ITERATIVE SOLUTION OF LINEAR PROGRAMS

by

O. L. Mangasarian

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O. L. Mangasarian[3]

ABSTRACT

By perturbing a linear program to a quadratic program it is possible to solve the latter in its dual variable space by iterative techniques such as successive over-relaxation (SOR) methods. This provides a solution to the original linear program.

Key Words: Linear program, quadratic program, linear complementarity problem, SOR methods, relaxation methods.

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[3] Computer Sciences Department and Department of Industrial Engineering, University of Wisconsin, 1210 West Dayton Street, Madison, Wisconsin 53706.

1. Introduction

We shall be concerned here with <u>iterative</u> methods for solving the linear program

$$\text{Minimize } p^T x \qquad \text{subject to } Ax \geq b \qquad (1)$$

where p and b are given vectors in R^n and R^m respectively, and A is a given $m \times n$ real matrix with no rows that are identically zero. The most popular methods for solving this problem are <u>direct</u> pivotal methods such as the simplex method and its variants (Refs. 1-2). However, more recently there have been a number of iterative procedures proposed (Refs. 3-8). Some of these (Refs. 3-5) consist of an iterative method for finding a feasible point of the Karush-Kuhn-Tucker inequalities that constitute the optimality conditions of the linear program (1). Others (Refs. 6 and 8) consist of minimizing a nonsmooth reformulated problem or (Ref. 7) finding stationary points of an augmented Lagrangian. Our approach here is different. We consider the following quadratic programming perturbation of (1)

$$\text{Minimize } \frac{\varepsilon}{2} x^T x + p^T x \qquad \text{subject to } Ax \geq b \qquad (2)$$

It has been shown (Ref. 9) that (2) has a unique solution $\bar{x}$, for all ε in $(0,\bar{\varepsilon}]$ for some $\bar{\varepsilon} > 0$, which is independent of ε and which also solves the linear program (1). By working in the dual variable space of (2) we can utilize the iterative techniques developed in (Ref. 10) for solving the symmetric linear complementarity problem to solve (2). It turns out that a sufficient condition

for the iterative method to lead to a solution is that the constraints of the linear program (1) be stable (Ref. 11) which in this case means that they satisfy the Slater constraint qualification (Ref. 12). It is interesting to note that in order to obtain convergence of the present iterative method use is made of various recent results concerning linear programs, namely (a) nonlinear perturbation of linear programs (Ref. 9) which originated with the uniqueness characterization of linear programming solutions (Ref. 13), (b) stability of systems of linear inequalities (Ref. 11) and (c) general sufficient conditions for the convergence of iterative techniques for the solution of the symmetric linear complementarity problem (Ref. 10).

The outline of the paper is as follows. In Section 2 we describe how our proposed iterative procedure is applied to a general quadratic program with a positive definite Hessian. This procedure may be used also in finding the projection of a point on a polytope. In Section 3 we adapt the procedure of Section 2 specifically for solving the linear program (1) by solving the perturbed quadratic program (2). In Section 4 we give an algorithm for the solution of more general linear programs. In Section 5 we present some numerical results which show that when the perturbation and relaxation parameters are properly chosen our proposed iterative method is competitive with the revised simplex method (Refs. 1-2) and may even be more robust in that it can solve problems for which a revised simplex code fails.

We briefly describe now the notation used in this paper. All matrices and vectors are real. For an $m \times n$ matrix A, row i is

denoted by A_i and the element in row i and column j by A_{ij}. For x in the real n-dimensional Euclidean space R^n, element j is denoted by x_j. The superscript T denotes the transpose. All vectors are column vectors unless transposed. Superscripts such as K^i, u^i refer to specific matrices and vectors and usually denote iteration numbers. If u is in R^m, u_+ denotes the vector in R^m with elements

$$(u_+)_i = \max \{0, u_i\}, \quad i = 1, \ldots, m$$

The vector e will denote a vector of ones in R^m or R^n and I denotes the $m \times m$ identity matrix. The Euclidean norm $(x^T x)^{\frac{1}{2}}$ of a vector x in R^n will be denoted by $\|x\|$.

2. Iterative Solution of the Quadratic Programming Problem

We shall consider in this section the quadratic program

$$\text{Minimize } \frac{1}{2} x^T Q x + p^T x \qquad \text{subject to } Ax \geq b \tag{3}$$

where p, b, Q and A are respectively a vector in R^n, a vector in R^m, a real symmetric positive definite $n \times n$ matrix and a real $m \times n$ matrix. We shall develop an iterative algorithm for solving the dual of (3) (Ref. 12)

$$\underset{x,u}{\text{Maximize}} \ -\frac{1}{2} x^T Q x + b^T u \qquad \text{subject to } Qx - A^T u + p = 0, u \geq 0 \tag{4}$$

which under the positive definite assumption on Q is, upon eliminating x, equivalent to

$$\underset{u \geq 0}{\text{Minimize}} \ \frac{1}{2} u^T A Q^{-1} A^T u - (b + A Q^{-1} p)^T u \tag{5}$$

The proposed iterative procedure will solve (5), and from the solution u the solution x of (3) is then obtained from

$$x = Q^{-1}(A^T u - p) \tag{6}$$

Because the proposed procedure involves the inversion of the matrix Q, it is not, in general, a suitable procedure for solving (3) with a general Q. However, for certain applications such as those requiring the solution of (2), Q is the diagonal matrix εI which is easily inverted. Another such problem is that of projecting a point c in R^n on the polytope $\{x | Ax \geq b\}$ using the Euclidean norm, in which case $Q = I$ and $p = -c$.

We are now ready to apply the results of Ref. 10 to solve (5) iteratively once we realize that, since $AQ^{-1}A^T$ is positive semi-definite, (5) is equivalent to the following symmetric linear complementarity problem

$$v = AQ^{-1}A^Tu - (b+AQ^{-1}p) \geq 0, \; u \geq 0, \; u^Tv = 0 \tag{7}$$

We shall use the following special case of Algorithm 2.1 and Remark 2.4 of Ref. 10.

__Algorithm 2.1__ Let u^0 be an arbitrary nonnegative vector in R^m. Having u^i compute u^{i+1} as follows

$$u^{i+1} = (u^i - \omega E(AQ^{-1}A^Tu^i - b - AQ^{-1}p + K^i(u^{i+1} - u^i)))_+ \tag{8}$$

where E is a positive diagonal matrix, K^i is either the strict lower triangular part L or the strict upper triangular part U of $AQ^{-1}A^T$, G is the diagonal of $AQ^{-1}A^T$ and

$$0 < \omega < 2 / \max_{G_{jj}>0} G_{jj}E_{jj} \tag{9}$$

Note that for computing purposes u^{i+1} is computed in the order $u_1^{i+1}, \; u_2^{i+1}, \ldots \ldots, u_m^{i+1}$ when $K^i = L$, and in the reverse of that order when $K^i = U$.

The following theorem is a direct consequence of Theorem 2.1 of Ref. 10.

<u>Theorem 2.1</u> Let Q be symmetric and positive definite. Then, each accumulation point u of the sequence $\{u^i\}$ generated by Algorithm 2.1 solves (5) and the corresponding x determined by (6) is the unique solution of (3).

This theorem does not guarantee the existence of an accumulation of the sequence $\{u^i\}$ whereas the following one does under the slightly more demanding condition of a constraint qualification.

<u>Theorem 2.2</u> Let Q be symmetric and positive definite, and let the constraints of (3) satisfy the Slater constraint qualification, that is $A\hat{x} > b$ for some $\hat{x}$ in R^n. Then the sequence $\{u^i\}$ generated by Algorithm 2.1 is bounded and has at least one accumulation point. Each accumulation point u of $\{u^i\}$ solves (5) and the corresponding x determined by (6) is the unique solution of (3).

<u>Proof</u> Because $A\hat{x} > b$, there exists a $\delta > 0$ such that the set $\{x|Ax \geq b+\delta e\}$ is nonempty. Let $\tilde{x}$ be the solution of the quadratic program

$$\text{Minimize } \frac{1}{2} x^T Q x + p^T x \qquad \text{subject to } Ax \geq b + \delta e$$

A solution $\tilde{x}$ to this problem exists because Q is positive definite, and together with a $\tilde{u}$ in R^m satisfies the following Karush-Kuhn-Tucker conditions (Ref. 12)

$$Q\tilde{x} + p - A^T\tilde{u} = 0, \quad \tilde{u} \geq 0, \quad A\tilde{x} \geq b + \delta e, \quad \tilde{u}^T(A\tilde{x}-b-\delta e) = 0$$

Hence

$$(AQ^{-1}A^T)\tilde{u} - (b+AQ^{-1}p) \geqq \delta e > 0$$

By Theorem 2.2 and condition (10) of Ref. 10 the sequence $\{u^i\}$ is

bounded and has at least one accumulation point. By Theorem 2.1

above, each accumulation of $\{u^i\}$ solves (5) and the corresponding

x determined by (6) is the unique solution of (3). ☐

Remark 2.1 The Slater constraint qualification is equivalent to

the stability condition (Ref. 11) that for each d in R^m there

exists (x,ε) in R^{m+1} satisfying

$$Ax \geqq b + \varepsilon d, \quad \varepsilon > 0.$$

3. Iterative Solution of the Linear Programming Problem

We now turn our attention back to the linear program (1) and state a result which is a direct consequence of Theorem 1 of Ref. 9.

Theorem 3.1 Let the linear program (1) have a solution. Then there exists a real positive number $\bar{\varepsilon}$ such that for each ε in the interval $(0,\bar{\varepsilon}]$ the unique solution $\bar{x}$ of (2) is independent of ε and is also a solution of the linear program (1).

From the proof of Theorem 1 of Ref. 9 we can obtain an a posteriori upper bound on $\bar{\varepsilon}$ of the above theorem, namely $\bar{\varepsilon} \leq \frac{1}{\bar{\gamma}}$ where $\bar{\gamma}$ is the positive optimal Lagrange multiplier associated with the last constraint of the problem

$$\text{Minimize } \frac{1}{2} x^T x \qquad \text{subject to } Ax \geq b, \ p^T x \leq \bar{\theta}$$

and where $\bar{\theta}$ is the minimum of problem (1). If $\bar{\gamma} = 0$ then $\bar{\varepsilon}$ can be any nonnegative number. There is also another interesting interpretation of ε (Ref. 14). If we take the dual (Ref. 12) of the convex quadratic program (2) we obtain the problem

$$\text{Maximize } b^T u - \frac{1}{2\varepsilon} \| A^T u - p \|^2 \qquad \text{subject to } u \geq 0$$

This is precisely the exterior penalty problem associated with the dual linear program of (1)

$$\text{Maximize } b^T u \qquad \text{subject to } A^T u = p, \ u \geq 0$$

with penalty parameter $\alpha = \frac{1}{\varepsilon}$. Results of ordinary exterior penalty

methods (Ref. 15) require that $\alpha \to \infty$ and hence $\varepsilon \to 0$. However, sharper results that take advantage of linearity of the problem (Ref. 16) require merely that $\alpha \geq \bar{\alpha}$ for some $\bar{\alpha} > 0$ or equivalently $\varepsilon \leq \bar{\varepsilon}$ for some $\bar{\varepsilon} > 0$. These sharper results correspond to the cited results of (Ref. 9).

We can now combine Theorem 3.1 with Algorithm 2.1 to solve problem (2) with any ε in $(0,\bar{\varepsilon}]$ and thus obtain a solution to the linear program (1). In particular we set in Algorithm 2.1, $Q = \varepsilon I$, $E = \varepsilon D^{-1}$ where D is the diagonal of AA^T, and obtain the following.

Algorithm 3.1   Choose a positive number ε and any nonnegative vector u^0 in R^m. Having u^i compute u^{i+1} as follows

$$u^{i+1} = (u^i - \omega D^{-1}(AA^T u^i - Ap - \varepsilon b + K^i(u^{i+1} - u^i)))_+ \qquad (10)$$

where D is the diagonal of AA^T that is

$$D_{jj} = A_j(A_j)^T, \; j = 1, \ldots, m$$

K^i is either the strict lower triangular part L or the strict upper triangular part U of AA^T, and $0 < \omega < 2$.

Combining Theorems 3.1, 2.1 and 2.2 we obtain the following two convergence theorems for Algorithm 3.1.

Theorem 3.2   There exists a real positive number $\bar{\varepsilon}$ such that for each ε in the interval $(0,\bar{\varepsilon}]$, each accumulation point u of the

sequence $\{u^i\}$ generated by Algorithm 3.1 solves

$$\text{Minimize } \frac{1}{2} u^T A A^T u - (\varepsilon b + Ap)^T u \qquad (11)$$
$$u \geq 0$$

and the corresponding x which is independent of ε and determined by

$$x = \varepsilon^{-1}(A^T u - p) \qquad (12)$$

is the unique solution of (2) and is also a solution of the linear program (1).

We again note that Theorem 3.2 does not guarantee the existence of an accumulation point whereas the following theorem does under the additional assumption that the constraints are stable.

Theorem 3.3 Let $A\hat{x} > b$ for some $\hat{x}$ in R^n. There exists a real positive number $\bar{\varepsilon}$ such that for each ε in the interval $(0,\bar{\varepsilon}]$, the sequence $\{u^i\}$ generated by Algorithm 3.1 is bounded and has at least one accumulation point. Each accumulation point u of $\{u^i\}$ solves (11) and the corresponding x which is independent of ε and determined by (12) is the unique solution of (2) and is also a solution of the linear program (1).

4. More General Linear Programs

We outline in this part of the paper the corresponding results for the case of more general constraints and omit the proofs which are similar to those of Ref. 10 and of Section 3 of this paper. In particular we consider here instead of (1) the linear program

$$\text{Minimize } p^T x \qquad \text{subject to } Ax \geq b, \; Cx = d \qquad (13)$$

where the additional equality constraint is specified through the $k \times n$ matrix C and the vector d in R^k. We note that this problem encompasses linear programs of a very general type. We shall again assume, for simplicity, that no rows of A or C are identically zero and associate with (13) the following quadratic program for some positive ε

$$\text{Minimize } \frac{\varepsilon}{2} x^T x + p^T x \qquad \text{subject to } Ax \geq b, \; Cx = d \qquad (14)$$

and the corresponding dual problem

$$\underset{\substack{(u,v) \in R^{m+k} \\ u \geq 0}}{\text{Minimize}} \; \frac{1}{2} \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} A \\ C \end{pmatrix} (A^T \; C^T) \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} Ap + \varepsilon b \\ Cp + \varepsilon d \end{pmatrix}^T \begin{pmatrix} u \\ v \end{pmatrix} \qquad (15)$$

where the relation between x, u and v is given by

$$x = \varepsilon^{-1} (A^T u + C^T v - p) \qquad (16)$$

The iterative procedure associated with (15) is as follows.

Algorithm 4.1 Choose a positive number ε , an arbitrary non-negative vector u^0 in R^m and an arbitrary vector v^0

in R^k . Having $\begin{pmatrix} u^i \\ v^i \end{pmatrix}$ compute $\begin{pmatrix} u^{i+1} \\ v^{i+1} \end{pmatrix}$ as follows

$$\begin{pmatrix} u^{i+1} \\ v^{i+1} \end{pmatrix} = \left(\begin{pmatrix} u^i \\ v^i \end{pmatrix} - \omega D^{-1} \left(\begin{pmatrix} A \\ C \end{pmatrix} (A^T \ C^T) \begin{pmatrix} u^i \\ v^i \end{pmatrix} - \begin{pmatrix} Ap \\ Cp \end{pmatrix} - \varepsilon \begin{pmatrix} b \\ d \end{pmatrix} + K^i \begin{pmatrix} u^{i+1} - u^i \\ v^{i+1} - v^i \end{pmatrix} \right) \right)_*$$

where D is the diagonal of $\begin{pmatrix} A \\ C \end{pmatrix} (A^T C^T)$, K^i is the strictly lower or

upper triangular part of the same matrix, $0 < \omega < 2$ and

$$\begin{pmatrix} u \\ v \end{pmatrix}_* = \begin{pmatrix} u_+ \\ v \end{pmatrix} .$$

Theorem 4.1 There exists a real positive number $\bar{\varepsilon}$ such that for

each ε in the interval $(0,\bar{\varepsilon}]$, each accumulation point $\begin{pmatrix} u \\ v \end{pmatrix}$ of the

sequence $\left\{ \begin{pmatrix} u^i \\ v^i \end{pmatrix} \right\}$ generated by Algorithm 4.1 solves (15) and the

corresponding x determined by (16), which is independent of ε , is

the unique solution of (14) and in addition is a solution of the linear

program (13).

Theorem 4.2 If in addition to the assumptions of Theorem 4.1 the

constraints of (13) are stable, that is there exists an $\hat{x}$ in R^n

such that $A\hat{x} > b$, $C\hat{x} = d$, and the rows of C are linearly independent,

then for ε in $(0,\bar{\varepsilon}]$ the sequence $\left\{ \begin{pmatrix} u^i \\ v^i \end{pmatrix} \right\}$ of Algorithm 4.1 is

bounded and hence has at least one accumulation $\begin{pmatrix} u \\ v \end{pmatrix}$. Each such

accumulation point solves (15) and the corresponding x which is

independent of ε and determined by (16) is the unique solution of (14) and is also a solution of the linear program (13).

5. Numerical Results

Some test results were obtained using the iterative SOR Algorithm 3.1 starting with $u^0 = 0$ to solve the linear program (1) on the Argonne National Laboratory IBM 3033 computer running under VM release 5 PLC 11 and the Conversational Monitoring System. For comparative purposes a revised simplex code was also used (Ref. 17). The test problems were generated as follows. The matrix A was a fully dense matrix with random elements A_{ij} uniformly distributed in the interval $[-100,400]$. The vectors b and p were chosen such that

$$b_i = \begin{cases} \sum_{j=1}^{n} A_{ij} & \text{if } \sum_{j=1}^{n} A_{ij} > 0 \\ \\ -1 + 2\sum_{j=1}^{n} A_{ij} & \text{if } \sum_{j=1}^{n} A_{ij} \leq 0 \end{cases}$$

$$i = 1, \ldots, m$$

and

$$p_j = \sum_{i \in J} A_{ij} \quad \text{where} \quad J = \{i \mid \sum_{j=1}^{n} A_{ij} > 0\}, \, j = 1, \ldots, n$$

These choices for b and p made the point $x = 2e$ satisfy the constraint qualification $A(2e) > b$ and the point $x = e$ primal optimal with a minimum value of $\sum_{j=1}^{n} \sum_{i \in J} A_{ij}$. A dual optimal variable is given by $u_i = 1$ for $i \in J$ and $u_i = 0$ for $i \notin J$. Results for six cases are summarized in Table 1. Note that for cases 1, 2, 3 and 5 because $n > m$ the linear program (1) does not have a unique

solution (Ref. 13). Cases 4 and 6 have a unique solution if and only if the matrix with rows A_i, $i \epsilon J$, has linearly independent columns (Ref. 13). Thus the accuracy of the solutions described in Table 1 is measured by (a) the number of figures in agreement between the calculated objective function and the theoretical minimum: $\sum_{j=1}^{n} \sum_{i \epsilon J} A_{ij}$, and (b) the ∞-norm of the infeasibility of the calculated primal solution x, that is the maximum $\sum_{1 \leq i \leq m} \sum_{j=1} (b_i - A_{ij}x_j)_+$.

We now make the following observations regarding Table 1:

(i) Except for cases 4 and 6, the computing times for the two methods are quite similar. For case 6, the revised simplex method failed and for case 4 the iterative method took 2.5 times as long as the revised simplex method.

(ii) The revised simplex solutions, when obtained, are more accurate than those of the iterative method.

(iii) The iterative method is more robust in the sense that it never fails to provide some answer when the constraints are stable.

(iv) The values of the perturbation parameter ϵ and the relaxation parameter were obtained after some experimenting, but are not necessarily optimal. Table 2 gives a typical range of answers obtained by the iterative method for various values of ϵ and ω which led to the values $\epsilon = 10^5$ and $\omega = 0.5$ given in Table 1 for the case of m = 250 and n = 100.

TABLE 1

Numerical Results for Min $p^T x$ s.t. $Ax \geqq b$ where A is an m×n matrix

Case	m	n	Revised Simplex Method			ε	ω	Iterative Method			
			Seconds of Virtual Machine Time	Accuracy				No. of Iterations	Seconds of Virtual Machine Time	Accuracy	
				No. of Correct Figures in Objective Function	∞-Norm of Primal Infeasibility					No. of Correct Figures in Objective Function	∞-Norm of Primal Infeasibility
1	10	100	<2	15	0.324×10^{-11}	10^3	0.8	136	<2	14	0.374×10^{-10}
2	50	200	6	14	0.902×10^{-10}	10^4	0.8	862	<2	9	0.365×10^{-4}
3	50	850	20	14	0	10^5	0.1	642	20	12	0.257×10^{-5}
4	100	98	28	15	0.218×10^{-9}	10^5	0.5	1300	70	4	0.400×10^{-5}
5	100	850	94	16	0.144×10^{-8}	10^6	0.1	915	106	13	0.969×10^{-7}
6	250	100	700	Failed: Problem Declared Unbounded		10^5	0.5	1114	130	10	0.484×10^{-6}

(v) The values of ε in Table 1 lie in the range $mn \leq \varepsilon \leq 12mn$.

(vi) Even though the relaxation factors ω in Table 1 are all less than 1, we still (Ref. 18, p. 215) refer to our iterative method as an over-relaxation rather than an under-relaxation method.

The above numerical results indicate that the proposed iterative method is a viable one and when ε and ω are properly chosen it is competitive with the revised simplex method. These parameters may be chosen experimentally by making a few short test runs starting with $\varepsilon \cong 5mn$ and $\omega \cong 0.5$ and picking those values for which $\| x^{i+1} - x^i \|$ approach zero fastest. The main advantages of the method are its robustness, simplicity and ability to handle large problems.

TABLE 2

Numerical Results for Iterative Method for Min $p^T x$ s.t. $Ax \geq b$, $m = 250$, $n = 100$

ε	ω	No. of Iterations	Seconds of Virtual Machine Time	Accuracy	
				No. of Figures in Objective Function	∞-Norm of Primal Infeasibility
10^4	0.5	500	62	5	0.860×10^{-2}
10^4	0.2	500	64	4	0.196×10^{-1}
10^4	0.8	500	62	4	0.436×10^{-1}
10^5	0.5	500	64	5	0.725×10^{-2}
10^6	0.5	500	64	4	0.164
10^5	0.2	500	64	4	0.116
10^5	0.8	500	62	4	0.859×10^{-1}
10^5	0.5	1114	130	10	0.484×10^{-6}

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TABLE 1

Numerical Examples: Min $p^T x$ s.t. $Ax \geq b$ where A is an m×n Matrix

Case No. & Formulation P:Primal D:Dual	m	n	Revised Simplex Method No. of Steps Phase I+Phase II	Virtual Time Min:Sec	Accuracy No. of Correct Figures of Min Value	∞-Norm of Infeasibility	Iterative SOR Method ε	ω	No. of Iterations	Virtual Time Min:Sec	Accuracy No. of Correct Figure of Min Value	∞-Norm of Infeasibility	$\frac{\varepsilon}{m}$	$\frac{\varepsilon}{n}$
1 P	100	20	83+19	2:20	6	$.260 \times 10^{-12}$	100	1.2	99	0:46	6	$.673 \times 10^{-3}$	1.0	1.0
1 D	100	20	27+29	0:14	7	0	20	.5	220	0:42	7	$.796 \times 10^{-1}$		1.0
2 P	100	80	113+50	5:53	7	$.243 \times 10^{-11}$	100	.5	708	5:01	6	$.275 \times 10^{-2}$	1.0	
2 D	100	80	200+0	5:10	Failed. Declared Infeasible		115	.4	500	5:47	5	.925		1.44
3 P	200	50	177+70	23:07	7	$.152 \times 10^{-10}$	1000	1.2	363	10:05	6	$.689 \times 10^{-3}$	5.0	
3 D	200	50	84+89	2:59	7	0	100	.5	481	6:52	6	$.368 \times 10^{-1}$		2.00
4 P	300	50	400+40	86:20	6	$.984 \times 10^{-12}$	3000	1.2	246	16:11	6	$.701 \times 10^{-3}$	10.0	
4 D	300	50	72+164	5:09	6	0	100	.5	268	6:14	5	$.400 \times 10^{-1}$		2.00
5 D	400	50	76+169	6:44	6	0	100	.5	204	6:41	4	$.277 \times 10^{-1}$		2.00
6 P	20	100	17+11	0:10	5*	$.568 \times 10^{-12}$	200	.5	45	0:07	6	$.887 \times 10^{-4}$	10.0	
7 P	80	100	77+162	6:12	7	$.230 \times 10^{-11}$	600	.8	63	1:00	8	$.141 \times 10^{-2}$	7.5	
8 P	50	200	41+77	2:10	5*	$.121 \times 10^{-9}$	200	.5	45	0:46	7	$.112 \times 10^{-2}$	4.0	
9 P	50	300	31+59	2:23	5*	$.270 \times 10^{-10}$	300	.8	24	1:05	6	$.156 \times 10^{-5}$	6.0	
10 P	50	400	34+54	2:51	4*	$.475 \times 10^{-6}$	400	.8	23	1:26	6	$.586 \times 10^{-6}$	8.0	

Beale's Example: Min $p^T x$ s.t. $Ax \geq b$, $x \geq 0$ where A is an m×n Matrix

Case No. & Formulation	m	n	No. of Steps Phase I+Phase II	Virtual Time Min:Sec			ε	ω	No. of Iterations	Virtual Time Min:Sec	Accuracy No. of Correct Figure of Min Value	∞-Norm of Infeasibility	$\frac{\varepsilon}{m}$	$\frac{\varepsilon}{n}$
11 P	3	4	10000	0:53	Failed. Reached Max Iteration Number		.6	1.9	487	0:02	5	$.126 \times 10^{-4}$	.200	

*Problem declared unbounded