_0} &\geqq 0
\end{aligned} \tag{3.13}$$

Because the first equation of (3.13) is solvable for each c_J it follows that the rows of $(M_{JJ}^T \;\; M_{K_0J}^T \;\; M_{K_1J}^T)$ are linearly independent which is equivalent to (3.9). By taking in (3.13) $c_J = -M_{K_0J}^T e_{K_0}$ $c_{K_1} = -M_{K_0K_1}^T e_{K_0} - e_{K_1}$ where e_{K_0} and e_{K_1} are vectors of ones it follows that the system (3.10) has a solution for each partition $\{K_0, K_1\}$ of K. We have thus shown that (3.13) having a solution for each partition of $\{K_0, K_1\}$ of K and each (c_J, c_{K_1}) implies that (3.9) holds and (3.10) has a solution for each partition $\{K_0, K_1\}$ of K. We now show the converse. Let $\{K_0, K_1\}$ be any partition of K and take any (c_J, c_{K_1}). Let $(\bar{u}_J, \bar{u}_{K_0}, \bar{u}_{K_1})$ be a solution of the first equation of (3.13), which by the linear independence of the columns of $\begin{pmatrix} M_{JJ} \\ M_{KJ} \end{pmatrix}$ must have a solution for each c_J. Because (3.10) has a solution for each partition $\{K_0, K_1\}$ of K it follows that there exists $(\hat{u}_J, \hat{u}_{K_0}, \hat{u}_{K_1})$ satisfying

$$\begin{aligned} M_{JJ}^T \hat{u}_J + M_{K_0J}^T \hat{u}_{K_0} + M_{K_1J}^T \hat{u}_{K_1} &= 0 \\ M_{JK_1}^T \hat{u}_J + M_{K_0K_1}^T \hat{u}_{K_0} + M_{K_1K_1}^T \hat{u}_{K_1} &< 0 \\ \hat{u}_{K_0} &> 0 \end{aligned}$$

Hence for sufficiently large positive λ, the point $(\bar{u}_J+\lambda\hat{u}_J, \bar{u}_{K_0}+\lambda\hat{u}_{K_0}, \bar{u}_{K_1}+\lambda\hat{u}_{K_1})$ solves (3.13). $\square$

We conclude by stating and proving a corollary to Theorem 3.5 which gives a sufficient condition for the local uniqueness of a solution to the linear complementarity problem. Robinson's strong regularity condition for the linear complementarity problem [17, Theorem 3.1] is that M_{JJ} is nonsingular and all the principal subdeterminants of the Schur complement of M_{JJ} in $\begin{pmatrix} M_{JJ} & M_{JK} \\ M_{KJ} & M_{KK} \end{pmatrix}$,

that is $M_{KK} - M_{KJ} M_{JJ}^{-1} M_{JK}$, be positive. Robinson's condition implies among other things the local uniqueness of a solution of the linear complementarity problem. It is obvious that Robinson's strong regularity condition ensures the satisfaction of the hypothesis of the following sufficient condition for local uniqueness of a linear complementarity problem solution.

3.6 Corollary (Sufficient condition for local uniqueness of a linear complementarity problem solution) Let $\bar{x}$ be a solution of the linear complementarity problem (3.1) and let J and K be defined by (3.5). If M_{JJ} and the principal submatrices of the Schur complement $M_{KK} - M_{KJ} M_{JJ}^{-1} M_{JK}$ are nonsingular then $\bar{x}$ a locally unique solution of (3.1).

Proof The nonsingularity of M_{JJ} ensures the satisfaction of condition (3.9). Condition (3.10) is satisfied by taking $u_{K_0} = e_{K_0}$, a vector of ones, and noting that

$$M_{JJ}^T u_J + M_{K_1J}^T u_{K_1} = -M_{K_0J}^T e_{K_0}$$

$$M_{JK_1}^T u_J + M_{K_1K_1}^T u_{K_1} = -e_{K_1} - M_{K_0K_1}^T e_{K_0}$$

where e_{K_1} is a vector of ones, has a solution for any $K_1 \subset K$ because M_{JJ} and $M_{K_1K_1}^T - M_{JK_1}^T (M_{JJ}^T)^{-1} M_{K_1J}^T$ are nonsingular. Hence by Theorem 3.5, $\bar{x}$ is a locally unique solution of the linear complementarity problem. □

Corollary 3.6 can be stated in another equivalent and simpler way as follows.

3.7 <u>Corollary</u> (Sufficient condition for local uniqueness of a linear complementarity problem solution) Let $\bar{x}$ be a solution of the linear complementarity problem (3.1), let J and K be defined by (3.5) and let $H = J \cup K$. If all principal submatrices of M_{HH} containing M_{JJ} are nonsingular then $\bar{x}$ is a locally unique solution of (3.1).

Note that Corollary 3.7 is stronger then Corollary 3.4 because in Corollary 3.7 M_{JJ} itself need not have nonsingular principal submatrices nor does M_{II}.

We conclude this section by giving a characterization of uniqueness when M_{JJ} is nonsingular. This result follows easily from Theorem 3.1 and extends a result of Kaneko [4] for positive semi-definite matrices to arbitrary matrices.

3.8 <u>Theorem</u> (Characterization of locally unique solutions in terms of principal pivotal transforms) Let $\bar{x}$ be a solution of the linear complementarity problem (3.1), let J and K be defined by (3.5) and let M_{JJ} be nonsingular. Then $\bar{x}$ is a locally unique solution of (3.1) if and only if $x_K = 0$ is the only solution of the following linear complementarity problem

$$\begin{aligned} (M_{KK} - M_{KJ} M_{JJ}^{-1} M_{JK})\ x_K &\geqq 0 \\ x_K &\geqq 0 \\ x_K(M_{KK} - M_{KJ} M_{JJ}^{-1} M_{JK})\ x_K &= 0 \end{aligned} \tag{3.14}$$

Proof That (3.14) has only zero as a solution is a paraphrase of implication (3.6) when M_{JJ} is nonsingular. This theorem then follows from Theorem 3.1. $\square$

4. Sufficient Conditions for the Local Uniqueness of Solutions of the Nonlinear Complementarity Problem

The nonlinear complementarity problem [1] is that of finding an x in R^n such that

$$F(x) \geqq 0,\ x \geqq 0,\ xF(x) = 0 \tag{4.1}$$

where F is a given function from R^n into itself. A solution $\bar{x}$ is a locally unique solution of (4.1) if there exists an open Euclidean ball around $\bar{x}$ which contains no other solution of (4.1). Locally unique solutions are required in the application of certain computational algorithms such as Newton or quasi-Newton methods [12]. We shall follow here the same approach as that of the previous section and note that $\bar{x}$ is a solution of the nonlinear complementarity problem (4.1) if and only if $F(\bar{x}) \geqq 0$, $\bar{x} \geqq 0$ and

$$0 = \bar{x}F(\bar{x}) = \text{minimum } \{xF(x)\,|\,F(x) \geqq 0,\ x \geqq 0\} \tag{4.2}$$

We can now apply McCormick's [8,3] second order sufficient optimality conditions for the local uniqueness of $\bar{x}$ as a solution to (4.2) and consequently obtain the equivalent local uniqueness of $\bar{x}$ as a solution of (4.1). Because the problem (4.2) is not quadratic these second order sufficient conditions are not necessary for local uniqueness anymore. The Karush-Kuhn-Tucker conditions for the nonlinear program (4.2) are

$$\begin{aligned} &F(\bar{x}) + \bar{x}\nabla F(\bar{x}) - \bar{u}\nabla F(\bar{x}) - \bar{v} = 0 \\ &F(\bar{x}) \geqq 0,\ \bar{u} \geqq 0,\ \bar{u}F(\bar{x}) = 0 \\ &\bar{x} \geqq 0,\ \ \bar{v} \geqq 0,\ \ \bar{v}\bar{x} = 0 \end{aligned} \tag{4.3}$$

for some $(\bar{u},\bar{v})$ in R^{2n}, where $\nabla F(\bar{x})$ is the Jacobian of F at $\bar{x}$, row i of which is given by $\nabla F_i(\bar{x})$. It follows from (4.2) that $\bar{u}=\bar{x}$ and $\bar{v}=F(\bar{x})$ satisfy (4.3) by rendering it into

$$F(\bar{x}) \geqq 0,\ \bar{x} \geqq 0,\ \bar{x}F(\bar{x}) = 0 \tag{4.4}$$

which is the nonlinear complementarity problem (4.1) satisfied by $\bar{x}$. We can now apply McCormick's [8,3] second order sufficient optimality conditions to (4.2) and (4.3) with $\bar{u}=\bar{x}$ and $\bar{v}=F(\bar{x})$ to obtain a local uniqueness result. First we define the index sets

$$\begin{aligned} I &= \{i \,|\, F_i(\bar{x})>0\} = \{i \,|\, \bar{v}_i>0\} \\ J &= \{i \,|\, \bar{x}_i>0\} = \{i \,|\, \bar{u}_i>0\} \\ K &= \{i \,|\, F_i(\bar{x})=0,\ \bar{x}_i=0\} = \{i \,|\, \bar{v}_i=0,\ \bar{u}_i=0\} \end{aligned} \tag{4.5}$$

We further define submatrices of the Jacobian $\nabla F(\bar{x})$ such as $\nabla F_J(\bar{x})$ as that submatrix of $\nabla F(\bar{x})$ with rows $\nabla F_i(\bar{x})$, $i \in J$. We also define $\nabla_K F_J(\bar{x})$ as that submatrix of $\nabla F(\bar{x})$ with elements $\dfrac{\partial F_i(\bar{x})}{\partial x_j}$, $i \in J$, $j \in K$, and $\nabla^2 F_i(x)$ as the $n \times n$ Hessian of F_i at x. If we let $L(x,u,v)$ denote the Lagrangian of the minimization problem (4.2), that is

$$L(x,u,v) = xF(x) - uF(x) - vx$$

with (u,v) in R^{2n}, then the $n \times n$ Hessian $\nabla_{11}L(x,u,v)$ of $L(x,u,v)$ with respect to x is given by

$$\nabla_u L(x,u,v) = \nabla F(x) + \nabla F(x)^T + \sum_{i=1}^{n} (x_i - u_i)\ \nabla^2 F_i(x)$$

and consequently if we set $x = \bar{x}$, $u = \bar{u} = \bar{x}$ and $v = \bar{v} = F(\bar{x})$ we have that

$$\nabla_u L(\bar{x},\bar{u},\bar{v}) = \nabla F(\bar{x}) + \nabla F(\bar{x})^T$$

Note that this is an expression that does not contain second partial derivatives of F. Now application of McCormick's second order sufficient optimality conditions [8,3] to the nonlinear program (4.2) gives upon noting that $\bar{u} = \bar{x}$ and $\bar{v} = F(\bar{x})$

$$\left.\begin{aligned} \nabla F_J(\bar{x})x &= 0 \\ \nabla F_K(\bar{x})x &\geqq 0 \\ x_I &= 0 \\ x_K &\geqq 0 \\ x &\neq 0 \end{aligned}\right\rangle \Longrightarrow x \nabla F(\bar{x}) x > 0$$

which is equivalent to

$$\left.\begin{aligned} \nabla_J F_J(\bar{x})x_J + \nabla_K F_J(\bar{x})x_K &= 0 \\ \nabla_J F_K(\bar{x})x_J + \nabla_K F_K(\bar{x})x_K &\geqq 0 \\ x_K &\geqq 0 \\ (x_J,\ x_K) &\neq 0 \end{aligned}\right\rangle \Longrightarrow x_K(\nabla_J F_K(\bar{x})x_J + \nabla_K F_K(\bar{x})x_K) > 0 \qquad (4.6)$$

We thus have the following sufficient condition for the local uniqueness of $\bar{x}$ as a solution of (4.2) or equivalently of (4.1).

4.1 Theorem (Sufficient conditions for local uniqueness of solutions of the nonlinear complementarity problem) If a point $\bar{x}$ in R^n satisfies the conditions (4.4)-(4.6) and if F is twice differentiable at $\bar{x}$, then $\bar{x}$ is a locally unique solution of the nonlinear complementarity problem (4.1).

The following corollaries and theorems follow from Theorem 4.1 in a very similar manner as the corresponding corollaries and theorems of Section 3 follow from Theorem 3.1 and hence we shall omit their proofs.

4.2 Corollary (Sufficient conditions for local uniqueness of nondegenerate solutions of the nonlinear complementarity problem) A nondegenerate solution $\bar{x}$ of the nonlinear complementarity problem (4.1) (that is K in (4.5) is empty) is locally unique if F is twice differentiable at $\bar{x}$ and $\nabla_J F_J(\bar{x})$ is nonsingular, where J is defined in (4.5).

4.3 Corollary (Sufficient conditions for local uniqueness of the origin as a solution of the nonlinear complementarity problem) The origin in R^n is a locally unique solution of the nonlinear complementarity problem (4.1) if F is twice differentiable at the origin, $F(0) \geqq 0$ and there exists no x_K such that

$$\nabla_K F_K(0) x_K \geqq 0, \quad 0 \neq x_K \geqq 0, \quad x_K \nabla_K F_K(0) x_K = 0 \tag{4.7}$$

where

$$K = \{i \mid F_i(0)=0\}$$

If F is coninuous at the origin and $F(0) > 0$ then the origin is a locally unique solution of the nonlinear complementarity problem.

4.4 Corollary Let $\bar{x}$ be a solution of the nonlinear complementarity problem (4.1) and let F be twice differentiable at $\bar{x}$. If $\nabla F(\bar{x})$ has nonsingular principal submatrices then $\bar{x}$ is a locally unique solution of the nonlinear complementarity problem (4.1).

Megiddo and Kojima [13, Proposition 3.6] show that each of the sufficient conditions for global uniqueness of Cottle, Karamardian and Moré [1,5,14,15] implies one of their conditions [13, Theorem 3.4], which in turn imples that $\nabla F(\bar{x})$ has positive principal subdeterminants. Hence under the assumption that F is twice differentiable at $\bar{x}$ all these conditions imply the weaker local uniqueness condition of Corollary 4.4 that $\nabla F(\bar{x})$ has nonsingular principal submatrices.

In [12] local uniqueness of a solution $\bar{x}$ to the nonlinear complementarity problem (4.1) is implied by a nondegeneracy assumption, that is $\bar{x} + F(\bar{x}) > 0$ and nonsingularity of the principal submatrices of $\nabla F(\bar{x})$. Corollary 4.4 replaces the nondegeneracy assumption by the twice differentiability assumption of F at $\bar{x}$.

4.5 Theorem (Sufficient conditions for local uniqueness of solutions of the nonlinear complementarity problem) Let $\bar{x}$ be a solution of the nonlinear complementarity problem (4.1), let F be twice differentiable at $\bar{x}$, let J and K be defined as in (4.5), let

$$\text{Columns of } \begin{pmatrix} \nabla_J F_J(\bar{x}) \\ \nabla_J F_K(\bar{x}) \end{pmatrix} \text{ be linearly independent} \tag{4.9}$$

and for each partition $\{K_0, K_1\}$ of K let the following system have a solution (u_J, u_{K_0}, u_{K_1})

$$\begin{aligned} \nabla_J F_J(\bar{x})^T u_J + \nabla_J F_{K_0}(\bar{x})^T u_{K_0} + \nabla_J F_{K_1}(\bar{x})^T u_{K_1} &= 0 \\ \nabla_{K_1} F_J(\bar{x})^T u_J + \nabla_{K_1} F_{K_0}(\bar{x})^T u_{K_0} + \nabla_{K_1} F_{K_1}(\bar{x})^T u_{K_1} &< 0 \\ u_{K_0} &> 0 \end{aligned} \tag{4.10}$$

Then $\bar{x}$ is a locally unique solution of the nonlinear complementarity problem (4.1).

4.6 Corollary Let $\bar{x}$ solve the nonlinear complementarity problem (4.1), let F be twice differentiable at $\bar{x}$ and let J and K be defined by (4.5). If $\nabla_J F_J(\bar{x})$ and the principal submatrices of the Schur complement

$$\nabla_K F_K(\bar{x}) - \nabla_J F_K(\bar{x}) \nabla_J F_J(\bar{x})^{-1} \nabla_K F_J(\bar{x})$$

are nonsingular then $\bar{x}$ is a locally unique solution of the nonlinear complementarity problem (4.1).

4.7 Corollary Let $\bar{x}$ solve the nonlinear complementarity problem (4.1), let F be twice differentiable at $\bar{x}$, let J and K be defined by (4.5) and let $H = J \cup K$. If all the principal submatrices of $\nabla_H F_H(\bar{x})$ containing $\nabla_J F_J(\bar{x})$ are nonsingular then $\bar{x}$ is a locally unique solution of the nonlinear complementarity problem (4.1).

We note that Kojima's Theorem 5-6(b) [7] shows that a certain function is locally one-to-one which in turn guarantees that a solution to the nonlinear complementarity problem is locally unique. Kojima's theorem requires that all the principal subdeterminants of $\nabla_H F_H(\bar{x})$ containing $\nabla_J F_J(\bar{x})$, as defined in Corollary 4.7, be positive. This is a stronger requirement than that of Corollary 4.7 which requires that these same subdeterminants be nonzero.

4.8 Theorem Let $\bar{x}$ be a solution of the nonlinear complementarity problem (4.1), let J and K be defined by (4.5) and let $\nabla_J F_J(\bar{x})$ be nonsingular. Then $\bar{x}$ is a locally unique solution of (4.1) if $x_K = 0$ is the only solution of the following linear complementarity problem

$$\begin{aligned} (\nabla_K F_K(\bar{x}) - \nabla_J F_K(\bar{x})\ \nabla_J F_J(\bar{x})^{-1} \nabla_K F_J(\bar{x}))\ x_K &\geqq 0 \\ x_K &\geqq 0 \\ x_K(\nabla_K F_K(\bar{x}) - \nabla_J F_K(\bar{x})\ \nabla_J F_J(\bar{x})^{-1} \nabla_K F_J(\bar{x}))\ x_K &= 0. \end{aligned}$$

ACKNOWLEDGEMENT

I am grateful to Ken Kaneko and Wayne Hallman for their comments on this paper.

References

1. R. W. Cottle: "Nonlinear programs with positively bounded Jacobians", SIAM Journal on Applied Mathematics 14, 1966, 147-158.

2. R. W. Cottle & G. B. Dantzig: "Complementary pivot theory of mathematical programming", Linear Algebra and Its Applications, 1, 1968, 103-125.

3. A. V. Fiacco & G. P. McCormick: "Nonlinear programming: sequential unconstrained minimization techniques", Wiley, New York, 1968.

4. I. Kaneko: "The number of solutions of a class of linear complementarity problems", forthcoming in Mathematical Programming.

5. S. Karamardian: "The nonlinear complementarity problem with applications I and II", Journal of Optimization Theory and Applications 4, 1969, 87-98 and 167-181.

6. W. Karush: "Minima of functions of several variables with inequalities as side conditions", Master of Science Dissertation, Department of Mathematics, University of Chicago, Chicago, December 1939.

7. M. Kojima: "Studies on piecewise-linear approximations of piecewise - C^1 mappings in fixed points and complementarity theory", Mathematics of Operations Research 3, 1978, 17-36.

8. G. P. McCormick: "Second order conditions for constrained minima", SIAM Journal on Applied Mathematics 15, 1967, 641-652.

9. A. Majthay: "Optimality conditions for quadratic programming", Mathematical Programming 1, 1971, 359-365.

10. O. L. Mangasarian: "Nonlinear programming", McGraw-Hill, New York, 1969.

11. O. L. Mangasarian: "Uniqueness of solution in linear programming", University of Wisconsin, Computer Sciences Department Technical Report #316, February 1978, forthcoming in Linear Algebra and Its Applications.

12. O. L. Mangasarian: "Equivalence of the complementarity problem to a system of nonlinear equations", SIAM Journal on Applied Mathematics 31, 1976, 89-92.

13. N. Megiddo & M. Kojima: "On the existence and uniqueness of solutions in nonlinear complementarity theory", Mathematical Programming 12, 1977, 110-130.

14. J. J. Moré: "Coercivity conditions in nonlinear complementarity problems", SIAM Review 16, 1974, 1-15.

15. J. J. Moré: "Classes of functions and feasibility conditions in nonlinear complementarity problems", Mathematical Programming 6, 1974, 327-338.

16. K. G. Murty: "On the number of solutions to the complementarity problem and spanning properties of complementary cones", Linear Algebra and Its Applications 5, 1972, 65-108.

17. S. M. Robinson: "Strongly regular generalized equations", University of Wisconsin, Mathematics Research Center Technical Report #1877, September 1978.